Shear Modulus of the Neutron Star Crust

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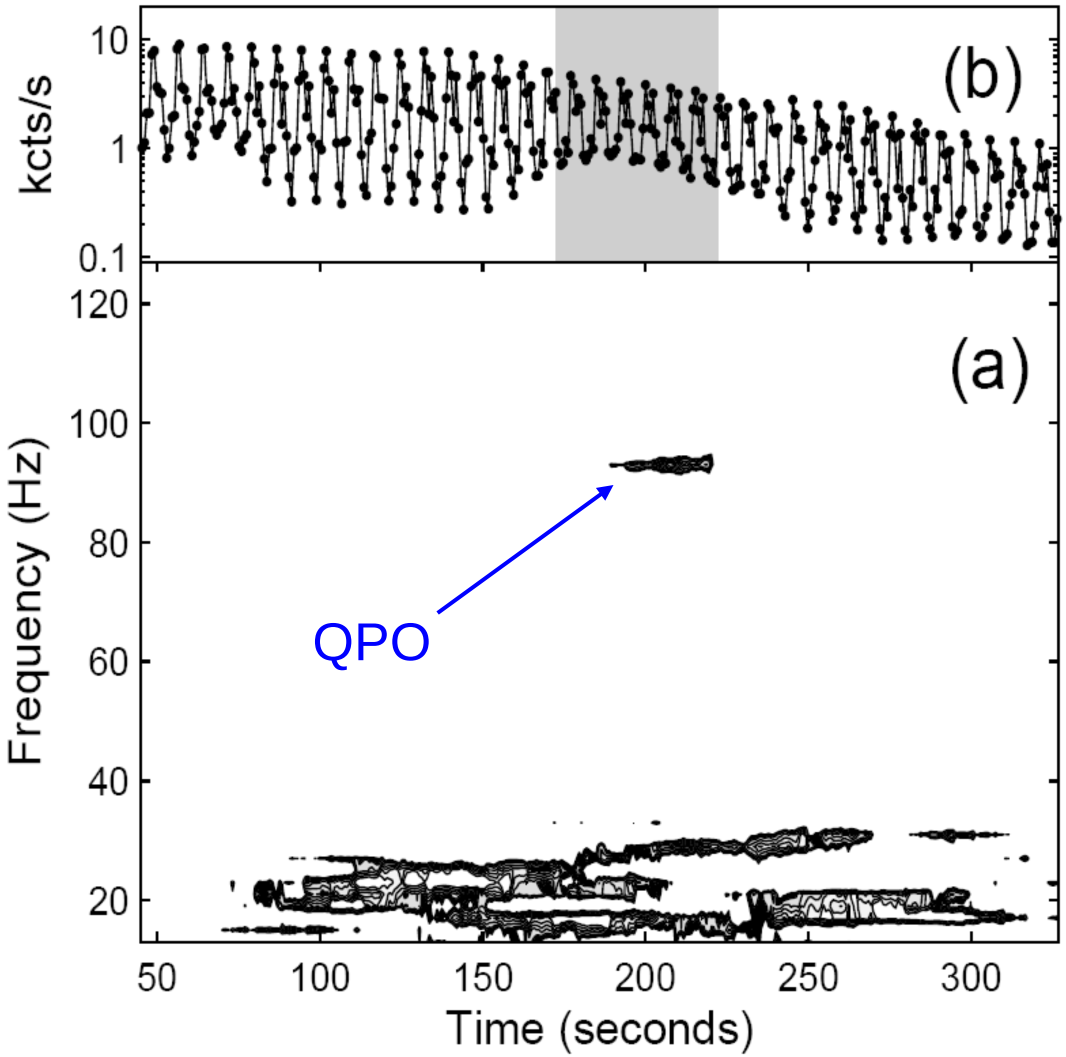
Part I: Why do we want to know the shear modulus?

QPOs in SGRs



SGR 1806-20
Giant Flare Tail

RXTE
(Israel et al., 2005)

QPOs in SGRs

- SGR **1806-20: 18, 26, 30, 50, 84, 92, 103, 150, 625, 648, 1840, ... Hz**

- SGR **1900+14: 28, 53, 84, 155 Hz**

- SGR **0526-66: 43 Hz**

Barat et al (1983, Prognoz, Venera)
Israel et al (2005, RXTE)
Strohmayer & Watts (2005,2006, RXTE, RHESSI)
Terasawa et al (2006, Geotail)
El-Mezeini & Ibrahim (2010, RXTE)

Toroidal Shear Modes

- The main candidate is the *starquakes* which cause the *torsional oscillations* (toroidal shear modes) of the neutron star crust.

- These modes were observed on Earth after the Chilean earthquake (1960). The period of the fundamental mode (oscillations of the whole hemispheres in the opposite directions) is 43 min.

- In neutron stars (as opposed to completely solid bodies) these modes must be *easier to excite* because the crust is relatively thin and the friction against the superfluid core is relatively weak. There is no compression of dense matter and vertical motions. The elasticity is associated with weak Coulomb forces only.

- These modes have significant amplitudes right *at the observable surface*.

- Duncan (1998): realistic period estimate 33.6 ms (30 Hz). As we shall see shortly, the torsional oscillation frequencies are determined by the *shear modulus of the neutron star crust.*

Equation of Motion

velocity **stress tensor**

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \frac{1}{\rho}\,\nabla \cdot \boldsymbol{\sigma} - \nabla\Phi + \frac{1}{\rho}\,\frac{1}{c}\,[\boldsymbol{j} \times \boldsymbol{B}]$$

mass density **gravitational potential** **current density**

$$\boldsymbol{E} = -\frac{1}{c}\,[\boldsymbol{v} \times \boldsymbol{B}]$$

frozen-in magnetic field
infinite electric conductivity

$$\frac{4\pi}{c}\,\boldsymbol{j} = \nabla \times \boldsymbol{B} + \frac{1}{c^2}\,\frac{\partial}{\partial t}\,[\boldsymbol{v} \times \boldsymbol{B}]$$

Maxwell equation

Linearization

displacement

$$\boldsymbol{r} \to \boldsymbol{x} = \boldsymbol{r} + \boldsymbol{u}(\boldsymbol{r}, t) \qquad \boldsymbol{v} = \frac{\partial \boldsymbol{u}}{\partial t}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times [\boldsymbol{v} \times \boldsymbol{B}]$$ **Maxwell equation**

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}^{\text{equil}} + \sigma'_{\alpha\beta} \qquad \sigma_{\alpha\beta}^{\text{equil}} = -P\delta_{\alpha\beta}$$

$$\sigma' = \delta\sigma - (\boldsymbol{u} \cdot \nabla)\sigma^{\text{equil}}$$

pressure

Lagrange variation

Euler variation

Stress Tensor Variation

$$\delta\sigma_{\alpha\beta} = \Gamma_1 P \varepsilon_{\gamma\gamma}\,\delta_{\alpha\beta} + 2\mu\left(\varepsilon_{\alpha\beta} - \frac{1}{3}\varepsilon_{\gamma\gamma}\,\delta_{\alpha\beta}\right)$$

(Landau and Lifshitz VII)

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha\beta} + u_{\beta\alpha}\right)$$ **deformation tensor**

$$u_{\alpha\beta} = \frac{\partial u_\alpha}{\partial r_\beta}$$ **deformation gradient**

$$\boxed{\boldsymbol{r} \rightarrow \boldsymbol{x} = \boldsymbol{r} + \boldsymbol{u}(\boldsymbol{r}, t)}$$

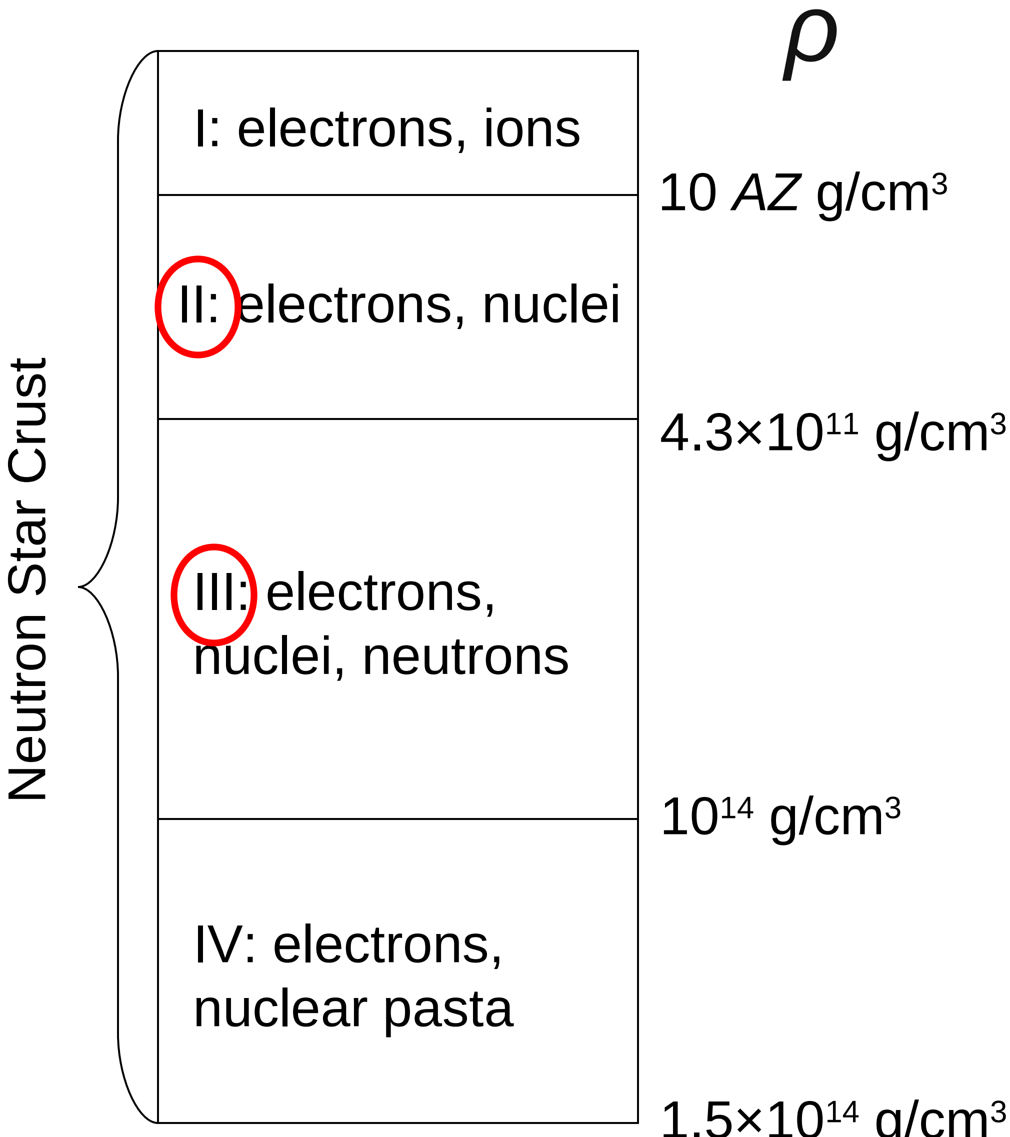
$$\boxed{\frac{\delta\rho}{\rho} = -u_{\gamma\gamma} = -\varepsilon_{\gamma\gamma}}$$

$$\Gamma_1 = \frac{\partial \ln P}{\partial \ln \rho}$$

The result is a linear equation for $\boldsymbol{u}(\boldsymbol{r},t)$, eigennumbers of which (given appropriate boundary conditions) are the frequencies of the shear modes. The shear modulus enters as a parameter.

Part II: How do we find the shear modulus?

Neutron Star Crust



- We study regions II and III
- Electrons are in the form of a strongly degenerate gas; they are ultrarelativistic at $\rho \gg 10^6$ g/cm^3
- Atoms are fully pressure-ionized and form crystals
- At any density we have *bcc* crystal of nuclei of one type (charge number Z, number density n_{ion}, mass M)
- Electrons provide perfectly uniform charge-compensating background
- This is the *Coulomb crystal* model
- Dripped neutrons play no role

What was already known about the shear modulus?

Shear modulus of the *static* Coulomb lattice
(Fuchs, 1936)

Shear modulus of the vibrating *classic* Coulomb lattice by Monte Carlo simulations
(Ogata and Ichimaru, 1990)

We shall include *quantum* effects and do everything analytically (almost)

Nondeformed Hamiltonian

$$\hat{H}_0 = U_0 + \hat{H}_2$$

static lattice energy

$$U_0 = -\zeta N_{\rm ion} \frac{Z^2 e^2}{a_{\rm ion}}$$

$$a_{\rm ion} = \left(\frac{3}{4\pi n_{\rm ion}} \right)^{1/3} \qquad \zeta \approx 0.9$$

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sum over phonon modes

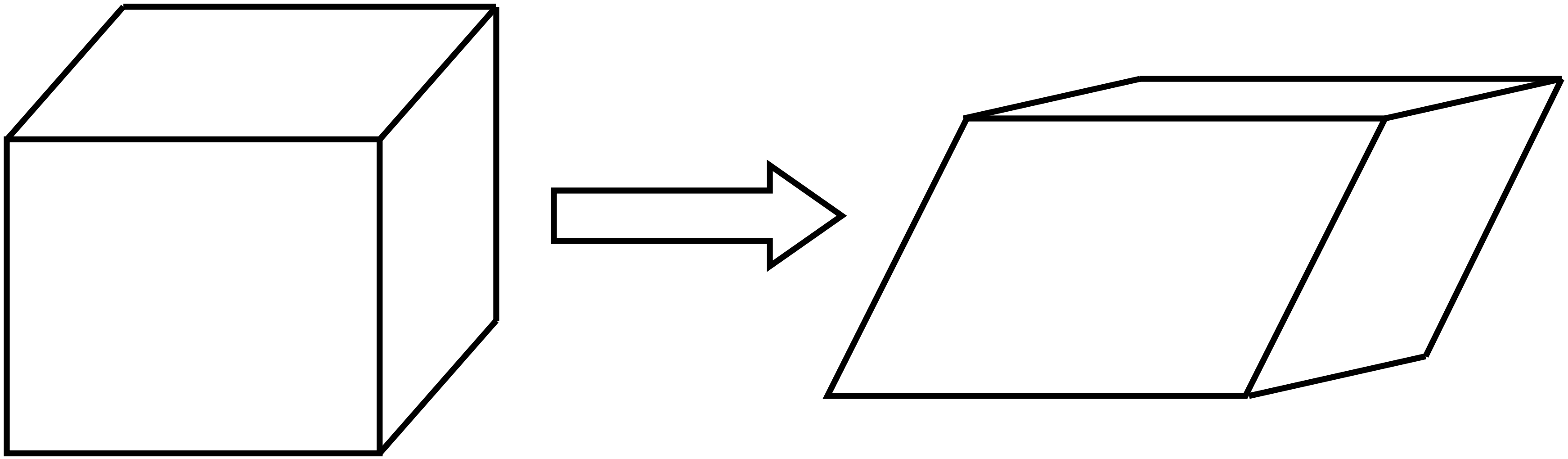
$$\hat{H}_2 = \sum_{\boldsymbol{k}s} \hbar\omega_{\boldsymbol{k}s} \left(\frac{1}{2} + a^\dagger_{\boldsymbol{k}s} a_{\boldsymbol{k}s} \right)$$

Hamiltonian of a collection of harmonic oscillators

phonon frequencies

phonon creation and annihilation operators

Deformation



$$X_\alpha = R_\alpha + u_{\alpha\beta} R_\beta$$

infinitely small uniform deformation

Variation of the Hamiltonian

$$\hat{H}_0 \to \hat{H}_0 + \delta\hat{H} \qquad \delta\hat{H} = \delta U_0 + \delta\hat{H}_2$$

$$\frac{\delta U_0}{V} = -P^{\text{static}} u_{\alpha\alpha} + \frac{1}{2} S^{\text{static}}_{\alpha\beta\gamma\lambda} u_{\alpha\beta} u_{\gamma\lambda}$$

volume

electrostatic pressure

static elastic moduli (Fuchs)

$$\delta\hat{H}_2 = \frac{1}{2}\sum_{\boldsymbol{k}ss'} \left(\Phi^{\boldsymbol{k}ss'}_{\alpha\beta} u_{\alpha\beta} + \frac{1}{2}\Phi^{\boldsymbol{k}ss'}_{\alpha\beta\gamma\lambda} u_{\alpha\beta} u_{\gamma\lambda} \right) (a_{\boldsymbol{k}s} + a^{\dagger}_{-\boldsymbol{k}s})(a_{-\boldsymbol{k}s'} + a^{\dagger}_{\boldsymbol{k}s'})$$

Calculation of phonon $S_{\alpha\beta\gamma\lambda}$

Thermodynamic perturbation theory (Landau and Lifshitz V)

$$\delta F = \sum_n (\delta \hat{H}_2)_{nn} w_n + \frac{1}{2T}\left(\left[\sum_n (\delta \hat{H}_2)_{nn} w_n\right]^2 - \sum_n [(\delta \hat{H}_2)_{nn}]^2 w_n\right) + \sum_{n,m}{}' \frac{|(\delta \hat{H}_2)_{nm}|^2 w_n}{E_n^{(0)} - E_m^{(0)}} + \dots$$

probability of state n

$$\Downarrow$$

$$\frac{\delta F}{V} = -P^{\mathrm{phon}} u_{\alpha\alpha} + \frac{1}{2} S^{\mathrm{phon}}_{\alpha\beta\gamma\lambda} u_{\alpha\beta} u_{\gamma\lambda}$$

phonon pressure

elastic moduli due to ion motion (what we're looking for)

Anisotropic Stress Tensor

in anisotropic material instead of

$$\delta\sigma_{\alpha\beta} = \Gamma_1 P \varepsilon_{\gamma\gamma}\,\delta_{\alpha\beta} + 2\mu\left(\varepsilon_{\alpha\beta} - \frac{1}{3}\varepsilon_{\gamma\gamma}\,\delta_{\alpha\beta}\right)$$

we have

$$\delta\sigma_{\alpha\beta} = B_{\alpha\beta\gamma\lambda}\varepsilon_{\gamma\lambda}$$

for cubic symmetry only

$$B_{1111}, \quad B_{1122}, \quad B_{1212} = B_{1221}$$

are independent

$$S_{\alpha\beta\gamma\lambda} = B_{\alpha\beta\gamma\lambda} + P(\delta_{\alpha\lambda}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\lambda})$$

stress tensor expansion coefficient

free energy expansion coefficient

Isotropisation

for isotropic material we can formally introduce B^{isotr}

$$\Gamma_1 P \varepsilon_{\gamma\gamma}\,\delta_{\alpha\beta} + 2\mu\left(\varepsilon_{\alpha\beta} - \frac{1}{3}\,\varepsilon_{\gamma\gamma}\,\delta_{\alpha\beta}\right) = B^{\text{isotr}}_{\alpha\beta\gamma\lambda}\varepsilon_{\gamma\lambda}$$

$$\mu = B^{\text{isotr}}_{1212} \to B_{1212} = S_{1212}$$

so we can simply use S_{1212} **as the shear modulus**

$$\Gamma_1 P = \frac{1}{3}\,(B^{\text{isotr}}_{1111} + 2B^{\text{isotr}}_{1122}) = B^{\text{isotr}}_{1111} - \frac{4}{3}\,B^{\text{isotr}}_{1212}$$

(only two B^{isotr} **are independent)**

Another Option: Ogata & Ichimaru

$$\rho \ddot{u}_\alpha = B_{\alpha\beta\gamma\lambda} \frac{\partial^2 u_\gamma}{\partial r_\beta \partial r_\lambda}$$

linearized equation of motion

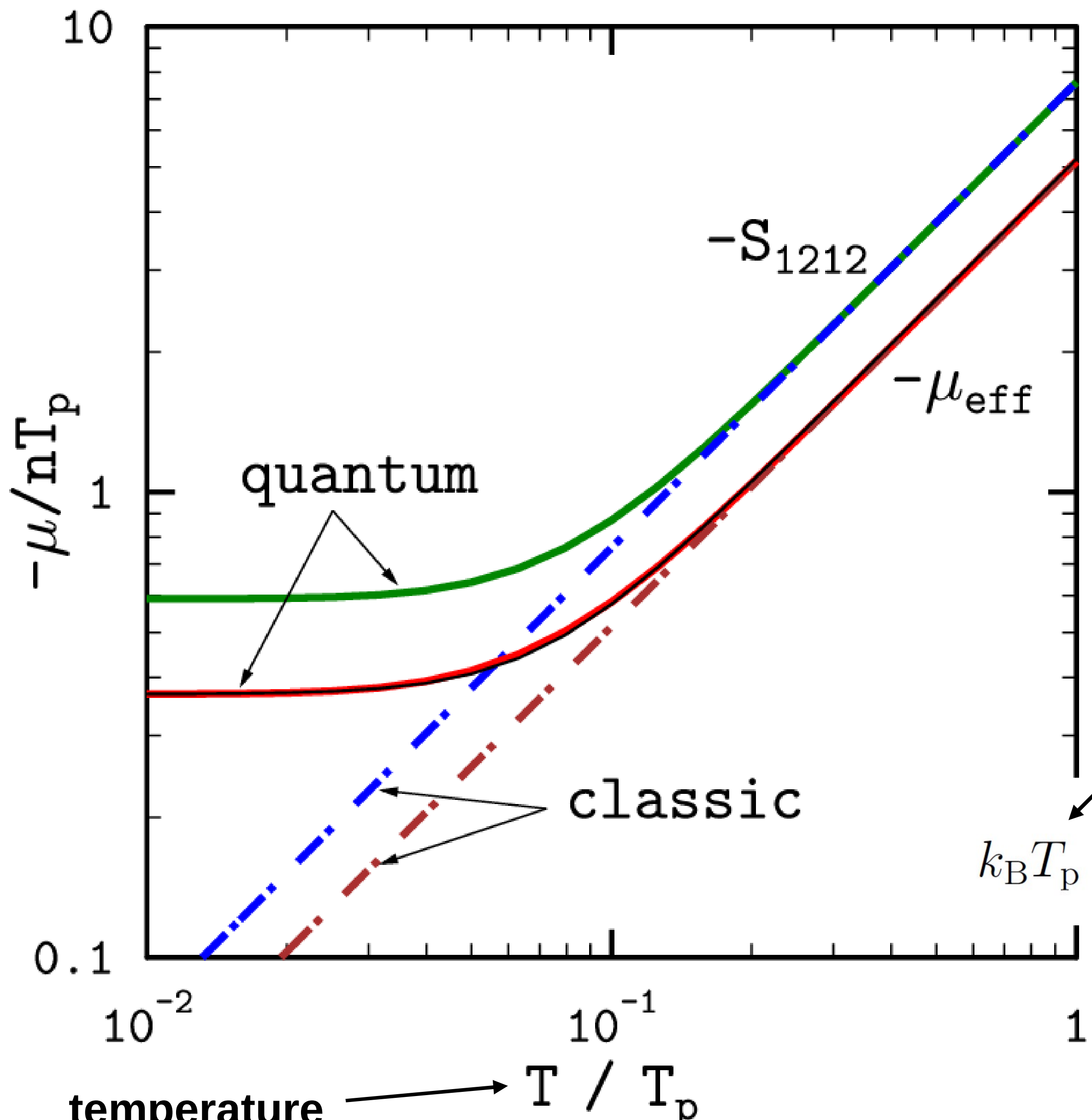
Substitute plane waves and average over all *u* transverse to *k* and over all directions of *k*

$$\Downarrow$$

$$\boxed{\mu_{\text{eff}} = \frac{1}{5}(B_{1111} - B_{1122} + 3B_{1212}) = \frac{1}{5}(S_{1111} - S_{1122} - S_{1221} + 4S_{1212})}$$

In principle, a model of anisotropic matter must be assumed and anisotropic equation of motion must be solved

Phonon Contribution to the Shear Modulus



temperature

ion plasma temperature

$$k_\mathrm{B}T_\mathrm{p} = \hbar\sqrt{\frac{4\pi Z^2 e^2 n_\mathrm{ion}}{M}}$$

Full Shear Modulus

$$S_{1212} = S_{1212}^{\text{static}} + S_{1212}^{\text{phon}} = 0.1827\, n_{\text{ion}} \frac{Z^2 e^2}{a_{\text{ion}}} + S_{1212}^{\text{phon}}$$

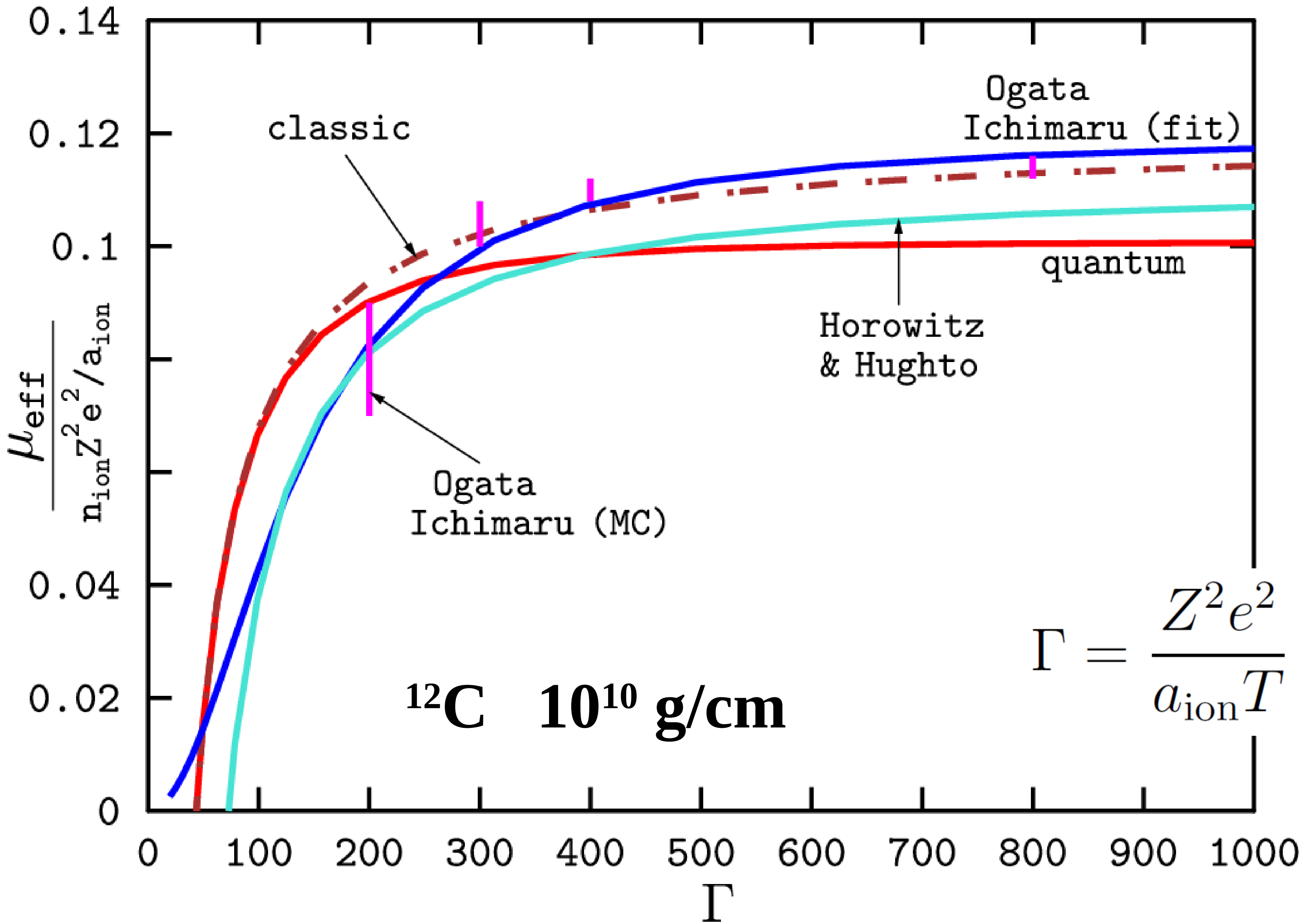
$$\mu_{\text{eff}} = \mu_{\text{eff}}^{\text{static}} + \mu_{\text{eff}}^{\text{phon}} = 0.1194\, n_{\text{ion}} \frac{Z^2 e^2}{a_{\text{ion}}} + \mu_{\text{eff}}^{\text{phon}}$$

static lattice part was calculated by Fuchs (1936)

$$\boxed{-\mu_{\text{eff}}^{\text{phon}} = n_{\text{ion}} T_{\text{p}} \left[0.368^3 + 140 \left(\frac{T}{T_{\text{p}}} \right)^3 \right]^{1/3}}$$

main result

Full Effective Shear Modulus



Thank you!