Electrical conductivity of the neutron star crust at low temperatures

A.I. Chugunov

Ioffe Institute

Physics of Neutron Stars - 2011

July 11-15, 2011, St. Petersburg, Russia

Motivation

The electrical conductivity at low temperatures ($\lesssim 10^7$ K) is an important parameter for modeling:

- Evolution of pulsar magnetic field
- Size and life time of magnetically confined mountains (see Vigelius&Melatos, ApJ **717** (2010), 404)
- ...

Widely used model [Gnedin et al., MNRAS **324**, 725 (2001)] may lead to overestimation of the electrical conductivity for orders of magnitude at $T \lesssim 10^7$ K.

The electrical conductivity

$$\sigma = \frac{e^2 n_e \tau}{m_e^*}$$

$$\tau^{-1} = \tau_{\rm ee}^{-1} + \boxed{\tau_{\rm ei}^{-1}} + \tau_{\rm imp}^{-1}$$

$$\tau_{\rm ei} \begin{cases} T \gtrsim 10^7 \text{ K — Gnedin et al. [MNRAS } \mathbf{324}\text{, 725 (2001)]} \\ T \lesssim 10^7 \text{ K — at this talk} \end{cases}$$

Electron-phonon scattering

The momentum conservation

$$\mathbf{p} + \hbar\mathbf{q} + \hbar\mathbf{g} = \mathbf{p}'$$

$\mathbf{g}$ — reciprocal lattice vector

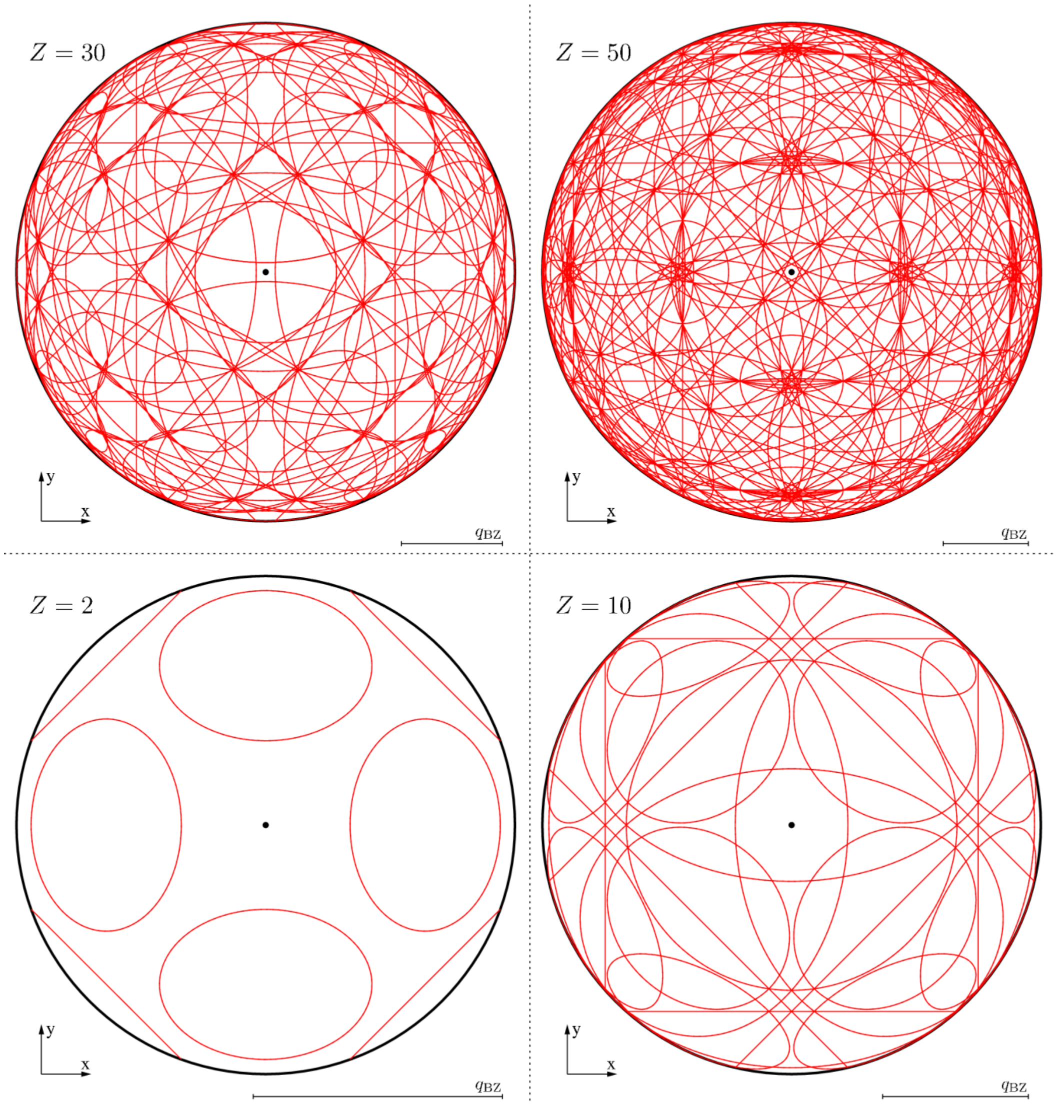
Low temperatures:

$$q \sim T/V_{\rm ph}\hbar \ll q_{\rm BZ}$$

$\Downarrow$

Scattering is efficient only in a vicinity of intersection of Brillouin zones boundaries with Fermi sphere :

$$\mathbf{p} - \mathbf{p}' \approx \hbar\mathbf{g}$$



Electrons in a vicinity of boundaries

$$E^{\pm} = \epsilon_{\rm F} + V_{\perp}\, c\Delta p_{\perp} \pm \sqrt{\left(V_{||}\, c\,\Delta p_{||}\right)^2 + V_{\mathbf{g}}{}^2}$$

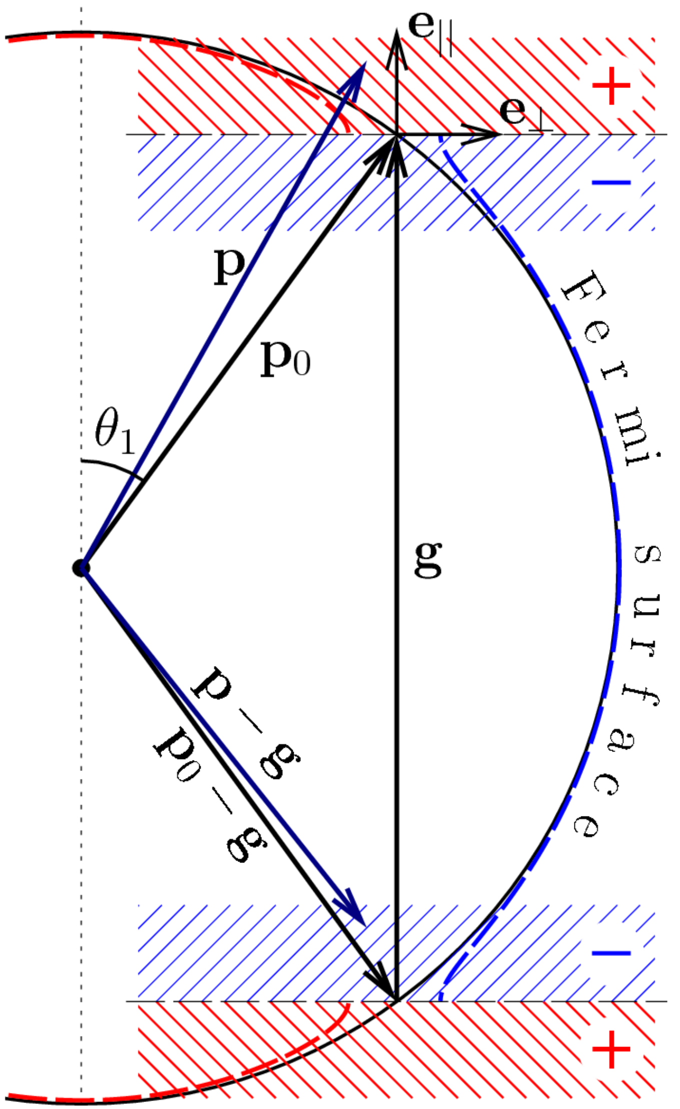
$$\Psi^{+}_{\sigma,\,\mathbf{p}} = u_{\mathbf{k}} \Psi^{\rm free}_{\sigma}(\mathbf{p}) + v_{\mathbf{k}}\, \Psi^{\rm free}_{\sigma}(\mathbf{p} - \hbar\,\mathbf{g})$$

$$\Psi^{-}_{\sigma,\,\mathbf{p}} = v_{\mathbf{k}}\, \Psi^{\rm free}_{\sigma}(\mathbf{p}) - u_{\mathbf{k}}\, \Psi^{\rm free}_{\sigma}(\mathbf{p} - \hbar\,\mathbf{g})$$

$$u_{\mathbf{k}} = \frac{V_{\mathbf{g}}}{\sqrt{2\,\epsilon_{\mathbf{k}}\ (\epsilon_{\mathbf{k}} - \xi_{\mathbf{k}})}} \qquad v_{\mathbf{k}} = \sqrt{\frac{\epsilon_{\mathbf{k}} - \xi_{\mathbf{k}}}{2\,\epsilon_{\mathbf{k}}}}$$

$$\xi_{\mathbf{k}} = V_{||}\, c\,\Delta p_{||}, \quad \epsilon_{\mathbf{k}} = \sqrt{\left(V_{||}\, c\,\Delta p_{||}\right)^2 + V_{\mathbf{g}}{}^2}$$

$$V_{||} = \frac{g}{2\,k_{\rm F}}, \quad V_{\perp} = \sqrt{1 - V_{||}^2}$$



Are gaps important?

$$\epsilon_{\mathbf{k}} = \sqrt{\left(V_{||}\, c\, \Delta p_{||}\right)^2 + V_{\mathbf{g}}{}^2}$$

Yes, if

$$\Delta p \lesssim 2\, V_{\mathbf{g}}/c$$

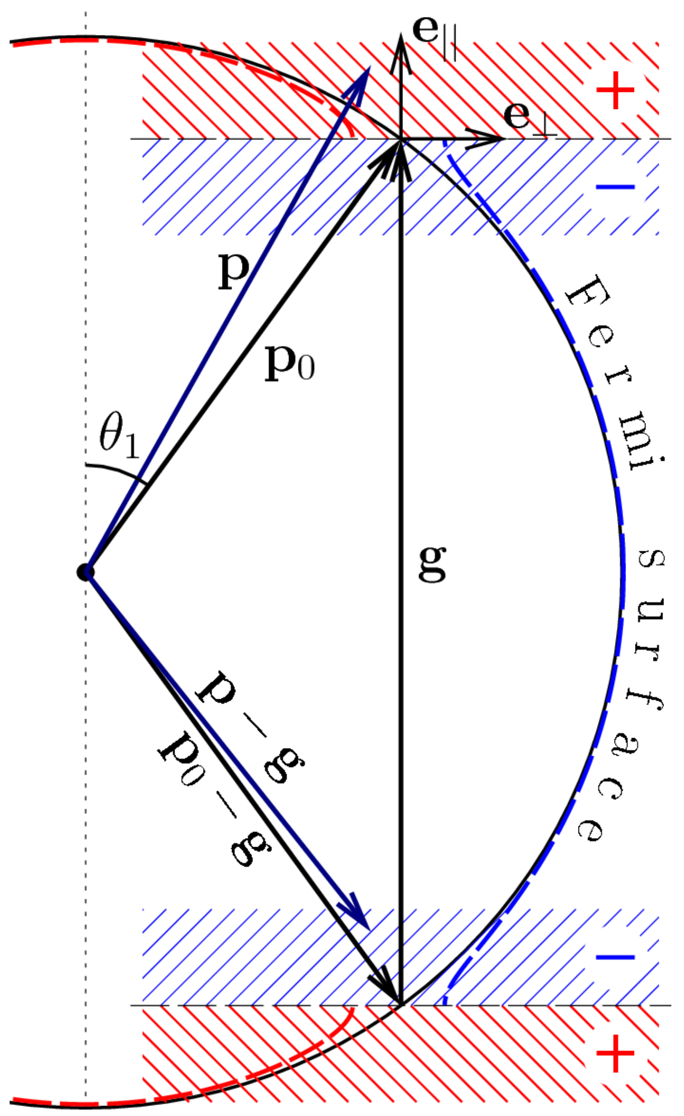
Wave numbers of thermal phonons

$$q_T \sim T/V_{\mathrm{ph}}$$

Scattering in a gap region

$$q_T \sim \Delta p \;\Rightarrow T \sim 2V_{\mathbf{g}}\, V_{\mathrm{ph}}/c$$

Raikh, Yakovlev, Ap&SS, **87** (1982), 193



Typical temperatures

Ion plasma temperature:

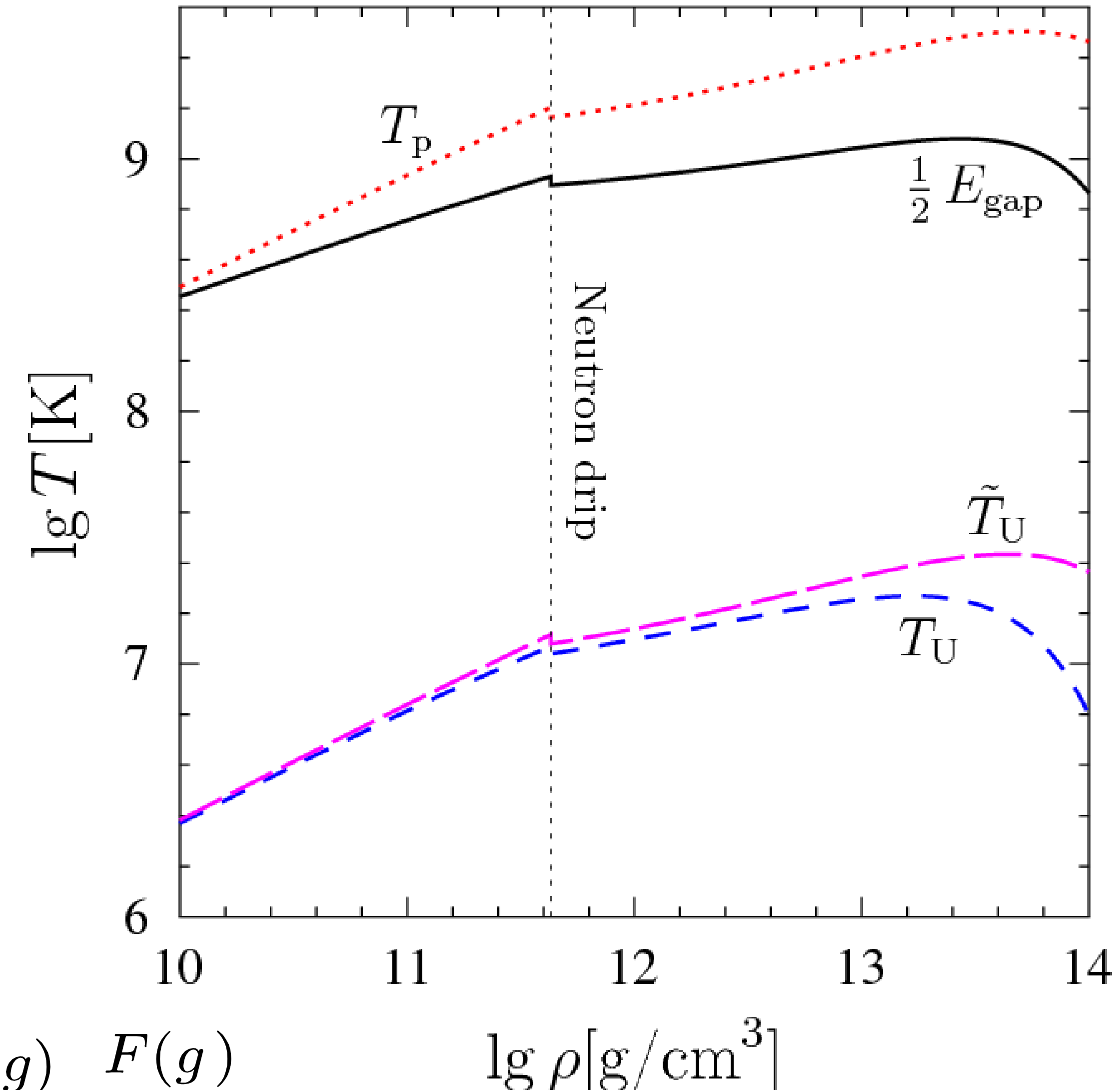
$$T_{\rm p} = \hbar\omega_{\rm p} = \hbar\left(\frac{4\pi Z^2 e^2 n_{\rm i}}{m_{\rm i}}\right)^{1/2}$$

The gap energy:

$$E_{\rm gap} = 2\,V_{g=k_{\rm F}}$$

Gaps are important if:

$$T_{\rm U} = \frac{V_{\rm ph}}{c} E_{\rm gap}$$



$$V_{\mathbf{g}} = 4\,\pi\,n_{\rm i}\,Z\,e^2\,\mathrm{e}^{-W(g)}\,\frac{F(g)}{g^2\,\epsilon(g)}$$

If $T \lesssim T_{\mathrm{U}}$

Free electron approximation is invalid

Raikh, Yakovlev (1982): Umklapp processes are switched off

Gnedin et al. (2001):

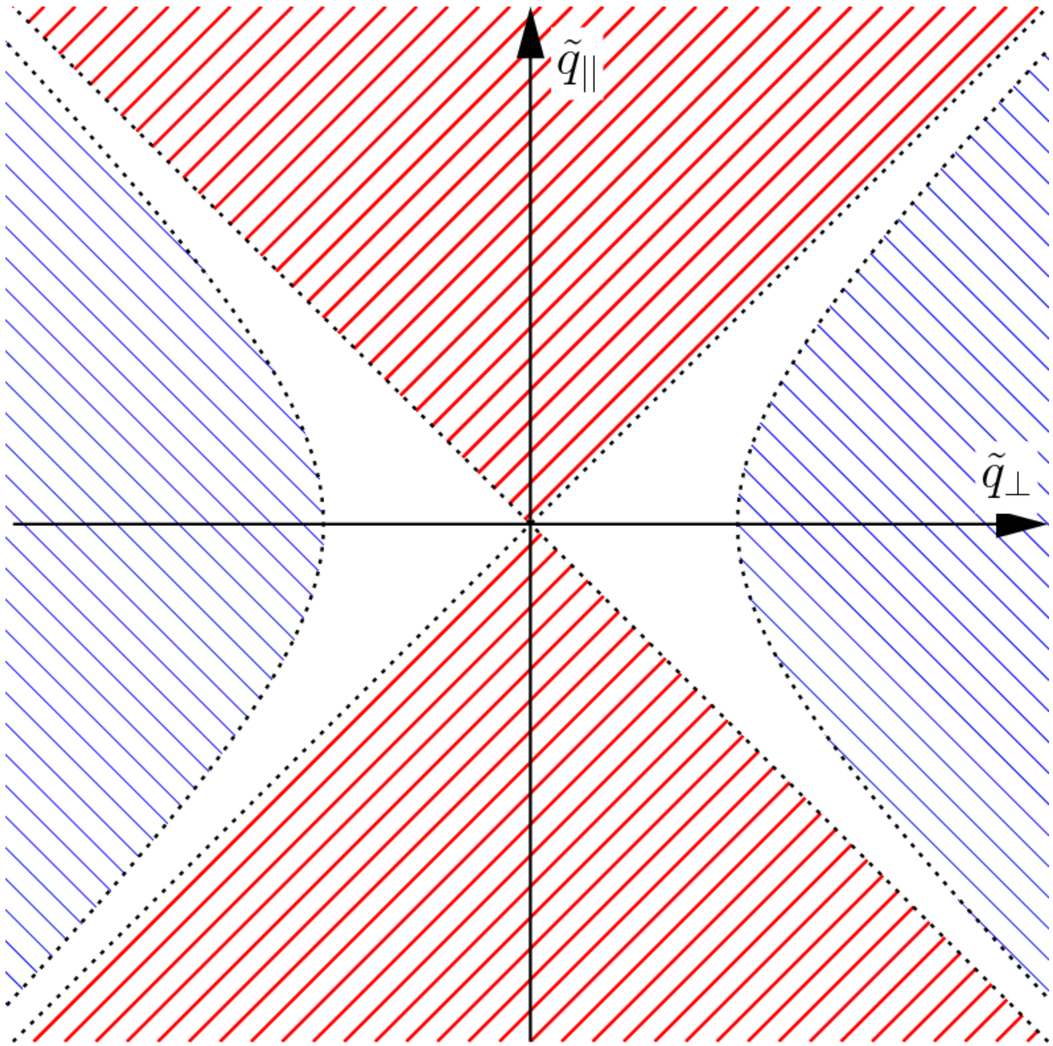
$$\Lambda_{\mathrm{ei}}^{\sigma,\kappa} = \Lambda_{\mathrm{ei,high}}^{\sigma,\kappa} \exp(-T_{\mathrm{u}}/T) + \Lambda_{\mathrm{ei,low}}^{\sigma,\kappa}[1 - \exp(-T_{\mathrm{u}}/T)]$$

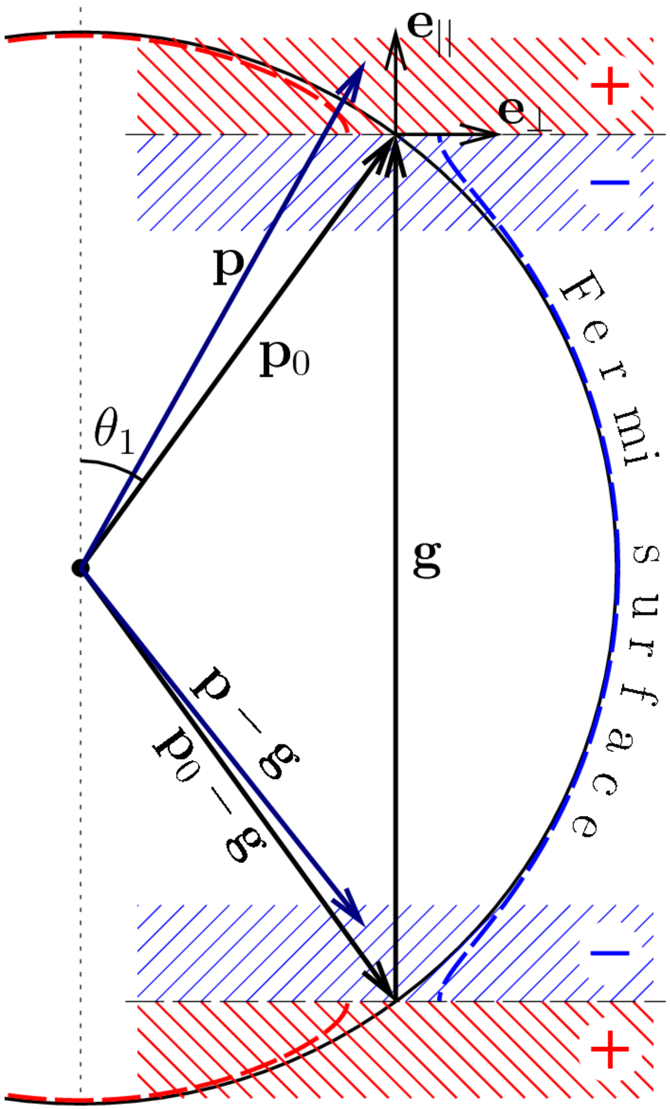
Accurate calculation should be based on the wave functions:

$$\Psi_{\sigma,\,\mathbf{p}}^{+} = u_{\mathbf{k}}\Psi_{\sigma}^{\mathrm{free}}(\mathbf{p}) + v_{\mathbf{k}}\Psi_{\sigma}^{\mathrm{free}}(\mathbf{p} - \hbar\,\mathbf{g})$$

$$\Psi_{\sigma,\,\mathbf{p}}^{-} = v_{\mathbf{k}}\Psi_{\sigma}^{\mathrm{free}}(\mathbf{p}) - u_{\mathbf{k}}\Psi_{\sigma}^{\mathrm{free}}(\mathbf{p} - \hbar\,\mathbf{g})$$

Types of scattering





Scattering probability

$$P_{\mathbf{p},\,\mathbf{q},\,\nu}^{\mathbf{p}\,'} = \frac{2\pi}{\hbar}\left|M_{\mathbf{q}}^{(i)(i')}(\mathbf{p},\mathbf{p}\,')\right|^2 f_{\mathbf{p}}\,(1-f_{\mathbf{p}\,'})\,\delta\,(E_{\mathbf{p}}+\hbar\omega_{\mathbf{q},\,\nu}-E_{\mathbf{p}\,'})$$

$$M_{\mathbf{q}}^{(i)(i')}(\mathbf{p},\mathbf{p}') = \frac{n_{\mathrm{i}}\,n_{\mathbf{q},\,\nu}{}^{1/2}}{2\epsilon_{\mathbf{k}}}\left(\frac{\hbar}{m_{\mathrm{i}}n_{\mathrm{i}}\,V\,\omega_{\mathbf{q},\,\nu}}\right)^{1/2} \times J_{\mathbf{q},\,\nu}^{(i)(i')}(\mathbf{p}\,,\mathbf{p}\,')\,(2\pi)^4\,\delta_{\mathbf{k}-\mathbf{k}\,'+\mathbf{q}}$$

$$J_{\mathbf{q},\,\nu}^{(i)(i')}(\mathbf{p},\mathbf{p}') = -\mathbf{e}_{\mathbf{q},\,\nu}\left\langle\Psi_{\sigma,\mathbf{p}}^{(i)}\middle|\nabla U_a\middle|\Psi_{\sigma,\,\mathbf{p}\,'}^{(i')}\right\rangle$$

$$P_{\mathbf{p},\,\mathbf{q},\nu}^{\mathbf{p}\,'} = (2\pi)^4\,\frac{n_{\mathrm{i}}\,n_{\mathbf{q},\,\nu}}{2m_{\mathrm{i}}\omega_{\mathbf{q},\,\nu}}\,(\mathbf{e}_{\mathbf{q},\nu}\cdot\mathbf{g})^2\,\left(F_{uv}^{\mathbf{g}}\right)^2\,U_g^2 \times f_{\mathbf{p}}\,(1-f_{\mathbf{p}'})\,\delta\,(E_{\mathbf{p}}+\hbar\omega_{\mathbf{q},\nu}-E_{\mathbf{p}'})\,\delta(\mathbf{k}'-\mathbf{k}-\mathbf{q})$$

"++" processes: $F_{uv}^{\mathbf{g}} = v_{\mathbf{k}}u_{\mathbf{k}\,'} - u_{\mathbf{k}}v_{\mathbf{k}\,'}$

Conductivity: variational estimate

$$\sigma = \left|\mathbf{I}_1\right|^2 / I_2$$

$$\mathbf{I}_1 = \int e\mathbf{V}_{\mathbf{k}}\, \Phi_{\mathbf{k}} \frac{\partial f}{\partial E_{\mathbf{p}}} \frac{\mathrm{d}^3 k}{(2\pi)^3}$$

$$I_2 = \frac{1}{T} \sum_{\nu} \int\int\int (\Phi_{\mathbf{k}} - \Phi_{\mathbf{k}'})^2 \; P_{\mathbf{p},\mathbf{q},\nu}^{\mathbf{p}'} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{\mathrm{d}^3 q}{(2\pi)^3}$$

For simple trial function $\Phi_{\mathbf{k}} = \mathbf{V}_{\mathbf{k}} \cdot \mathbf{u}$, $\mathbf{V}_{\mathbf{k}} = \partial E_{\mathbf{k}} / \partial \mathbf{p}$

$$\sigma = \frac{e^2 n_e \tau}{m_e^*}, \quad \tau = \frac{{p_{\mathrm{F}}}^2\, v_{\mathrm{F}}}{4\pi\, Z^2 e^4\, n_i\, \Lambda_\sigma}$$

$$\Lambda_\sigma = \frac{p_{\mathrm{F}}}{8\pi^2} \frac{\hbar}{T\, m_{\mathrm{i}}} \sum_{\mathbf{g}} \phi(g)^2\, I_3^{\mathbf{g}}$$

$$I_3^{\mathbf{g}} = 2\, V_{||}\, {V_\perp}^2 \sum_{\nu} \int \frac{\left(\Delta\widetilde{q^2}\right)^{7/2} \left(\Delta\widetilde{q^2} + 4\,\widetilde{V}_{\mathbf{g}}^2\right)^{1/2}}{\left(4\,\widetilde{q}_{||}^2\,\widetilde{V}_{\mathbf{g}}^2 + \left(\Delta\widetilde{q^2}\right)^2\right)^2} \frac{e^{-z_\nu}}{(1-e^{-z_\nu})^2} (\mathbf{e}_{\mathbf{q},\nu} \cdot \mathbf{g})^2 \mathrm{d}\varphi_{\mathbf{k}}\, \mathrm{d}^3 q$$

Low temperature limit

$$\Lambda_\sigma^{\mathrm{low}} \approx 5{\alpha_\mathrm{f}}^2 \, \frac{Z^{2/3}}{C^6} \, \frac{{p_\mathrm{F}}^2}{m_\mathrm{i}\, k_\mathrm{B} T_\mathrm{p}} \left(\frac{T}{\widetilde{T}_\mathrm{U}}\right)^5 \mathrm{e}^{0.4w}$$

$$w \approx w_{\mathrm{form}} = 43\, r_\mathrm{p}^2/a^2, \qquad C = 0.36$$

All temperatures

$$\Lambda_\sigma^{\mathrm{int}} \approx \left[\frac{1}{\Lambda_\sigma^{\mathrm{high}}} + \frac{1}{\Lambda_\sigma^{\mathrm{low}}}\right]^{-1}$$

$\Lambda_\sigma^{\mathrm{high}}$ — Gnedin et al., MNRAS **324**, 725 (2001)

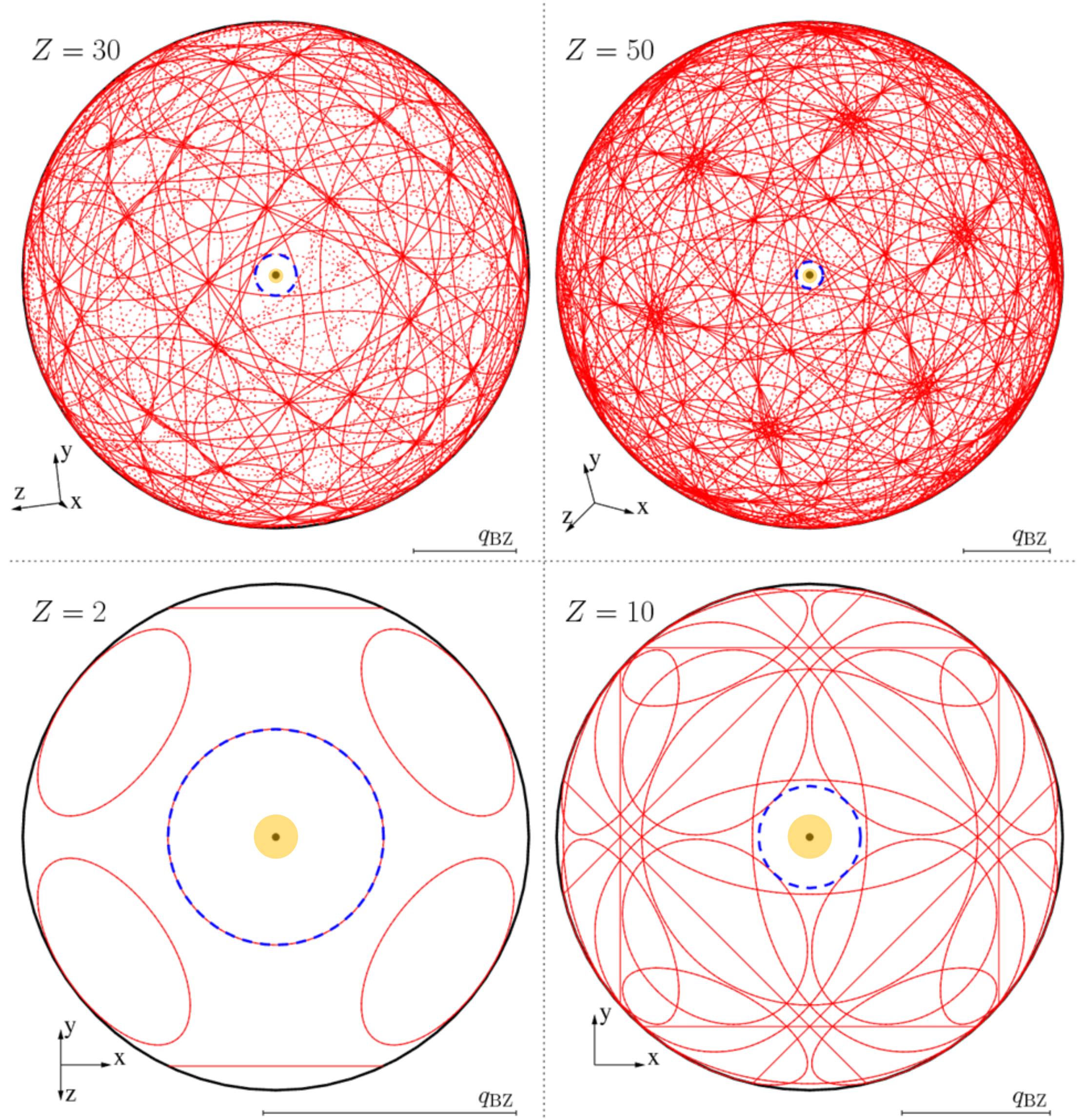
On the choice of the trial function

$$\Phi_{\mathbf{k}} = \frac{\partial E_{\mathbf{k}}}{\partial \mathbf{k}} \cdot \mathbf{u}$$

$$f_{\mathbf{k}} = f_{\mathbf{k}}^{0} - \Phi_{\mathbf{k}} \frac{\partial f_{\mathbf{k}}^{0}}{\partial E_{\mathbf{k}}}$$

$$\Phi_{\mathbf{k}}^{\exp} = \begin{cases} 0, & \Delta k \le \Delta k_0 \\ 1, & \Delta k > \Delta k_0 \end{cases}$$

$$\tau \propto \exp\left(\frac{\Delta k_0 V_{\rm ph}}{T}\right)$$

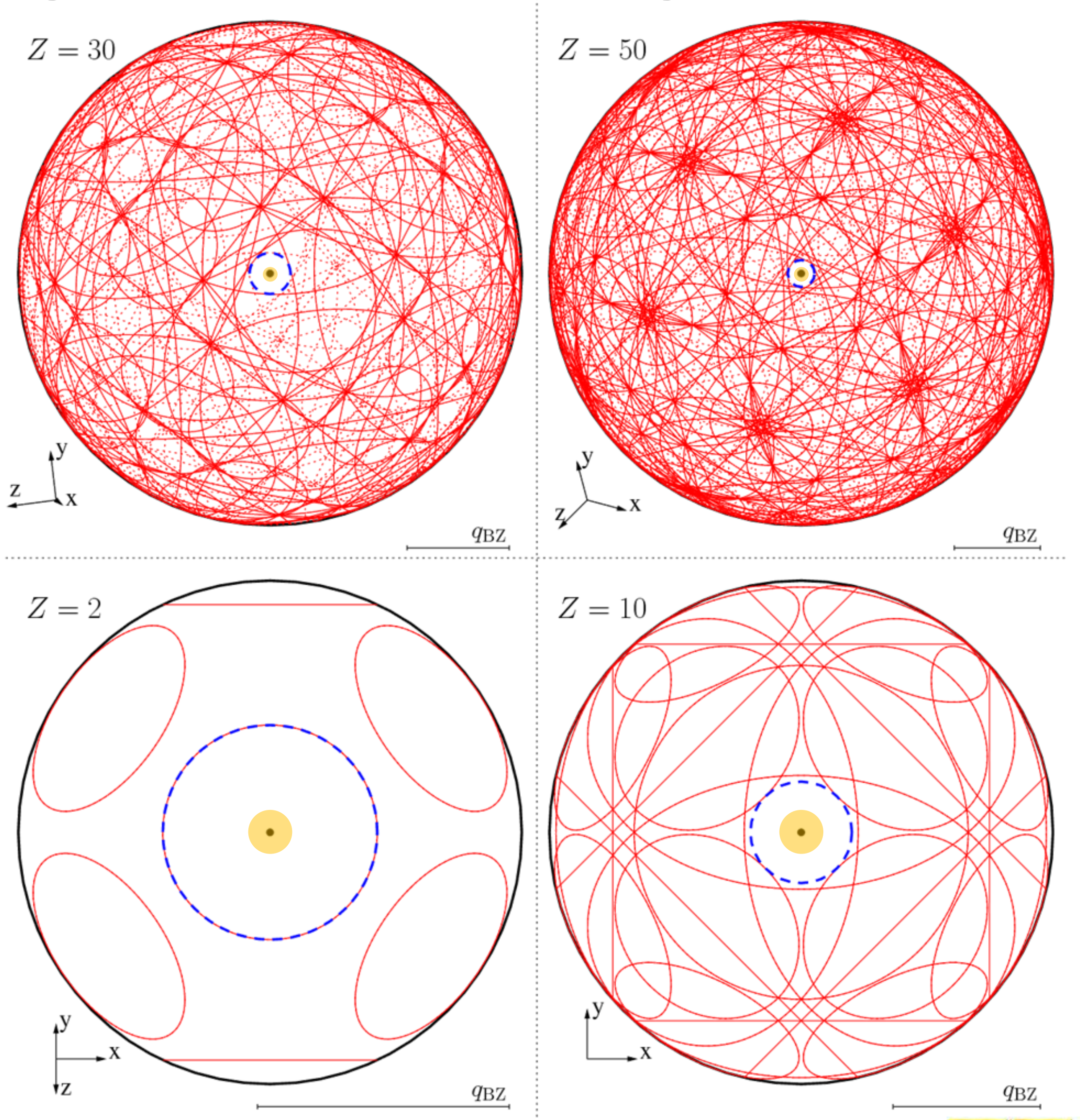


Small angle scattering

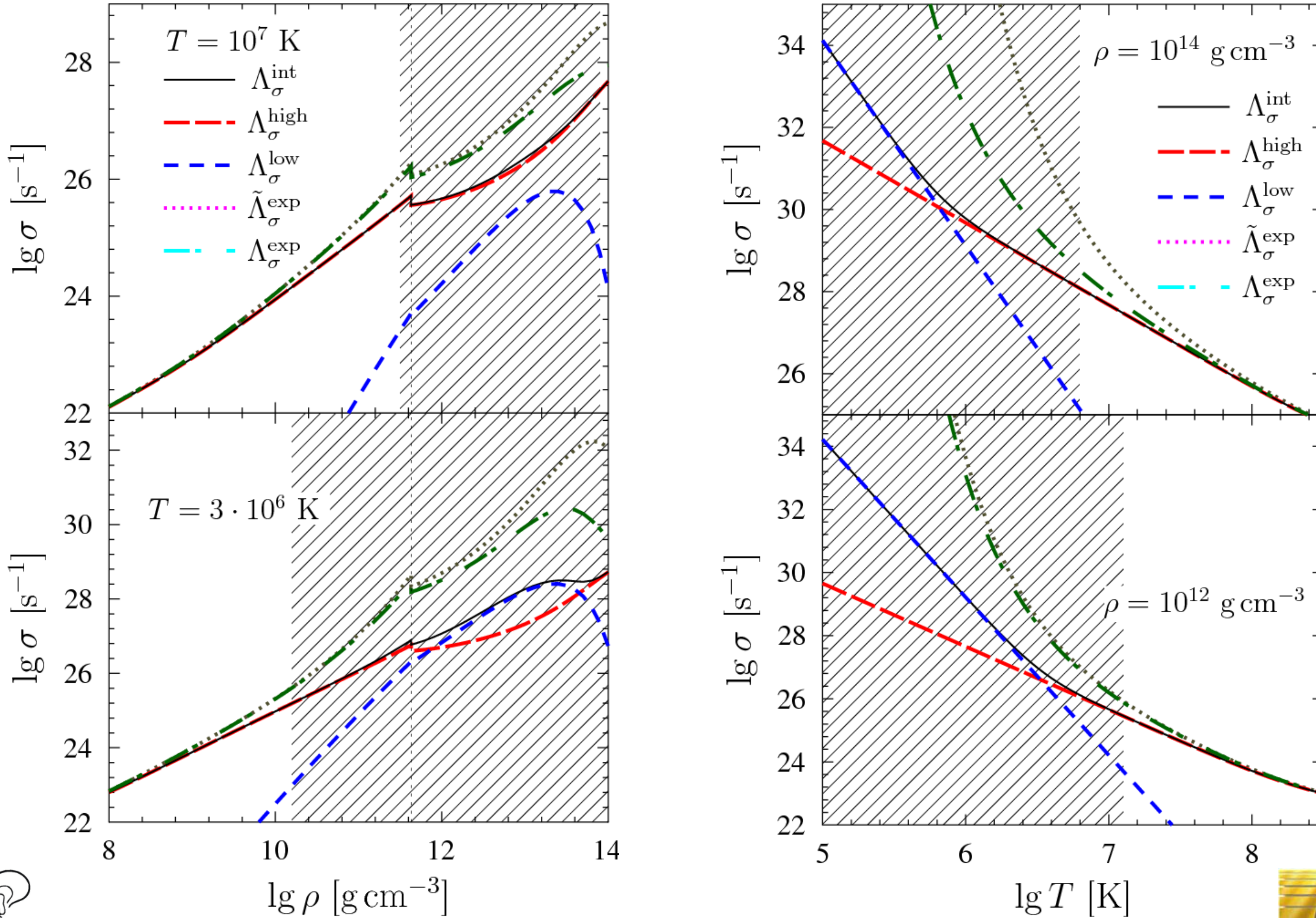
- **Electron-electron scattering**
- Normal electron-phonon scattering

Such processes take place over all Fermi surface and smoothes distribution fuction

$$\tau^{-1} \neq \tau_{\mathrm{ee}}^{-1} + \tau_{\mathrm{ei}}^{-1} + \tau_{\mathrm{imp}}^{-1}$$



Numerical results



Results

- The low temperature asymptote of electrical conductivity is derived
- This asymptote shown to be power low ($\propto T^{-5}$), without any exponential freezing out
- To describe electrical conductivity at all temperatures an interpolation is suggested
- Important role of low angle scattering at low temperatures is demonstrated

(submitted to Astronomy Letters)