The relativistic entrainment matrix of a superfluid nucleon-hyperon mixture at zero temperature

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We calculate the relativistic entrainment matrix Y_{ik} at zero temperature for nucleon-hyperon mixture composed of neutrons, protons, Λ- and Σ^--hyperons, as well as of electrons and muons. This matrix is analogous to the entrainment matrix (also termed mass-density matrix or Andreev-Bashkin matrix) of non-relativistic theory. It is an important ingredient for modelling the pulsations of massive neutron stars with superfluid nucleon-hyperon cores. The calculation is done in the frame of the relativistic Landau Fermi-liquid theory generalized to the case of superfluid mixtures; the matrix Y_{ik} is expressed through the Landau parameters of nucleon-hyperon matter. The results are illustrated with a particular example of the σ-ω-ρ mean-field model with scalar self-interactions. Using this model we calculate the matrix Y_{ik} and the Landau parameters. We also analyze stability of the ground state of nucleon-hyperon matter with respect to small perturbations.

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I. INTRODUCTION

An analysis of electromagnetic [1, 2, 3, 4] and (in the future) gravitational [5, 6, 7] radiation from pulsating neutron stars can shed light on the properties of superdense matter in their interiors. The most interesting is the question about the composition of massive neutron-star cores (nucleons? hyperons? quarks? exotic matter?) as well as about the properties of superfluid baryon matter (the dependence of baryon critical temperatures on density, the type of pairing of various baryon species).

To interpret correctly the observational data it is necessary to have realistic theoretical models of pulsating neutron stars. For that, one needs to formulate a hydrodynamics which can be used to describe pulsations. Clearly, the ordinary relativistic hydrodynamics (see, e.g., [8]), describing a liquid composed of identical particles, is not suitable for this purpose. The neutron-star cores are composed of a mixture of various species of particles with baryons (nucleons and hyperons) that can be in superfluid state ([9, 10, 11, 12, 13]). The hydrodynamics of superfluid mixtures strongly differs from the ordinary one because it allows the superfluid components to move independently of the normal (non-superfluid) liquid component without any dissipation of energy [14, 15].

This paper is devoted to a study of the relativistic entrainment matrix, which is an important quantity in hydrodynamics of superfluid mixtures. We mainly focus on the *nucleon-hyperon* matter in the core of massive neutron stars. Notice that, until now only the superfluid hydrodynamics of nucleon matter, composed of neutrons (n), protons (p), and electrons (e) with a possible admixture of muons (μ), has been considered in astrophysical literature. Let us discuss the results of previous works in more detail.

Assume that the neutrons and protons are in superfluid state. In this case three independent velocities can exist in nucleon matter. Two of them are the velocities $\boldsymbol{V}_{sn}$ and $\boldsymbol{V}_{sp}$ of neutron and proton superfluid components, respectively. The other is the velocity $\boldsymbol{V}_{\rm qp}$ of 'normal' (non-superfluid) neutrons and protons, as well as electrons and muons (it is assumed that, due to collisions, it is the same for all 'normal' particles). The physical meaning of the phenomenological superfluid velocities $\boldsymbol{V}_{sn}$ and $\boldsymbol{V}_{sp}$ can be understood on the basis of microphysics (see, e.g., [16] and Sec. IIB). It turns out that the velocity $\boldsymbol{V}_{si}$ is related to a Cooper pair momentum $2\boldsymbol{Q}_i$ of nucleon species $i=n,p$ by the equality

$$\boldsymbol{V}_{\rm s\it i} = \frac{\boldsymbol{Q}_i}{m_i}, \tag{1}$$

where m_i is the mass of a free paricle species i.

The non-relativistic expressions for the mass current density of neutrons $\boldsymbol{J}_n$ and protons $\boldsymbol{J}_p$ have the form (see, e.g., [16, 17, 18])

$$\boldsymbol{J}_n = (\rho_n - \rho_{nn} - \rho_{np})\boldsymbol{V}_{\rm qp} + \rho_{nn}\boldsymbol{V}_{\rm s\it n} + \rho_{np}\boldsymbol{V}_{\rm s\it p}, \tag{2}$$
$$\boldsymbol{J}_p = (\rho_p - \rho_{pp} - \rho_{pn})\boldsymbol{V}_{\rm qp} + \rho_{pp}\boldsymbol{V}_{\rm s\it p} + \rho_{pn}\boldsymbol{V}_{\rm s\it n}. \tag{3}$$

Here ρ_n and ρ_p are the neutron and proton density, respectively; $\rho_{ik}=\rho_{ki}$ is the symmetric 2×2 entrainment matrix, also termed Andreev-Bashkin or mass-density matrix ($i,\ k=n,\ p$). It follows from Eqs. (2) and (3) that superfluid motion of, for example, neutrons, contributes not only to $\boldsymbol{J}_n$ but also to $\boldsymbol{J}_p$ (and the same for protons). For the first time this effect was predicted, as applied to superfluid solutions of ^{3}He in ^{4}He, by Andreev and Bashkin [19]. The prefactors in front of $\boldsymbol{V}_{\rm qp}$ in Eqs. (2) and (3) can be interpreted as the densities of 'normal' neutrons and protons, respectively. Since at zero temperature ($T = 0$) all particles are paired, these densities vanish and we have [16, 17, 18]

$$\rho_n = \rho_{nn} + \rho_{np}, \tag{4}$$

$$\rho_p = \rho_{pp} + \rho_{pn}. \tag{5}$$

More strictly these conditions can be obtained from the requirement of Galilean invariance of the equations of superfluid hydrodynamics at $T = 0$ [17, 18].

The matrix ρ_{ik} was calculated for the case of $T = 0$ in Refs. [17, 18] and for arbitrary temperatures in Ref. [16]. In both cases, the authors used the non-relativistic Fermi-liquid theory of Landau. Though neutrons and especially protons in the cores of low-mass neutron stars can be (with a reasonable accuracy) considered as non-relativistic, more self-consistent (and necessary in the case of massive neutron stars) is the approach, in which nucleons are treated in the frame of relativistic theory. Following Refs. [21, 22, 23, 24], the relativistic analogue of Eqs. (2) and (3) can be presented in the form

$$\boldsymbol{j}_i = \left(n_i - \sum_k \mu_k \, Y_{ik} \right) \boldsymbol{u} + c^2 \sum_k Y_{ik} \, \boldsymbol{Q}_k. \tag{6}$$

Here $\boldsymbol{j}_i$ is the particle current density ($i = n, p$); c is the speed of light; $\boldsymbol{u}$ is the spatial component of the four-velocity u^μ, normalized by the condition $u^\mu u_\mu = -c^2$, and describing the motion of normal part of liquid; n_i and μ_i are, respectively, the number density and relativistic chemical potential of particle species i, measured in the frame where $u^\mu = (c, 0, 0, 0)$. Finally, the symmetric matrix Y_{ik} is the relativistic analogue of the entrainment matrix ρ_{ik}. In the non-relativistic limit Eq. (6) is equivalent to Eqs. (2) and (3) under conditions that

$$\boldsymbol{u} = \boldsymbol{V}_{\rm qp}, \qquad \rho_{ik} = m_i m_k \, c^2 \, Y_{ik}, \tag{7}$$

The prefactor in front of $\boldsymbol{u}$ in Eq. (6) can be interpreted (by analogy with the non-relativistic case) as the number density of 'normal' (non-superfluid) particle species i. At zero temperature this number density vanishes. This imposes a condition on the matrix Y_{ik} [22]

$$\sum_k \mu_k \, Y_{ik} = n_i. \tag{8}$$

Taking into account this condition, Eq. (6) can be rewritten (at $T = 0$) in the form

$$\boldsymbol{j}_i = c^2 \sum_k Y_{ik} \, \boldsymbol{Q}_k. \tag{9}$$

The matrix Y_{ik} for matter composed of neutrons and protons was calculated at $T = 0$ in Ref. [25] (the authors of Ref. [25] used a somewhat different formalism, see, e.g., the review [26] and references therein). The calculation was done in the frame of relativistic σ-ω mean-field model.

This paper is a natural continuation of the research described above. Our aim is to calculate the relativistic entrainment matrix Y_{ik} at zero temperature for matter composed not only of nucleons, electrons, and muons, but also of hyperons. We consider only two types of hyperons, appearing first in the neutron-star matter with the increasing density, namely Λ- and Σ^--hyperons (to be denoted by Λ and Σ, respectively). However, we wish to emphasize, that analytical results, obtained in this paper, can be (in principle) applied to any number of baryon species.

The phenomenological equations (1) and (6)–(9), which are discussed in this section in the context of nucleon matter, remain unchanged for nucleon-hyperon matter. The only difference is that now the indices i and k run over $i, k = n,\ p,\ \Lambda,\ \Sigma$ (see also Ref. [24]). Thus, now Y_{ik} is a 4×4 matrix.

The paper is organized as follows. In Sec. II the *relativistic* Landau Fermi-liquid theory [27] is generalized to the case of superfluid mixtures. In the frame of this theory we calculate the matrix Y_{ik} and express it through the Landau parameters f_1^{ik} of nucleon-hyperon matter. In Sec. III the general results of Sec. II are illustrated with a particular example of the σ-ω-ρ mean-field model with scalar self-interactions [28]. Namely, we (i) calculate the matrix Y_{ik}; (ii) determine *all* (spin-averaged) Landau parameters, corresponding to this model; (iii) analyze the stability of the ground state of nucleon-hyperon matter with respect to small perturbations. Section IV contains a summary of our results.

II. THE RELATIVISTIC ENTRAINMENT MATRIX AT ZERO TEMPERATURE FROM THE LANDAU FERMI-LIQUID THEORY

A. Relativistic Landau theory for mixtures of Fermi liquids

In this section we briefly discuss how to generalize the Landau Fermi-liquid theory to the case of nucleon-hyperon mixture composed of neutrons, protons, Λ-, and Σ^--hyperons. The original non-relativistic Landau Fermi-liquid theory (e.g., [29, 30]) was extended to the case of mixtures of protons and neutrons in the paper by Sjöberg [20] (see also [17]). The relativistic generalization of the Landau Fermi-liquid theory was given by Baym and Chin [27] (they considered a Fermi-liquid composed of identical particles).

The generalization of the Landau theory to the case of relativistic mixtures composed of more than one component can be made in the same way as in Refs. [20, 27]. Thus, we only briefly describe the main formulae of the theory which will be used subsequently. Notice, that the results obtained below in Section II can be applied to a Fermi-liquid composed of any number of baryon species (not necessarily four). In this case the particle indices in equations should run over all baryon species.

Unless is stated otherwise, throughout the rest of the paper we use the system of units in which the Planck constant $\hbar$, the Boltzmann constant $k_{\rm B}$, and the speed of light c equal unity, $\hbar = k_{\rm B} = c = 1$. We also imply that the subscripts i and k refer to baryons.

As it is demonstrated in Ref. [27], generally, the structure of the relativistic Landau theory of Fermi liquids is the same as that of the non-relativistic theory. Only those results of both theories are different which are obtained using Lorentz (Galilean) transformation properties of various quantities (in particular, the energy and momentum). For instance, we will show that the relativistic expression for the effective mass of particle species i differs from its non-relativistic analogue.

Let us consider a system in the ground state with the energy E_0 at temperature $T = 0$. The distribution function of quasiparticle species i is then a Fermi sphere,

$$n_{i0}(\boldsymbol{p}) = \theta(p_{\rm F\mathit{i}} - p), \tag{10}$$

where $\boldsymbol{p}$ is the quasiparticle momentum; $\theta(x)$ is the step function: $\theta(x) = 1$, if $x > 0$ and 0 otherwise. A small deviation $\delta n_i(\boldsymbol{p})$ of the distribution function from $n_{i0}(\boldsymbol{p})$ changes the system energy by

$$E - E_0 = \sum_{\boldsymbol{p} s i} \varepsilon_{i0}(\boldsymbol{p})\, \delta n_i(\boldsymbol{p}) + \frac{1}{2} \sum_{\boldsymbol{p}\boldsymbol{p}'\, ss'\, ik} f^{ik}(\boldsymbol{p}, \boldsymbol{p}')\, \delta n_i(\boldsymbol{p}) \delta n_k(\boldsymbol{p}'). \tag{11}$$

Addition of one more quasiparticle of a species i, with the momentum $\boldsymbol{p}$, to the system, increases the total energy E by the energy $\varepsilon_i(\boldsymbol{p})$ of the quasiparticle. From Eq. (11) it follows that

$$\varepsilon_i(\boldsymbol{p}) = \varepsilon_{i0}(\boldsymbol{p}) + \sum_{\boldsymbol{p}' s' k} f^{ik}(\boldsymbol{p}, \boldsymbol{p}')\, \delta n_k(\boldsymbol{p}'). \tag{12}$$

In Eqs. (11) and (12) $\boldsymbol{p}$ and $\boldsymbol{p}'$ are the particle momenta; s and s' are the spin indices; $i, k = n$, p, Λ, and Σ are the baryon species indices. Furthermore, $\varepsilon_{i0}(\boldsymbol{p})$ is the energy of a quasiparticle of a species i, corresponding to the distribution function $n_{i0}(\boldsymbol{p})$. It can be expanded into a series near the Fermi surface in powers of the quantity $p - p_{\rm F\mathit{i}}$ and presented in the linear form

$$\varepsilon_{i0}(\boldsymbol{p}) \approx \mu_i + v_{\rm F\mathit{i}}(p - p_{\rm F\mathit{i}}), \tag{13}$$

where $p_{\rm F\mathit{i}}$ is the Fermi momentum of (quasi)particle species i; $\mu_i = \varepsilon_{i0}(p_{\rm F\mathit{i}})$ is the relativistic chemical potential or, equivalently, the Fermi energy of quasiparticle species i; $\boldsymbol{v}_{{\rm F}i} = [\partial \varepsilon_{i0}(\boldsymbol{p})/\partial \boldsymbol{p}]_{p=p_{\rm F\mathit{i}}}$ is the velocity of quasiparticles on the Fermi surface. It can also be expressed as $v_{\rm F\mathit{i}} \equiv p_{\rm F\mathit{i}}/m_i^*$, where m_i^* is the effective mass of quasiparticle species i. Finally, the function $f^{ik}(\boldsymbol{p}, \boldsymbol{p}')$ in Eq. (12) is the spin-averaged Landau quasiparticle interaction (here and below we disregard the spin-dependence of this interaction since it does not affect our results). In the vicinity of the Fermi surface the arguments of the function $f^{ik}(\boldsymbol{p}, \boldsymbol{p}')$ can be approximately put equal to $p \approx p_{\rm F\mathit{i}}$ and $p' \approx p_{\rm F\mathit{k}}$, while the function itself can be expanded into Legendre polynomials $P_l(\cos\theta)$,

$$f^{ik}(\boldsymbol{p}, \boldsymbol{p}') = \sum_l f_l^{ik}\, P_l(\cos\theta), \tag{14}$$

where θ is the angle between $\boldsymbol{p}$ and $\boldsymbol{p}'$ and f_l^{ik} are the (symmetric) Landau parameters, $f_l^{ik} = f_l^{ki}$.

As in the non-relativistic case, the effective mass m_i^* in the relativistic theory can be expressed in terms of the Landau parameters f_1^{ik}. To find this relation let us consider, following Ref. [27], two frames K and $\overline{K}$, and assume that the frame $\overline{K}$ moves with the velocity $\boldsymbol{V}$ with respect to K. Below in this section the quantities marked with an overline will be referred to the frame $\overline{K}$, while those without the overline – to the frame K. The total energy of nucleon-hyperon mixture E ($\overline{E}$) and its momentum $\boldsymbol{P}$ ($\overline{\boldsymbol{P}}$) are related by the Lorentz transformation

$$E = (\overline{E} + \overline{\boldsymbol{P}}\boldsymbol{V})\,\gamma, \tag{15}$$

$$\boldsymbol{P} = \overline{\boldsymbol{P}} - \boldsymbol{e_V}\left(\boldsymbol{e_V}\overline{\boldsymbol{P}}\right)(1-\gamma) + \overline{E}\boldsymbol{V}\gamma. \tag{16}$$

In Eqs. (15) and (16) $\gamma = (1-V^2)^{-1/2}$, $\boldsymbol{e_V}$ is the unit vector along $\boldsymbol{V}$.

Now imagine that we add a quasiparticle of a species i, of momentum $\boldsymbol{p}$ and energy $\varepsilon_i(\boldsymbol{p})$, to the system. Then the total momentum and energy in the frame K become equal to $\boldsymbol{P}+\boldsymbol{p}$ and $E+\varepsilon_i(\boldsymbol{p})$, respectively. On the other hand, the momentum and energy in the frame $\overline{K}$ will be $\overline{\boldsymbol{P}}+\overline{\boldsymbol{p}}$ and $\overline{E}+\overline{\varepsilon}_i(\overline{\boldsymbol{p}})$. Consequently, using Eqs. (15) and (16) one obtains the transformation rules for the quasiparticle momentum and energy

$$\varepsilon_i(\boldsymbol{p}) = \left[\overline{\varepsilon}_i(\overline{\boldsymbol{p}}) + \overline{\boldsymbol{p}}\boldsymbol{V}\right]\gamma, \tag{17}$$

$$\boldsymbol{p} = \overline{\boldsymbol{p}} - \boldsymbol{e_V}\left(\boldsymbol{e_V}\overline{\boldsymbol{p}}\right)(1-\gamma) + \overline{\varepsilon}_i(\overline{\boldsymbol{p}})\boldsymbol{V}\gamma. \tag{18}$$

We need to know also how the distribution function of quasiparticle species i transforms from one frame to another. The answer is given by the standard formula

$$n_i(\boldsymbol{p}) = \overline{n}_i(\overline{\boldsymbol{p}}). \tag{19}$$

Assume now, that $\boldsymbol{V}$ satisfies the inequality, $V \ll v_{\mathrm{F}i}$. Then we have also $V \ll 1$ and, as follows from Eqs. (17)–(19), keeping linear terms in $\boldsymbol{V}$, one gets

$$\overline{\varepsilon}_i(\boldsymbol{p}) = \varepsilon_i(\boldsymbol{p}) + \frac{\partial \varepsilon_i(\boldsymbol{p})}{\partial \boldsymbol{p}}\,\varepsilon_i(\boldsymbol{p})\,\boldsymbol{V} - \boldsymbol{p}\boldsymbol{V}, \tag{20}$$

$$\overline{n}_i(\boldsymbol{p}) = n_i(\boldsymbol{p}) + \frac{\partial n_i(\boldsymbol{p})}{\partial \boldsymbol{p}}\,\varepsilon_i(\boldsymbol{p})\,\boldsymbol{V}. \tag{21}$$

In the case of non-interacting relativistic particles the sum of two last terms on the right-hand side of Eq. (20) equals zero, hence $\overline{\varepsilon}_i(\boldsymbol{p}) = \varepsilon_i(\boldsymbol{p})$.

In addition to Eq. (20), there is one more condition relating $\overline{\varepsilon}_i(\boldsymbol{p})$ and $\varepsilon_i(\boldsymbol{p})$. In fact, it follows from Eq. (12) that for any chosen momentum $\boldsymbol{p}$ the quasiparticle energy $\varepsilon_i(\boldsymbol{p})$ in the frame K will differ from the energy $\overline{\varepsilon}_i(\boldsymbol{p})$ in the frame $\overline{K}$ only to the extent that $n_i(\boldsymbol{p})$ differs from $\overline{n}_i(\boldsymbol{p})$. In other words,

$$\varepsilon_i(\boldsymbol{p}) = \overline{\varepsilon}_i(\boldsymbol{p}) + \sum_{\boldsymbol{p}' s' k} f^{ik}(\boldsymbol{p},\boldsymbol{p}')\left[n_k(\boldsymbol{p}') - \overline{n}_k(\boldsymbol{p}')\right]. \tag{22}$$

Substituting into Eq. (22) $\overline{\varepsilon}_i(\boldsymbol{p})$ and $\overline{n}_i(\boldsymbol{p})$ from Eqs. (20) and (21), respectively, one obtains

$$\left[\frac{\partial \varepsilon_i(\boldsymbol{p})}{\partial \boldsymbol{p}}\,\varepsilon_i(\boldsymbol{p}) - \boldsymbol{p}\right]\boldsymbol{V} - \sum_{\boldsymbol{p}' s' k} f^{ik}(\boldsymbol{p},\boldsymbol{p}')\,\frac{\partial n_k(\boldsymbol{p}')}{\partial \boldsymbol{p}'}\,\varepsilon_k(\boldsymbol{p}')\,\boldsymbol{V} = 0. \tag{23}$$

For a system in its ground state one has $n_i(\boldsymbol{p}) = n_{i0}(\boldsymbol{p})$ and $\varepsilon_i(\boldsymbol{p}) = \varepsilon_{i0}(\boldsymbol{p})$ (see Eqs. (10) and (13)). At $p = p_{\mathrm{F}i}$ Eq. (23) relates the effective mass m_i^* and the Landau parameters f_1^{ik}

$$\frac{\mu_i}{m_i^*} = 1 - \sum_k \frac{\mu_k G_{ik}}{n_i}. \tag{24}$$

Here the number density of particle species i is given by

$$n_i = \frac{p_{\mathrm{F}i}^3}{3\pi^2}, \tag{25}$$

while the symmetric matrix G_{ik} equals

$$G_{ik} = \frac{1}{9\pi^4}\,p_{\mathrm{F}i}^2 p_{\mathrm{F}k}^2\,f_1^{ik}. \tag{26}$$

For a liquid composed of identical particles, Eq. (24) transforms into the equation (13) of Ref. [27]. The non-relativistic limit of Eq. (24) can be obtained if one replaces μ_i by m_i. Applying then this formula to a mixture of two species of baryons, one reproduces the result of Sjöberg [20] (see also [17, 18]).

B. Calculation of the relativistic entrainment matrix

Let us employ the theory described above to calculate the relativistic entrainment matrix Y_{ik} at zero temperature. At first glance, this theory seems inappropriate for calculation of superfluid properties of nucleon-hyperon matter because it describes the normal Fermi fluid. However, as was demonstrated by Leggett [31, 32] in the context of superfluid ^{3}He, the particle current density $\boldsymbol{j}_i$ of particle species i in superfluid nucleon-hyperon matter is given by the same equation as in the case of normal (non-superfluid) matter, namely

$$\boldsymbol{j}_i = \sum_{\boldsymbol{p}s} \frac{\partial \varepsilon_i(\boldsymbol{p})}{\partial \boldsymbol{p}}\, n_i(\boldsymbol{p}). \tag{27}$$

All that we need to know is how the superfluid motions modify the distribution function $n_i(\boldsymbol{p})$ of quasiparticles. One also has to take into account that a change of $n_i(\boldsymbol{p})$ results in a change of the quasiparticle energy $\varepsilon_i(\boldsymbol{p})$. Leggett [31] showed that this energy can be calculated from the same formula (12) as for normal matter (see also [16, 17, 18, 25]).

As was already mentioned in Sec. I, the superfluid current is generated in the system when the Cooper pairs acquire a non-zero momentum $2\boldsymbol{Q}_i$. In this case they are formed by pairing of quasiparticles with momenta $(-\boldsymbol{p}+\boldsymbol{Q}_i)$ and $(\boldsymbol{p}+\boldsymbol{Q}_i)$ (rather than with strictly opposite momenta $-\boldsymbol{p}$ and $\boldsymbol{p}$, as it would be in the system without currents). The distribution function $n_i(\boldsymbol{p})$ for a system with currents can be especially easily found at zero temperature. In this case all quasiparticles are paired and, up to small terms of the order of $O[(\Delta/\mu)^2+(\boldsymbol{Q}_i/\mu)^2]$ (where Δ is some characteristic value of an energy gap in the dispersion relation for baryons; μ is the characteristic chemical potential of baryons), $n_i(\boldsymbol{p})$ is a Fermi-sphere, shifted by the vector $\boldsymbol{Q}_i$ in momentum space (see, e.g., [17, 18, 25]),

$$n_i(\boldsymbol{p}) = \theta(p_{\mathrm{F}i} - |\boldsymbol{p} - \boldsymbol{Q}_i|). \tag{28}$$

Here and below we assume that $Q_k \ll p_{\mathrm{F}i}$. In this case we may restrict ourselves to a linear in $\boldsymbol{Q}_k$ terms when calculating $\boldsymbol{j}_i$. Using the distribution function (28) as well as Eq. (12) for the energy of quasiparticle species i in which $\delta n_i(\boldsymbol{p}) \equiv \theta(p_{\mathrm{F}i} - |\boldsymbol{p}-\boldsymbol{Q}_i|) - n_{i0}(\boldsymbol{p}) \approx -\left[\partial n_{i0}(\boldsymbol{p})/\partial \boldsymbol{p}\right]\, \boldsymbol{Q}_i$, one gets from Eq. (27)

$$\boldsymbol{j}_i = -\sum_{\boldsymbol{p}s} \frac{\partial \varepsilon_{i0}(\boldsymbol{p})}{\partial \boldsymbol{p}} \left[\frac{\partial n_{i0}(\boldsymbol{p})}{\partial \boldsymbol{p}}\boldsymbol{Q}_i\right] - \sum_{\boldsymbol{p}s} \frac{\partial}{\partial \boldsymbol{p}} \left[\sum_{\boldsymbol{p}' s' k} f^{ik}(\boldsymbol{p},\boldsymbol{p}')\, \frac{\partial n_{k0}(\boldsymbol{p}')}{\partial \boldsymbol{p}'}\boldsymbol{Q}_k\right] n_{i0}(\boldsymbol{p}). \tag{29}$$

The first term in the right-hand side of Eq. (29) equals $I = n_i/(m_i^*)\,\boldsymbol{Q}_i$. Integrating by parts the second term, one has $II = \sum_k G_{ik}\,\boldsymbol{Q}_k$, where the matrix G_{ik} is defined by Eq. (26). Thus, one finds for the particle current density $\boldsymbol{j}_i$

$$\boldsymbol{j}_i = \frac{n_i}{m_i^*}\boldsymbol{Q}_i + \sum_k G_{ik}\,\boldsymbol{Q}_k. \tag{30}$$

Comparison of this result with Eq. (9) allows one to determine the expression for relativistic entrainment matrix Y_{ik} (δ_{ik} is the Kronecker symbol)

$$Y_{ik} = \frac{n_i}{m_i^*}\,\delta_{ik} + G_{ik}. \tag{31}$$

Using Eq. (24) we verified that the matrix Y_{ik} satisfies the condition (8).

The energy of nucleon-hyperon matter with superfluid currents can be also expressed through the matrix Y_{ik}. From Eq. (11) it follows that

$$E - E_0 = \frac{1}{2}\sum_{ik} Y_{ik}\,\boldsymbol{Q}_i\boldsymbol{Q}_k. \tag{32}$$

An analogous formula, valid for an arbitrary temperature (not only for $T=0$) was obtained for a mixture of two non-relativistic superfluids by Andreev and Bashkin [19]. Notice that, the difference $(E-E_0)$ can be interpreted as the energy of superfluid motion. For a stable superfluid ground state, $E-E_0>0$, and hence the quadratic form in the right-hand side of Eq. (32) should be positively defined. This leads to a set of conditions on the matrix Y_{ik} or, equivalently, on the Landau parameters f_1^{ik}. Here we will write only the simplest two of them (see also [18, 19])

$$Y_{ii} \geq 0, \qquad Y_{ii}Y_{kk} - Y_{ik}^2 \geq 0 \qquad (i \neq k). \tag{33}$$

III. THE RELATIVISTIC ENTRAINMENT MATRIX AT ZERO TEMPERATURE FROM THE σ-ω-ρ MODEL WITH SCALAR SELF-INTERACTIONS

Let us apply the general results obtained in Sec. II to a specific model describing the interaction of baryons, the σ-ω-ρ mean-field model with scalar self-interactions. Our aim will be to calculate in the frame of this model the relativistic entrainment matrix Y_{ik} as well as the Landau parameters f_l^{ik} of nucleon-hyperon mixture.

A. σ-ω-ρ mean-field model with scalar self-interactions: general equations

The σ-ω-ρ model with scalar self-interactions is described in detail in the monograph by Glendenning [33] (see also [28]). Here we briefly discuss its main equations which will be used below to calculate the relativistic entrainment matrix Y_{ik}. Let us consider a system of baryons n, p, Λ, and Σ in some uniform state. Interactions among those baryons are mediated by three different kinds of meson fields: scalar σ-field, vector ω-field and an isospin triplet of charged vector $\vec{\rho}$-fields. The mean-field approximation assumes that the σ-, ω-, and $\vec{\rho}$-fields are replaced by their mean expectation values in the chosen state. We denote these values by σ, ω^μ, and $\vec{\rho}^{\,\mu} = (\rho_1^\mu, \rho_2^\mu, \rho_3^\mu)$, respectively ($\mu$ is the space-time index). These mean values are to be calculated from the following (averaged) Euler-Lagrange equations [33]

$$\begin{aligned} m_\sigma^2 \sigma &= -b m_n\, g_{\sigma n}\, (g_{\sigma n}\sigma)^2 - c g_{\sigma n}\, (g_{\sigma n}\sigma)^3 \\ &+ \sum_{\boldsymbol{p} s i} g_{\sigma i}\, \frac{m_i - g_{\sigma i}\sigma}{\sqrt{(\boldsymbol{p} - g_{\omega i}\, \boldsymbol{\omega} - g_{\rho i}\, I_{3i}\, \boldsymbol{\rho}_3)^2 + (m_i - g_{\sigma i}\sigma)^2}}\, n_i(\boldsymbol{p}), \qquad (34) \\ \omega^\mu &= \sum_i \frac{g_{\omega i}}{m_\omega^2}\, j_i^\mu, \qquad (35) \\ \rho_1^\mu &= \rho_2^\mu = 0, \qquad (36) \\ \rho_3^\mu &= \sum_i \frac{g_{\rho i}}{m_\rho^2}\, I_{3i}\, j_i^\mu. \qquad (37) \end{aligned}$$

One sees that only the third isospin component ρ_3^μ of the $\vec{\rho}$-field, which corresponds to the neutral rho meson, has non-zero mean value. In Eqs. (34)–(37) the summation is performed over the baryon species $i = n$, p, Λ, and Σ; m_l is the mass of meson species $l = \sigma$, ω, or $\rho_{1,2,3}$; g_{li} is the coupling constant of meson l and baryon i; I_{3i} is the isospin projection for baryon species i. Furthermore, $n_i(\boldsymbol{p})$ is (as in Sect. IIA) a distribution function of particle species i; b and c are some dimensionless constants describing self-interaction of the scalar σ-field; $\boldsymbol{\omega}$ and $\boldsymbol{\rho}_3$ are the spatial components of four-vectors ω^μ and ρ_3^μ, respectively. The ω- and ρ_3-fields are generated by the baryon four-currents j_i^μ on the right-hand side of Eqs. (35) and (37). They are given by

$$\begin{aligned} j_i^0 &= n_i = \sum_{\boldsymbol{p} s} n_i(\boldsymbol{p}), \qquad (38) \\ \boldsymbol{j}_i &= \sum_{\boldsymbol{p} s} \frac{\partial E_i(\boldsymbol{p})}{\partial \boldsymbol{p}}\, n_i(\boldsymbol{p}), \qquad (39) \end{aligned}$$

where the number density n_i and the particle current density $\boldsymbol{j}_i$ are measured in the laboratory frame; $E_i(\boldsymbol{p})$ is the energy of a baryon species i

$$E_i(\boldsymbol{p}) = g_{\omega i}\, \omega^0 + g_{\rho i}\, I_{3i}\, \rho_3^0 + \sqrt{(\boldsymbol{p} - g_{\omega i}\, \boldsymbol{\omega} - g_{\rho i}\, I_{3i}\, \boldsymbol{\rho}_3)^2 + (m_i - g_{\sigma i}\sigma)^2}. \qquad (40)$$

In Eqs. (34), (38), and (39) the summation is performed over the momentum states occupied by the particles. If our system is not only uniform but also isotropic then (at zero temperature) the distribution function $n_i(\boldsymbol{p})$ is a Fermi sphere centered at $\boldsymbol{p} = 0$ in the momentum space, so that we have (see Eq. (10))

$$n_i(\boldsymbol{p}) = n_{i0}(\boldsymbol{p}). \qquad (41)$$

Substituting the distribution function (41) into Eq. (38) one obtains that the time component $j_i^0 = n_i$ is given by Eq. (25). Moreover, in this special case the spatial components of four-vectors ω^μ, ρ_3^μ, and j_i^μ vanish, $\boldsymbol{\omega} = \boldsymbol{\rho}_3 = \boldsymbol{j}_i = 0$ (there is no preferred direction!), while σ-field and the time components are still given by Eqs. (34), (35), and (37) with

$n_i(\boldsymbol{p})$ and j_i^0 taken from Eqs. (41) and (25), respectively. The chemical potential μ_i of baryon species i is presented in the form

$$\mu_i = g_{\omega i}\,\omega^0 + g_{\rho i}\,I_{3i}\,\rho_3^0 + \sqrt{p_{\mathrm{F}i}^2 + (m_i - g_{\sigma i}\sigma)^2}. \tag{42}$$

It is the energy of a particle on the Fermi surface.

B. The relativistic entrainment matrix from the σ-ω-ρ mean-field model

A derivation of the matrix Y_{ik} in the frame of the σ-ω-ρ mean-field model with scalar self-interactions is completely analogous to the derivation presented in Sec. IIB for the case of relativistic Landau Fermi-liquid theory. In nucleon-hyperon matter in which the superfluid currents are generated, the distribution function for baryon species i is approximately described by Eq. (28).

The superfluid current density $\boldsymbol{j}_i$ is given by Eq. (39) with the energy $E_i(\boldsymbol{p})$ calculated from Eq. (40) with the help of Eqs. (34), (35), and (37).

As it was mentioned in Sec. IIB, we restrict ourselves to a linear approximation when calculating $\boldsymbol{j}_i$ as a function of $\boldsymbol{Q}_k$. In this approximation the scalar σ-field as well as the time components ω^0 and ρ_3^0 remain the same (their variation $\sim \boldsymbol{Q}_i\boldsymbol{Q}_k$), whereas the spatial components $\boldsymbol{\omega}$ and $\boldsymbol{\rho}_3$ depend on some linear combinations of the vectors $\boldsymbol{Q}_k$. It follows from Eqs. (39) that

$$\begin{aligned}
\boldsymbol{j}_i &= \sum_{\boldsymbol{p}s} \frac{\partial E_i(\boldsymbol{p})}{\partial \boldsymbol{p}}\, \theta(p_{\mathrm{F}i} - |\boldsymbol{p} - \boldsymbol{Q}_i|) \\
&= \sum_{\boldsymbol{p}s} \frac{\partial E_i(\boldsymbol{p} + \boldsymbol{Q}_i)}{\partial \boldsymbol{p}}\, n_{i0}(\boldsymbol{p}) \\
&= \sum_{\boldsymbol{p}s} \frac{\partial}{\partial \boldsymbol{p}} \left[\sqrt{p^2 + (m_i - g_{\sigma i}\sigma)^2} + \frac{\boldsymbol{p}\,(\boldsymbol{Q}_i - g_{\omega i}\boldsymbol{\omega} - g_{\rho i} I_{3i}\boldsymbol{\rho}_3)}{\sqrt{p^2 + (m_i - g_{\sigma i}\sigma)^2}} \right] n_{i0}(\boldsymbol{p}) \\
&= \frac{n_i}{\sqrt{p_{\mathrm{F}i}^2 + (m_i - g_{\sigma i}\sigma)^2}}\, (\boldsymbol{Q}_i - g_{\omega i}\boldsymbol{\omega} - g_{\rho i} I_{3i}\boldsymbol{\rho}_3)\,.
\end{aligned} \tag{43}$$

This equation should be supplemented by the expressions (35) and (37) for $\boldsymbol{\omega}$ and $\boldsymbol{\rho}_3$, respectively

$$\boldsymbol{\omega} = \sum_i \frac{g_{\omega i}}{m_\omega^2}\, \boldsymbol{j}_i, \tag{44}$$

$$\boldsymbol{\rho}_3 = \sum_i \frac{g_{\rho i}}{m_\rho^2}\, I_{3i}\, \boldsymbol{j}_i. \tag{45}$$

Solving the system of six equations (43)–(45) one can find $\boldsymbol{j}_i$ and the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\rho}_3$ as functions of $\boldsymbol{Q}_k$. In this way the relativistic entrainment matrix Y_{ik} can be determined at zero temperature. The analytic expression for Y_{ik} is given in Appendix. It is easy to verify that the matrix Y_{ik} satisfies the condition (8).

Note that, in the limiting case considered by Comer and Joynt [25] our results for the matrix Y_{ik} do not reproduce theirs. Their results do not satisfy the condition (8). Let us remind that the authors of Ref. [25] considered *asymmetric* nuclear matter composed of neutrons, protons, and electrons. They assumed that nucleons interact through σ- and ω-fields (the neutral ρ_3-field and self-interactions of σ-field were neglected). The criticism of such assumption can be found in Ref. [34].

Fig. 1 presents the normalized elements Y_{ik}/Y of symmetric matrix Y_{ik}, calculated using Eq. (66), as functions of the baryon number density $n_b = n_n + n_p + n_\Sigma + n_\Lambda$ for the *third* equation of state of Glendenning [28]. The constant Y equals $Y = 3n_0/\mu_n(3n_0) = 2.48 \times 10^{41}$ erg^{-1} cm^{-3}, where $n_0 = 0.16$ fm^{-3} is the normal nuclear density; $\mu_n(3n_0) = 1.94 \times 10^{-3}$ erg is the neutron chemical potential at $n_b = 3n_0$. Each curve on the figure is plotted for some normalized element of the matrix Y_{ik} and marked with two particle species indices ik. For instance, symbols $n\Lambda$ on the figure (right panel) mark a curve plotted for the element $Y_{n\Lambda}/Y$ $(= Y_{\Lambda n}/Y)$. The chosen equation of state predicts first the appearance of Λ-hyperons at $n_b = n_{b\Lambda} = 0.310$ fm^{-3} and then Σ^--hyperons at $n_b = n_{b\Sigma} = 0.319$ fm^{-3}. One sees that at $n_b < n_{b\Lambda}$ (no hyperons) all the components of Y_{ik}, related to hyperons, become zero.

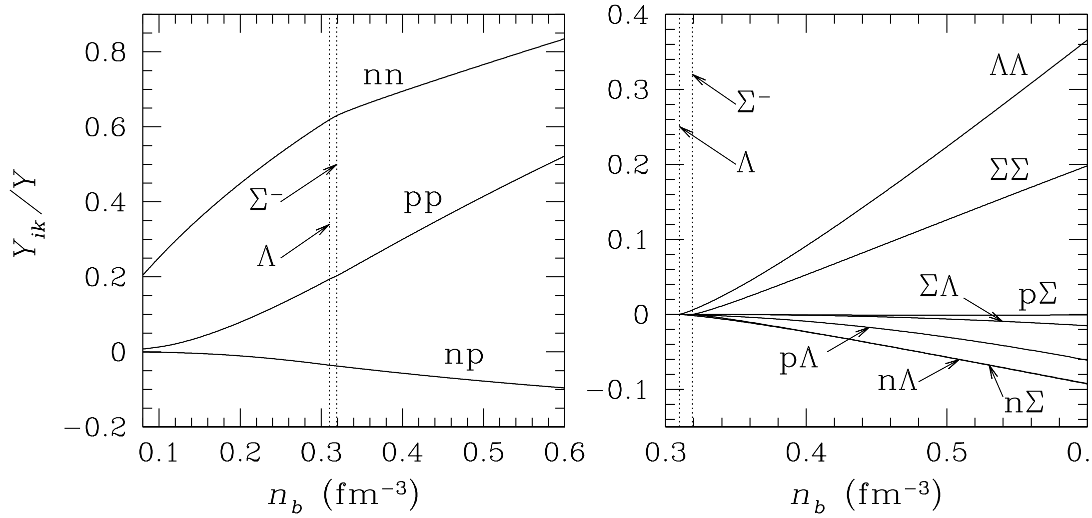

FIG. 1: Normalized symmetric matrix Y_{ik}/Y as a function of n_b for the third equation of state of Glendenning [28]. The normalization constant $Y = 3n_0/\mu_n(3n_0) = 2.48 \times 10^{41}$ erg^{-1} cm^{-3}. Solid lines show the elements of the matrix Y_{ik}/Y; each curve is marked by the corresponding symbol ik ($i, k = n$, p, Λ, Σ). Vertical dotted lines indicate the thresholds for the appearance of (from left to right) Λ- and Σ^--hyperons.

C. Calculation of Landau parameters

The σ-ω-ρ model described above can be reformulated in terms of the relativistic Landau theory of Fermi liquids (see Sec. II). For that, it is necessary to calculate the Landau parameters of nucleon-hyperon matter. In case of nucleon matter the Landau parameters were calculated for various relativistic mean-field models in a series of papers (see, e.g., [35, 36, 37, 38, 39, 40]) The derivation of these parameters for nucleon-hyperon matter is quite similar. The main idea of the derivation is to consider a small deviation of the distribution function of baryon species i from $n_{i0}(\boldsymbol{p})$ (see Eq. 10) and to analyze how it modifies the energy of baryon species k. Then the result should be compared with the corresponding Eq. (12) for the energy variation in the frame of the Landau theory. In this way one obtains the function $f^{ik}(\boldsymbol{p}, \boldsymbol{p}')$ or, equivalently, the parameters f_l^{ik}. In Refs. [35, 36, 37, 38, 39, 40], dealing with the case of nucleon matter, it is shown that only first two Landau parameters are non-zero: f_0^{ik} and f_1^{ik}. We checked that the same is true for nucleon-hyperon matter, $f_l^{ik} = 0$ at $l \geq 2$. In view of this observation it is enough to find only the parameters f_0^{ik} and f_1^{ik}.

Strictly speaking, the parameters f_1^{ik} have already been calculated in the previous section. Indeed, it follows from Eq. (31) that

$$f_1^{ik} = \frac{9\pi^4}{p_{\mathrm{F}i}^2 p_{\mathrm{F}k}^2} \left(Y_{ik} - \frac{n_i}{m_i^*}\, \delta_{ik} \right), \tag{46}$$

where Y_{ik} is given by Eq. (66) and the Landau effective masses m_i^* (not to be confused with the Dirac effective mass!) equal

$$m_i^* = \frac{p_{\mathrm{F}i}}{|\partial E_i(\boldsymbol{p})/\partial \boldsymbol{p}|_{p=p_{\mathrm{F}i}}} = \sqrt{p_{\mathrm{F}i}^2 + (m_i - g_{\sigma i}\sigma)^2}. \tag{47}$$

Fig. 2 illustrates the dependence of normalized Landau effective mass m_i^*/m_i ($i = n$, p, Λ, Σ) on n_b for the third equation of state of Glendenning [28].

Now let us calculate the parameters f_0^{ik}. For that we slightly vary the Fermi momentum $p_{\mathrm{F}i}$ by a small quantity $\Delta p_{\mathrm{F}i}$. This will alter $n_{i0}(\boldsymbol{p})$ by

$$\delta n_i(\boldsymbol{p}) = \theta(p_{\mathrm{F}i} + \Delta p_{\mathrm{F}i} - p) - n_{i0}(\boldsymbol{p}), \tag{48}$$

while the variation of the energy of baryon species i (on the Fermi surface) will be (see Eq. (12))

$$\delta\varepsilon_i(p_{\mathrm{F}i}) = \sum_k f_0^{ik}\, \delta n_k. \tag{49}$$

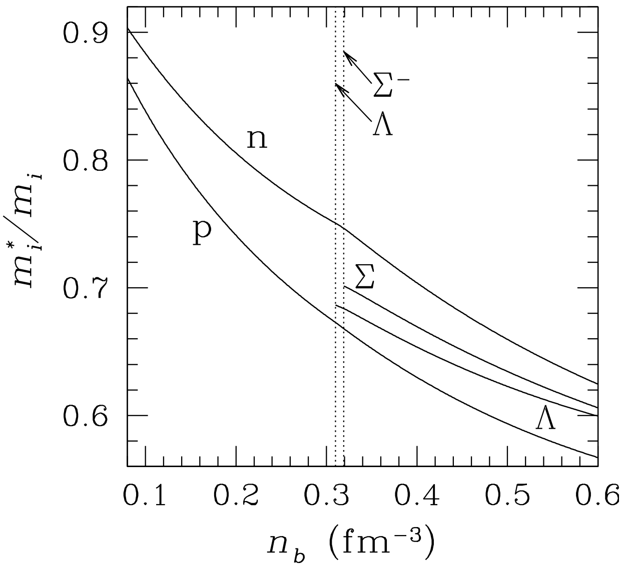

FIG. 2: The normalized Landau effective masses m_i^*/m_i ($i = n$, p, Λ, Σ) versus n_b for the third equation of state of Ref. [28]. Vertical dotted lines indicate thresholds for the appearance of (from left to right) Λ- and Σ^--hyperons.

Here $\delta n_k = p_{\mathrm{F}k}^2\, \Delta p_{\mathrm{F}k}/\pi^2$ is the variation of the number density of particle species i. On the other hand, if we consider the σ-ω-ρ model, the variation of the baryon energy on the Fermi surface will be (in the first approximation, see Eq. (40))

$$\delta E_i(p_{\mathrm{F}i}) = g_{\omega i}\, \delta\omega^0 + g_{\rho i} I_{3i}\, \delta\rho_3^0 - \frac{g_{\sigma i}(m_i - g_{\sigma i}\sigma)}{m_i^*}\delta\sigma, \tag{50}$$

The small terms $\delta\sigma$, $\delta\omega^0$, and $\delta\rho_3^0$ can be expressed through δn_k from Eqs. (34), (35), and (37), respectively

$$\delta\sigma = \frac{1}{L(\sigma)} \sum_k \frac{g_{\sigma k}\,(m_k - g_{\sigma k}\sigma)}{m_k^*}\, \delta n_k, \tag{51}$$

$$\delta\omega^0 = \sum_k \frac{g_{\omega k}}{m_\omega^2}\, \delta n_k, \tag{52}$$

$$\delta\rho_3^0 = \sum_k \frac{g_{\rho k}}{m_\rho^2}\, I_{3k}\, \delta n_k. \tag{53}$$

The function $L(\sigma)$ in Eq. (51) is given by

$$\begin{aligned} L(\sigma) = & \frac{\partial}{\partial\sigma}\Bigg[m_\sigma^2\sigma + b m_n\, g_{\sigma n}\,(g_{\sigma n}\sigma)^2 + c g_{\sigma n}\,(g_{\sigma n}\sigma)^3 \\ & - \sum_{\boldsymbol{p} s i} \frac{g_{\sigma i}(m_i - g_{\sigma i}\sigma)}{\sqrt{p^2 + (m_i - g_{\sigma i}\sigma)^2}}\, n_{i0}(\boldsymbol{p})\Bigg]. \end{aligned} \tag{54}$$

Substituting now Eqs. (51)–(53) into Eq. (50) and comparing the resulting expression with Eq. (49), one finds the Landau parameters f_0^{ik}

$$f_0^{ik} = \frac{g_{\omega i} g_{\omega k}}{m_\omega^2} + \frac{g_{\rho i} I_{3i}\, g_{\rho k} I_{3k}}{m_\rho^2} - \frac{1}{L(\sigma)}\, \frac{g_{\sigma i}(m_i - g_{\sigma i}\sigma)}{m_i^*}\, \frac{g_{\sigma k}(m_k - g_{\sigma k}\sigma)}{m_k^*}. \tag{55}$$

It follows from Eqs. (46) and (55) that the parameters f_0^{ik} and f_1^{ik} are indeed symmetric in the indices i and k.

Just as the parameters f_1^{ik} must guarantee the positive definiteness of the quadratic form (32), the parameters f_0^{ik} must satisfy a number of conditions. These conditions are related to stability of charged multi-component mixture

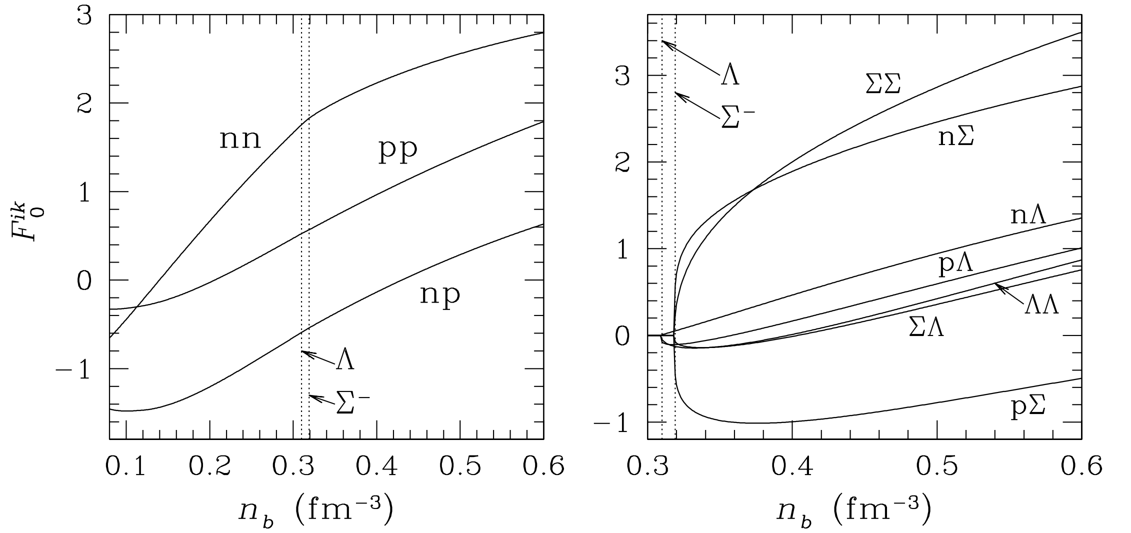

FIG. 3: Dimensionless Landau parameters F_0^{ik} versus n_b for the third equation of state of Ref. [28]. Other notations are the same as in Figs. 1 and 2.

with respect to density fluctuations and were carefully analysed for nucleon matter (see, e.g., [41, 42, 43, 44, 45]). They depend essentially on the matter composition and on the applied perturbation. Here we consider an equilibrated matter of massive neutron stars composed not only of nucleons (n and p) and hyperons (Λ and Σ) but also of electrons (e) and muons (μ). As an example, we analyse the stability of such matter with respect to long-wavelength density fluctuations.

The stability conditions follow from the requirement of minimum of the free energy $F \equiv E - \sum_j \mu_j n_j$ (at fixed μ_j; $j = n,\ p,\ \Lambda,\ \Sigma,\ e,\ \mu$) for the system in thermodynamic equilibrium, at $T = 0$. Using Eq. (11) for the variation of energy of baryons, it is easy to find a variation $\delta F = \delta E - \sum_j \mu_j\, \delta n_j$ caused by a small change of $\delta n_j(\boldsymbol{p})$ (see Eq. (48) with j instead of i):

$$\delta F = \frac{1}{2} \sum_{ik} \left(\frac{1}{N_i} \delta_{ik} + f_0^{ik} \right) \delta n_i\, \delta n_k + \frac{1}{2} \frac{\partial \mu_e}{\partial n_e} (\delta n_e)^2 + \frac{1}{2} \frac{\partial \mu_\mu}{\partial n_\mu} (\delta n_\mu)^2. \tag{56}$$

Here $N_i \equiv m_i^* p_{\mathrm{F}i}/\pi^2$ is the density of states of particle species i on the Fermi surface; μ_l and n_l are, respectively, the relativistic chemical potential and number density of electrons ($l = e$) and muons ($l = \mu$). To derive Eq. (56) we presented the variation δE_l of the energy E_l of leptons, in the form ($l = e,\ \mu$)

$$\delta E_l = \frac{\partial E_l}{\partial n_l}\, \delta n_l + \frac{1}{2} \frac{\partial^2 E_l}{\partial n_l^2} (\delta n_l)^2 = \mu_l\, \delta n_l + \frac{1}{2} \frac{\partial \mu_l}{\partial n_l} (\delta n_l)^2. \tag{57}$$

As it should be, the expansion of F begins with the terms of the second order in δn_j. The requirement of minimum of F means that $\delta F \geq 0$, that is the quadratic form in the right-hand side of Eq. (56) must be positively defined.

In Eq. (56) for the variation δF of the free energy, we neglected a positive term related to the Coulomb energy of the perturbed matter. However, it must be taken into account if the perturbed matter acquired a non-zero charge, which is the case when $\delta n_p - \delta n_e - \delta n_\mu - \delta n_\Sigma \neq 0$. The contribution of the Coulomb energy to δF is then $\sim q^{-2}$ (see, e.g., [41, 42, 44]), where q is the wave number of plane-wave density fluctuation. Here we are interested only in the limit of long wavelengths, for which $q \to 0$. In this limit, the positive Coulomb energy can be arbitrarily large, so that the matter is stable against the long-wavelength density perturbations at *any density*. To exclude the 'stabilizing' contribution of the Coulomb energy we consider only those variations δn_j of the number densities which preserve the charge neutrality,

$$\delta n_p - \delta n_e - \delta n_\mu - \delta n_\Sigma = 0. \tag{58}$$

Expressing δn_e using this equation and substituting it into Eq. (56), one finds

$$\delta F = \frac{1}{2} \sum_{jm} A_{jm}\, \delta n_j\, \delta n_m, \tag{59}$$

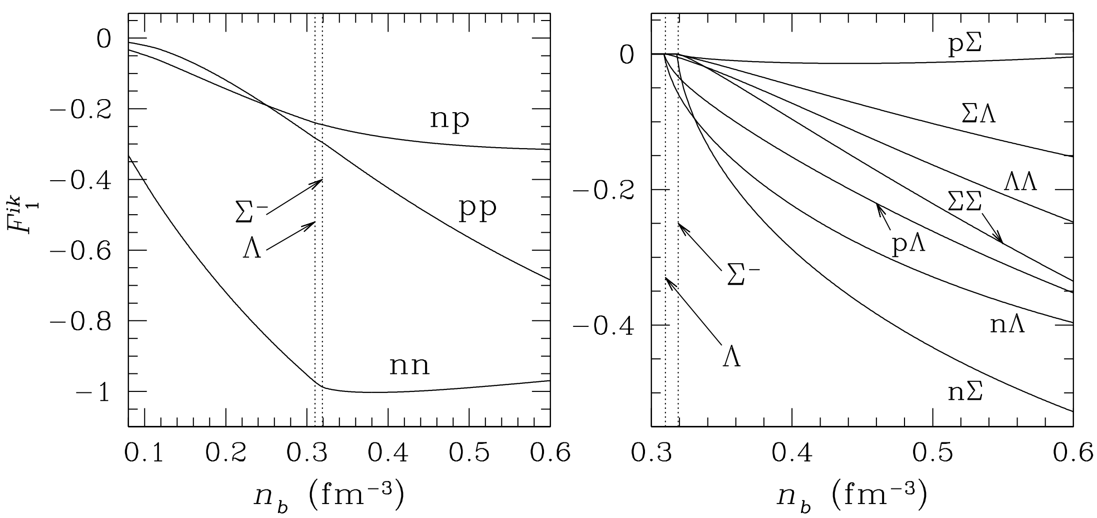

FIG. 4: The same as in Fig. 3 but for F_1^{ik}.

where the indices j and m run over all particle species except for electrons. The 5×5 matrix A_{jm} is given by

$$A_{jm} = \left(\frac{\delta_{jm}}{N_j} + f_0^{jm} \right) \delta_{jb}\delta_{mb} + \frac{\partial \mu_e}{\partial n_e} \, q_j q_m + \frac{\partial \mu_\mu}{\partial n_\mu} \, \delta_{j\mu}\delta_{m\mu}. \tag{60}$$

Here δ_{jb} and δ_{mb} equal 1 if j and $m = n$, p, Λ, Σ and 0 otherwise; q_j and q_m are, respectively, the electric charges of particle species j and m in units of proton charge (e.g., $q_e = -1$).

The requirement of positive definiteness of the quadratic form (59) imposes a set of conditions on the matrix elements A_{jm} or, equivalently, on the parameters f_0^{ik}; we write out only the simplest two of them

$$A_{jj} \geq 0, \tag{61}$$

$$A_{jj} A_{mm} - \left(A_{jm}\right)^2 \geq 0 \qquad (j \neq m). \tag{62}$$

These conditions are very well known in the literature devoted to stability of nucleon matter (see, e.g., [41, 42, 44]). For a mixture composed of *neutral* strongly interacting baryons they can be simplified and presented in the form (see, e.g., [43])

$$1 + F_0^{ii} \geq 0, \tag{63}$$

$$\left(1 + F_0^{ii}\right)\left(1 + F_0^{kk}\right) - \left(F_0^{ik}\right)^2 \geq 0 \qquad (i \neq k), \tag{64}$$

where the indices i and k refer to baryons and we introduced the dimensionless Landau parameters F_l^{ik},

$$F_l^{ik} \equiv \sqrt{N_i N_k} \, f_l^{ik}. \tag{65}$$

Our results are illustrated in Figs. 3 and 4, where the parameters F_0^{ik} and F_1^{ik} are presented for the third equation of state of Glendenning [28] as functions of n_b. The Landau parameters for neutrons and protons are plotted on the left panel in Figs. 3 and 4 (i, k= n, p). The right panel demonstrates the Landau parameters related to hyperons ($i = \Lambda$, Σ; $k = n$, p, Λ, Σ).

We checked that the nucleon-hyperon matter is stable down to baryon number density $n_b = 0.34 n_0 = 0.055$ fm^{-3} where the instability occurs (there are no hyperons and muons at such n_b). Mathematically, the occurence of instability means that the inequality (62) is not satisfied at $n_b < 0.34 n_0 = 0.055$ fm^{-3}. Thus, the matter is unstable with respect to long-wavelength density fluctuations. All other criteria, which are necessary for positive definiteness of the quadratic forms (32) and (59), are obeyed.

This type of instability is related to the crust-core phase transition and is carefully analyzed in the neutron-star literature (see, e.g., [41, 42, 44, 45, 46]). Since we study the stability of matter only in the extreme long-wavelength limit and under condition of microscopic charge neutrality, our result for the baryon number density of the crust-core interface is just the lower bound for the real value. Precise calculations would give a slightly higher value. For example, using extended Thomas-Fermi approach, Cheng et al. [46] found the crust-core boundary at $(0.058 - 0.073)$ fm^{-3}, depending on the choice of the σ-ω-ρ model parameters.

IV. SUMMARY

In this paper we calculated the relativistic entrainment matrix Y_{ik} at zero temperature for nucleon-hyperon mixture (see Eq. (31)). This matrix is a relativistic analogue of the entrainment matrix ρ_{ik} (also termed the mass-density matrix or Andreev-Bashkin matrix) and is related to ρ_{ik} in the non-relativistic limit by Eq. (7). The calculation is done in the frame of *relativistic* Landau Fermi-liquid theory [27], generalized to the case of mixtures. We show that, similarly to ρ_{ik} (see, e.g., [17, 18]), the matrix Y_{ik} can be expressed through the Landau parameters f_1^{ik} of nucleon-hyperon matter ($i, k = n$, p, Λ, Σ). If the number of baryon species is more than four, then the indices i and k in Eq. (31) should run over all these species.

The general results for Y_{ik}, following from the relativistic Landau Fermi-liquid theory, are illustrated with an example of the σ-ω-ρ mean-field model with scalar self-interactions. Using this model we obtain the analytic expression (66) for the matrix Y_{ik}. Comparison of this expression with Eq. (31) allows to determine the Landau parameters f_1^{ik} corresponding to the chosen mean-field model. Furthermore, we calculate the parameters f_0^{ik} and find that all other (spin-averaged) Landau parameters are equal zero, $f_l^{ik} = 0$ at $l \geq 2$.

In addition, we formulate a number of stability criteria for beta-equilibrated nucleon-hyperon matter (the positive definiteness of quadratic forms (32) and (59)). Employing the third equation of state of Glendenning [28], which is one of the versions of the σ-ω-ρ model with scalar self-interactions, we demonstrate that the nucleon-hyperon matter of neutron stars is stable down to the crust-core interface.

Our results can be used to model the pulsations of cold massive neutron stars with superfluid nucleon-hyperon cores. The generalization of these results to the case of finite temperatures will be given in a subsequent publication.

Appendix A

Using Eqs. (43)–(45), one can express the particle current densities $\boldsymbol{j}_i$ as functions of momenta $\boldsymbol{Q}_k$, and thus derive the coefficients of relativistic entrainment matrix Y_{ik} at zero temperature:

$$Y_{ik} = \frac{n_i}{m_i^*}\left[\delta_{ik} - \frac{g_{\omega i}}{A}\frac{n_k}{m_k^*}\left(\frac{g_{\omega k}}{m_\omega^2}a_{22} - \frac{g_{\rho k}I_{3k}}{m_\rho^2}a_{12}\right) - \frac{g_{\rho i}I_{3i}}{A}\frac{n_k}{m_k^*}\left(\frac{g_{\rho k}I_{3k}}{m_\rho^2}a_{11} - \frac{g_{\omega k}}{m_\omega^2}a_{21}\right)\right]. \tag{66}$$

Here m_i^* is given by Eq. (47) while the coefficients $a_{11}, a_{12}, a_{21}, a_{22}$, and A are given by

$$a_{11} = 1 + \sum_i \frac{g_{\omega i}^2}{m_\omega^2}\frac{n_i}{m_i^*}, \tag{67}$$

$$a_{12} = \sum_i \frac{g_{\omega i}g_{\rho i}I_{3i}}{m_\omega^2}\frac{n_i}{m_i^*}, \tag{68}$$

$$a_{21} = \sum_i \frac{g_{\omega i}g_{\rho i}I_{3i}}{m_\rho^2}\frac{n_i}{m_i^*}, \tag{69}$$

$$a_{22} = 1 + \sum_i \frac{g_{\rho i}^2 I_{3i}^2}{m_\rho^2}\frac{n_i}{m_i^*}, \tag{70}$$

$$A = a_{11}a_{22} - a_{12}a_{21}. \tag{71}$$

In formulae (67)–(70) the summation is assumed over all baryon species.

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