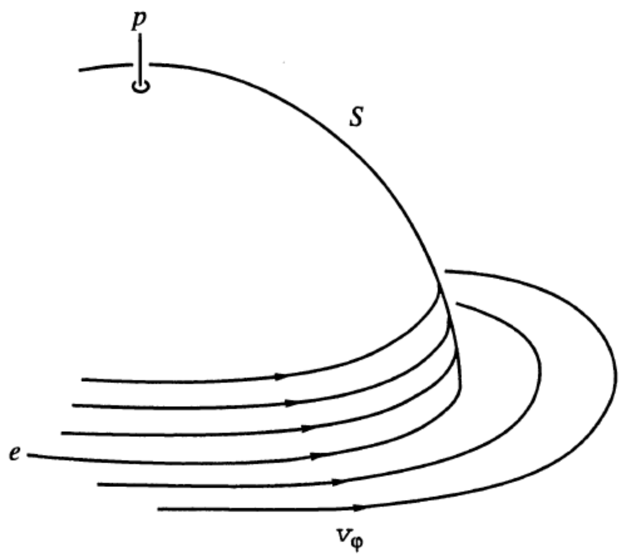
July 14, 2011

Physics of Neutron Stars
Sankt Peterburg



Disk accretion onto neutron stars
with a weak magnetic field

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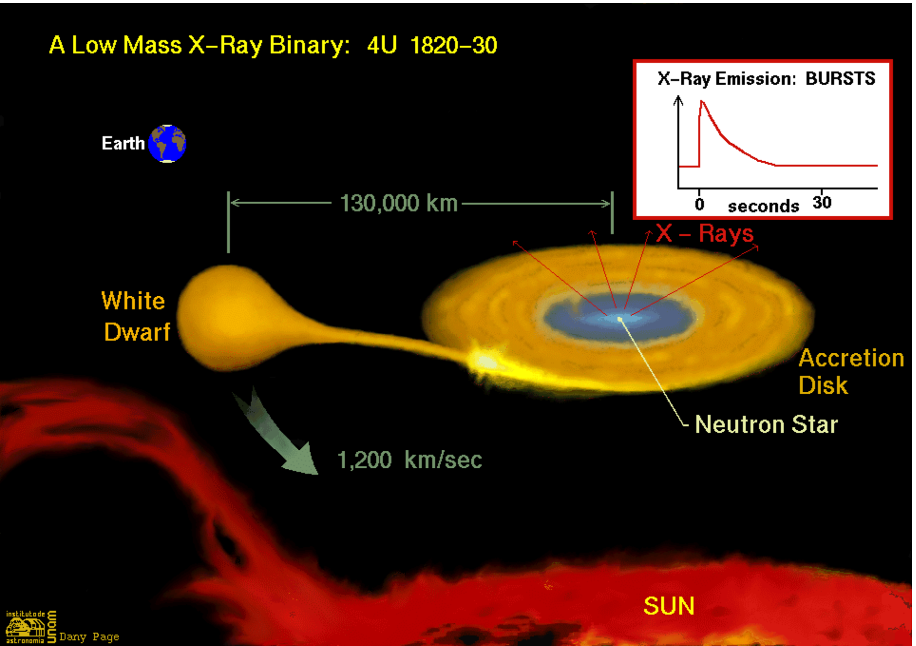


In collaboration with

Nail Inogamov

Landau Institute of Theoretical Physics

Inogamov, N. A.; Sunyaev, R. A.

Astronomy Letters, 2010, Vol. 36, p. 848-894

Spread of matter over a neutron-star surface during disk accretion: Deceleration of rapid rotation.

Inogamov, N. A.; Sunyaev, R. A.

Astronomy Letters, 1999, V. 25, pp.269-293

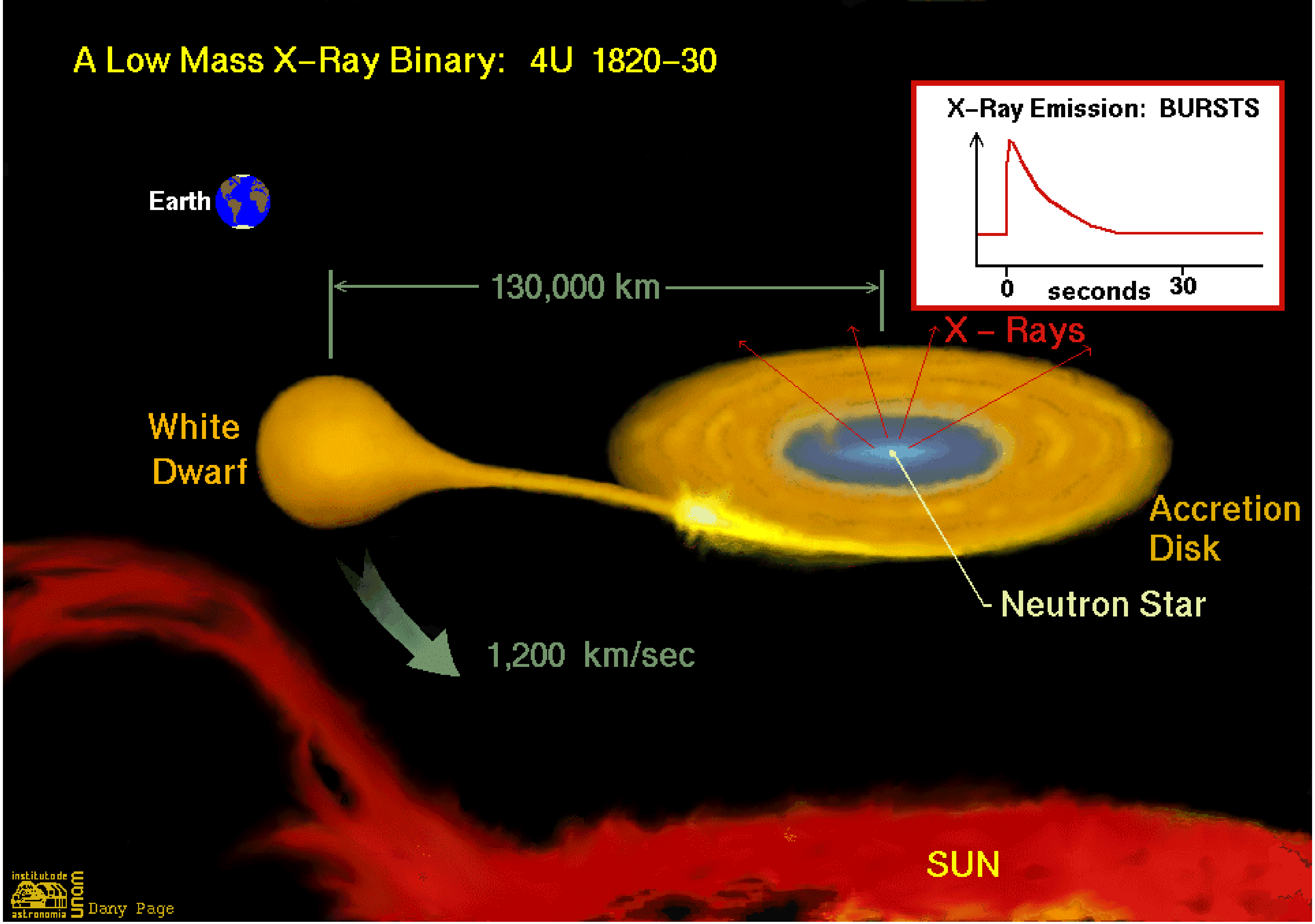
Spread of matter over a neutron-star surface during disk accretion

Let us start with the physics of levitating spreading layer which continues to be valid, it seems

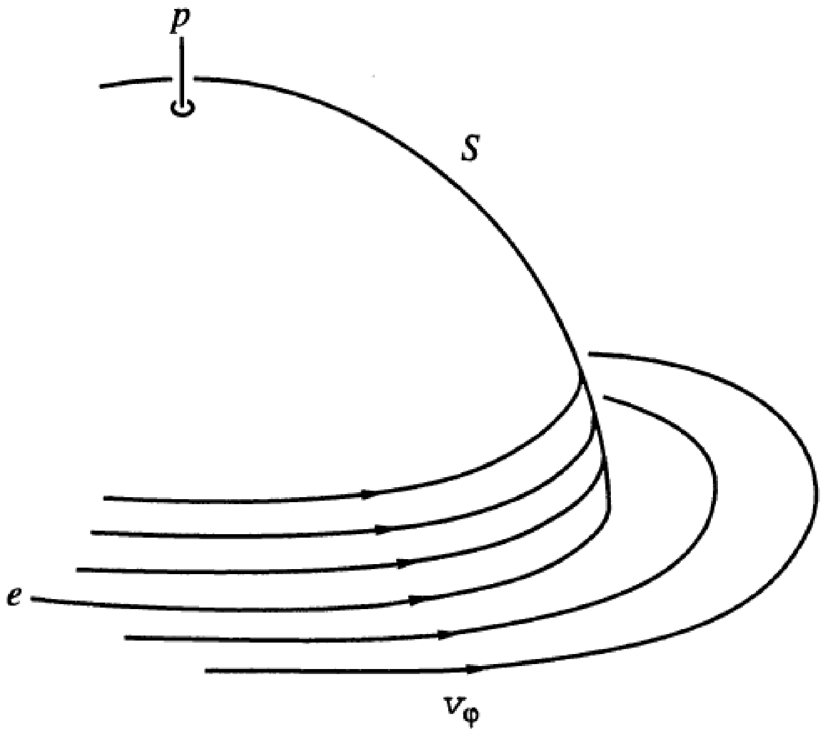
This is 12 year old result, I will try to explain it very fast without equations.

Papers of Sunyaev, Revnivtsev, 2000, Grebenev, 2002; Gilfanov, Revnivtsev and Molkov, 2003, 2006; Suleymanov and Poutanen, 2006 discussed important observational consequences of the model IS99:

radiation spectra and variability



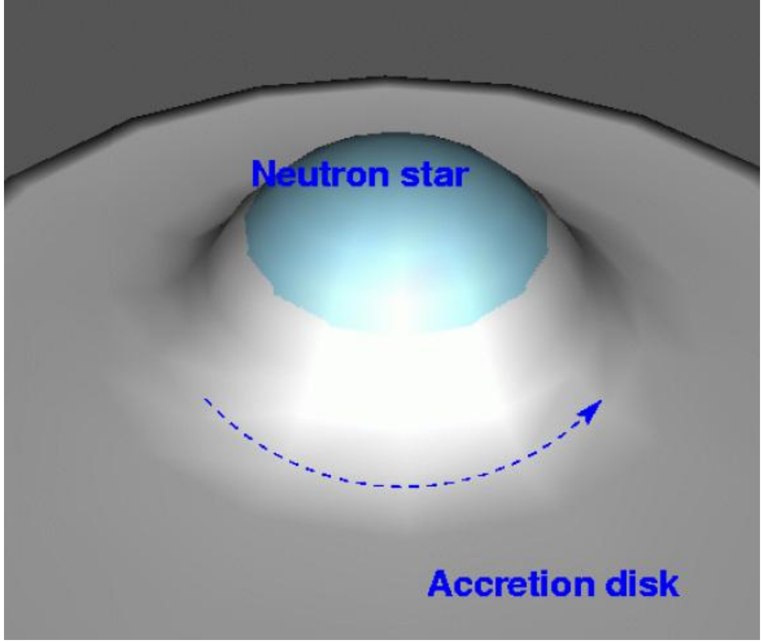
Second component (supplying mass) is a low mass star or a white dwarf. LMXBs are very old – billions of years. **They are of special intetest for Gravitational Wave astronomy.**



Let us try to consider the **disk accretion** onto **neutron star** “without” **magnetic filed**,

i.e. H < 2 10^8 Gauss, when at high accretion rates the magnetic field can not stop accretion far enough from the surface of the neutron star.

This is a beautiful problem: *flow of a collisional gas* (not particles !) with **V ~ 0.4 C** (velocity of light!!!) around **a carefully polished billiard ball** with only 10 km radius, but 1.4 Msun mass.



in the Newtonian mechanics **Ld=Ls**

(virial theorem)

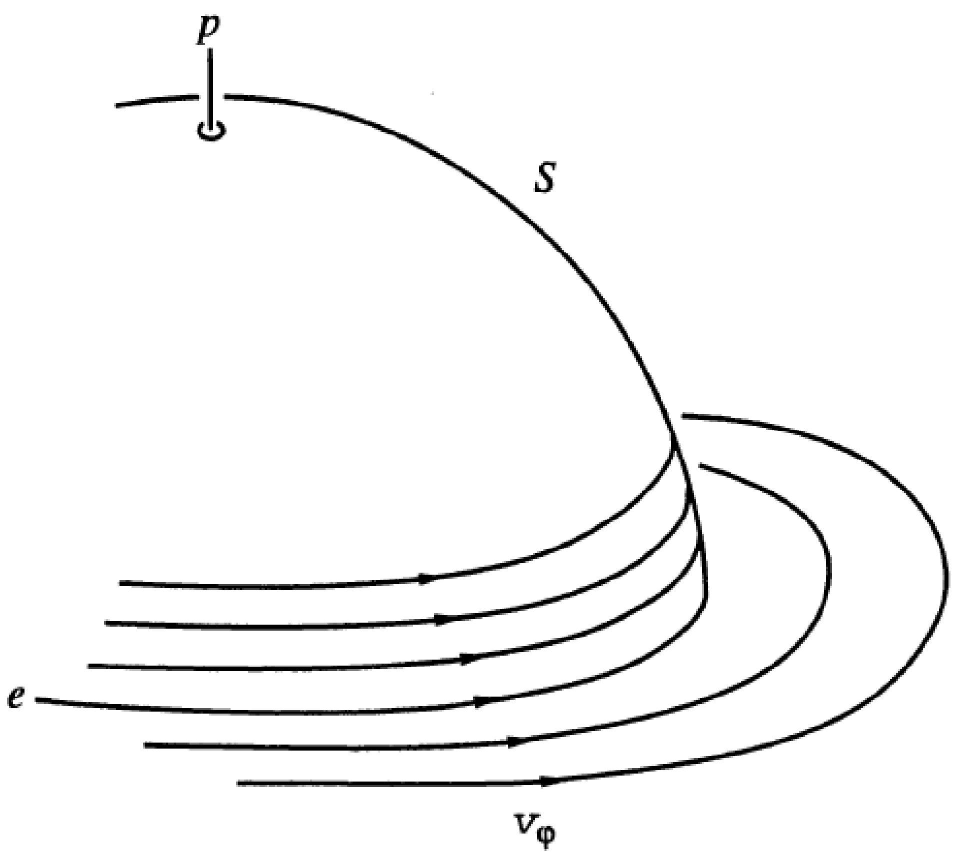
Half of the energy - released and radiated in extended accretion disk **Ld**

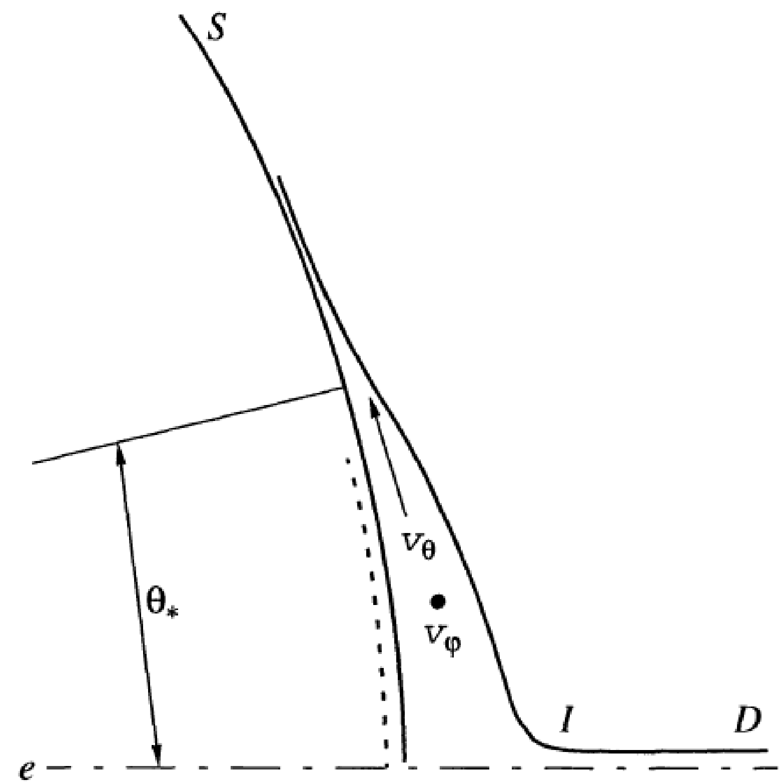
Second half - in the narrow spreading (boundary) layer
near the surface of the neutron star **Ls due to deceleration of the keplerian velocity**

Radiation from spreading layer should be more hard and variable expecially at short time scales

Two onedimensional problems:

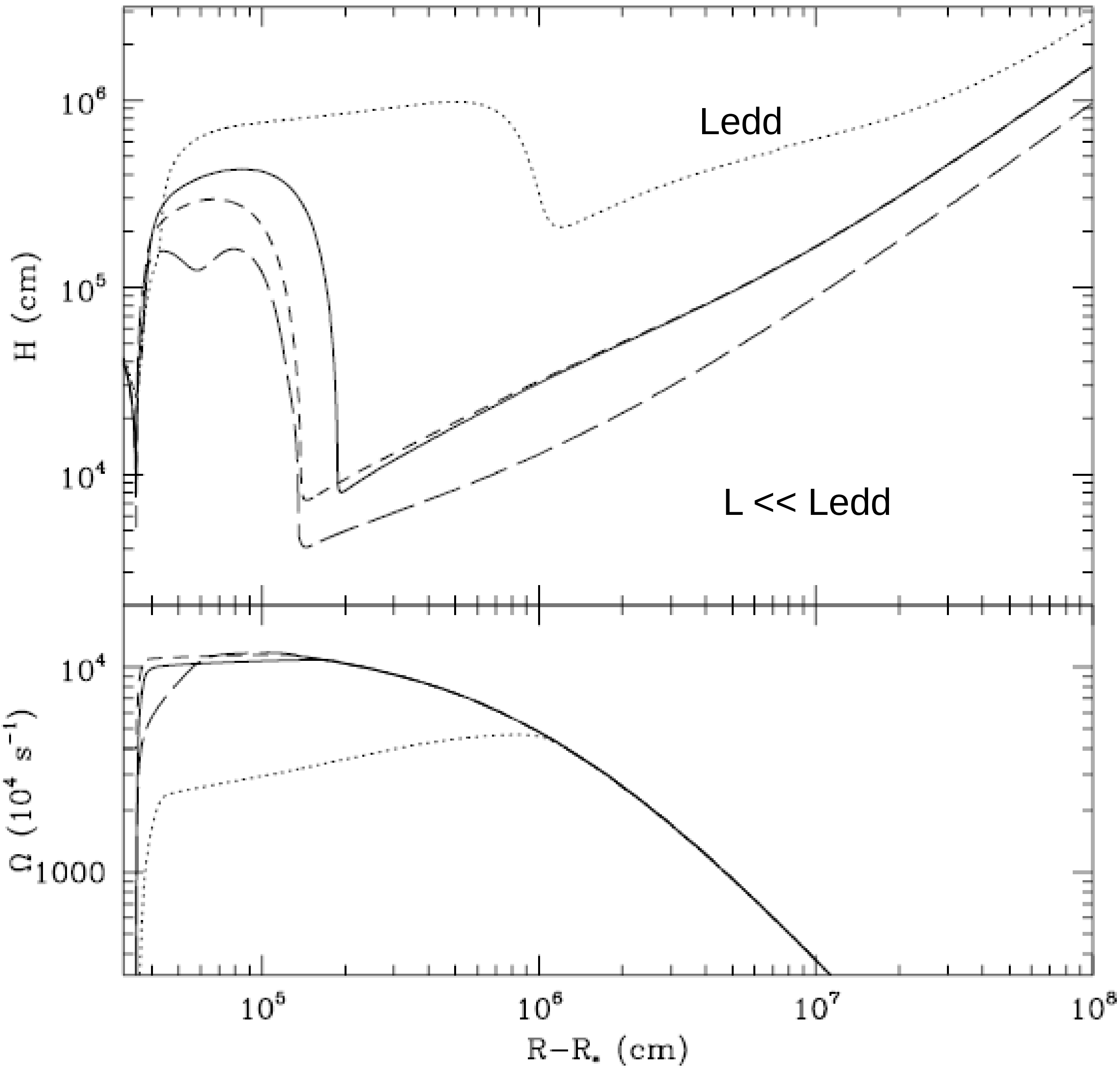
thin disk - slow radial motion, Keplerian rotation





spreading over the star surface

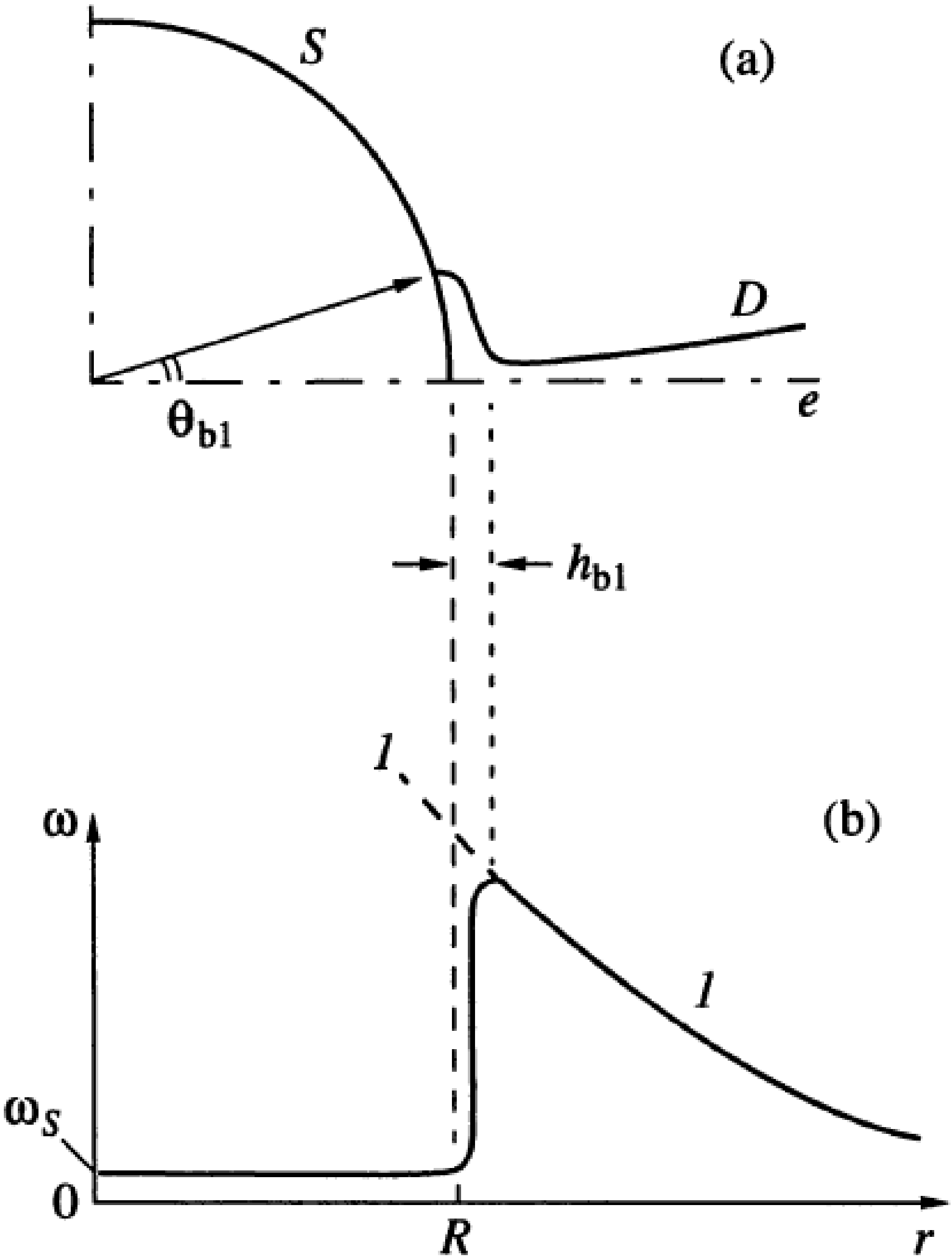
Huge rotational velocity, slow meridional flow, Vr = 0



Popham, Sunyaev, 2001

one dimensional problem like accretion disk:
there are azimuthal and radial velocities,
and
meridianal velocity
Is equal to zero.

Narrow neck !!!



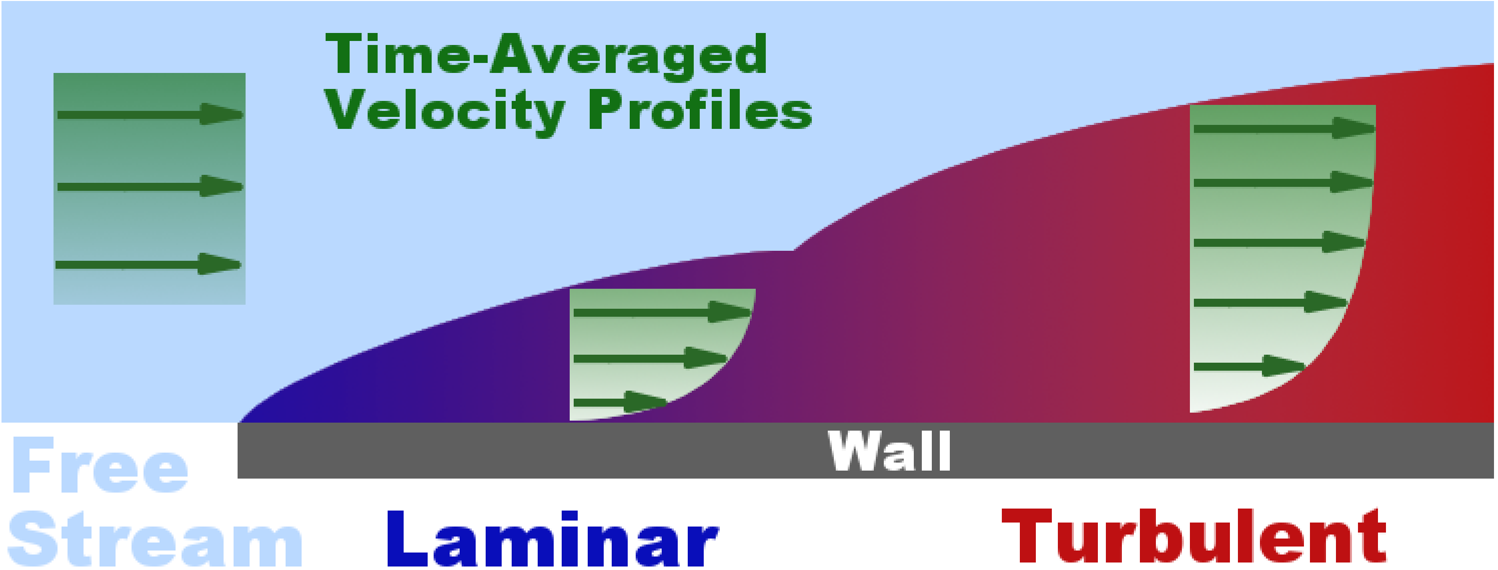
Boundary layer, where matter velocity drops from c/2 (in the disk) up to velocity of the star surface

Narrow neck only at high luminosities

$0.01 < L/L_{edd} < 1$

ADAF type flow can not exist at such luminosities – Comptonization cooling time near ns surface is shorter than revolution time in the disk

(Popham, Sunyaev, 2001)



The logarithmic law of the wall is a self similar solution for the mean velocity parallel to the wall, and is valid for flows at high Reynolds numbers — in an overlap region with approximately constant shear stress and far enough from the wall for (direct) viscous effects to be negligible:

In fluid dynamics, the law of the wall states that the **average velocity of a turbulent flow** at a certain point is **proportional** to the **logarithm of the distance from that point to the "wall**", or the boundary of the fluid region (von Kármán, 1930).

In fluid dynamics, the **law of the wall** states that the average velocity of a turbulent flow at a certain point is proportional to the logarithm of the distance from that point to the "wall", or the boundary of the fluid region (von Kármán, 1930).

The logarithmic law of the wall is a self similar solution for the mean velocity parallel to the wall, and is valid for flows at high Reynolds numbers — in an overlap region with approximately constant shear stress and far enough from the wall for (direct) viscous effects to be negligible:[

$$u^+ = \frac{1}{\kappa} \ln\, y^+ + C^+,$$

$$y^+ = \frac{y\, u_\tau}{\nu},$$

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

$$u^+ = \frac{u}{u_\tau}$$

From experiments, the Kármán constant is found to be $\kappa \approx 0.41$ and $C^+ \approx 5.0$ for a smooth wall

steady state one dimensional Spreading Layer model
in the spherical coordinate system (r , θ, ϕ)

continuity equation

Conservation of momentum

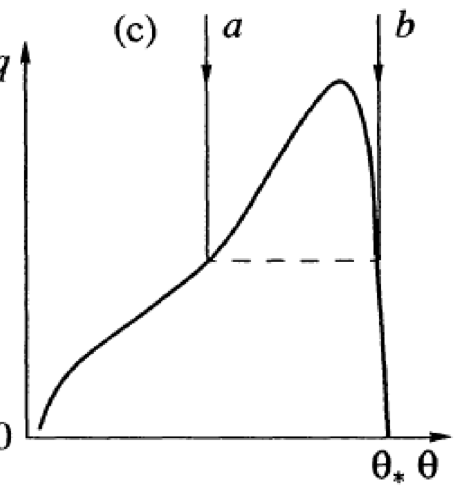
Energy equation

Critical Eddington flux

$$L_{\rm edd} = 4\pi R^2 q_0 = 1.26\times 10^{38}\frac{M}{M_\odot}, \text{erg s}^{-1},$$

$$q_0 = \frac{m_p g_0 c}{\sigma_T} = \Sigma_T g_0 c$$

$$= 1.00\times 10^{25}\frac{(M/M_\odot)}{R_6^2}, \text{erg s}^{-1}\ \text{cm}^{-2}.$$



$$\dot{M}\frac{(v_\varphi^k)^2}{2} = q_0 4\pi R^2 \sin\theta_0^*;$$

$$\theta_0^* = \arcsin(L_{\rm SL}/L_{\rm edd})$$

Physical conditions in the spreading layer

1. Radiation flux is huge. Radiation pressure dominates:
   $P_{rad} \gg 2\,N\,kT$

2. Thomson scattering dominates in the opacity

3. Effective gravity – difference between gravity g, centrifugal acceleration and radiation force acceleration

This is main reason why spreading layer levitates !!!

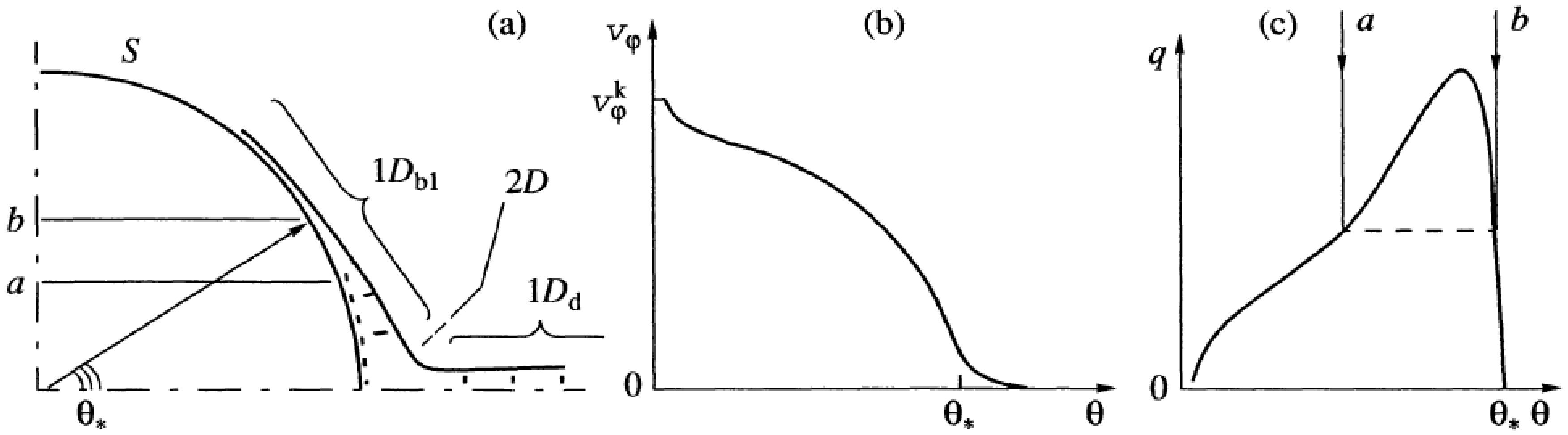
4. near equator - effective gravity close to zero even without radiation force or radiation flux

5. Meridional velocity is much less than rotational

$$g_{\mathrm{eff}} \approx g_0(1 - W^2), \text{ where } W = v_\varphi / v_\varphi^k$$

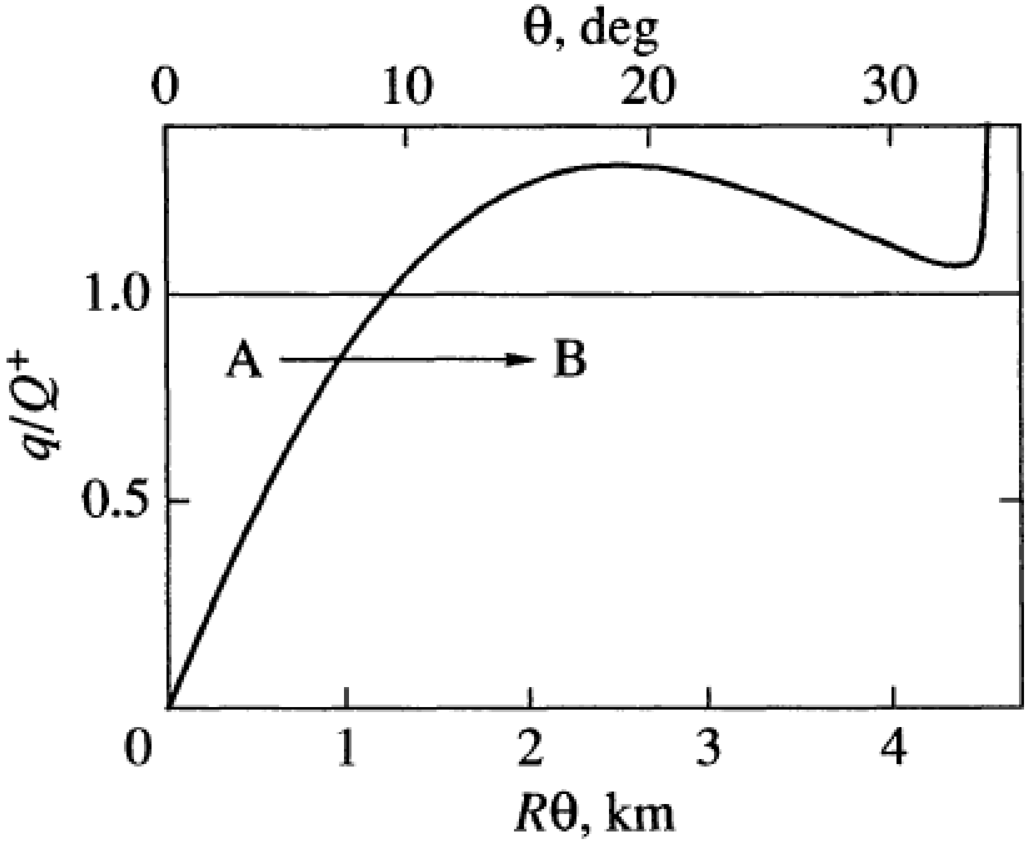
$$h \ll R, \quad |dh/dl| \ll 1$$

Shallow water approximation



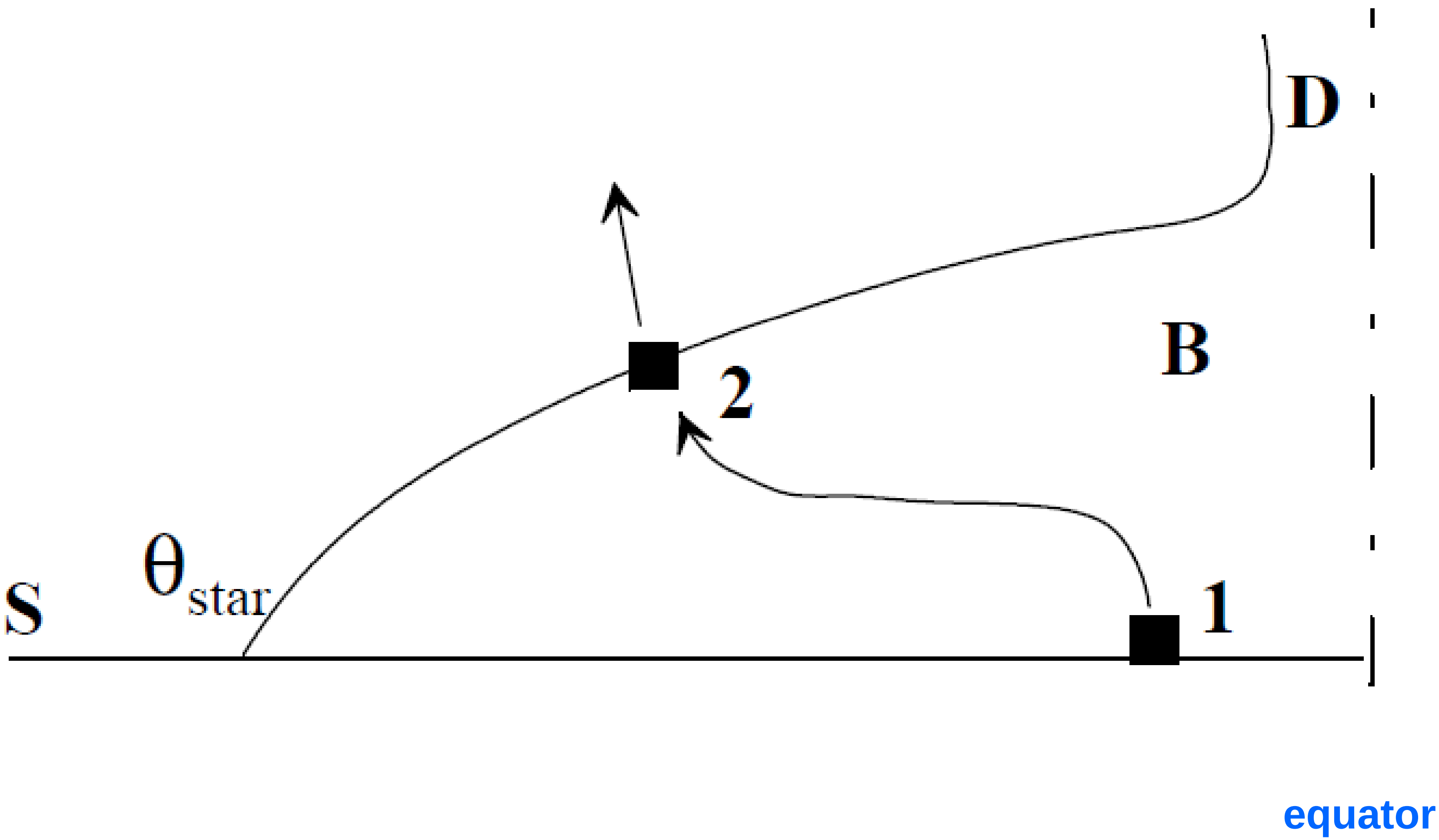
One of the main predictions: **zero radiation flux along equator**,

two bright belts above and below equator

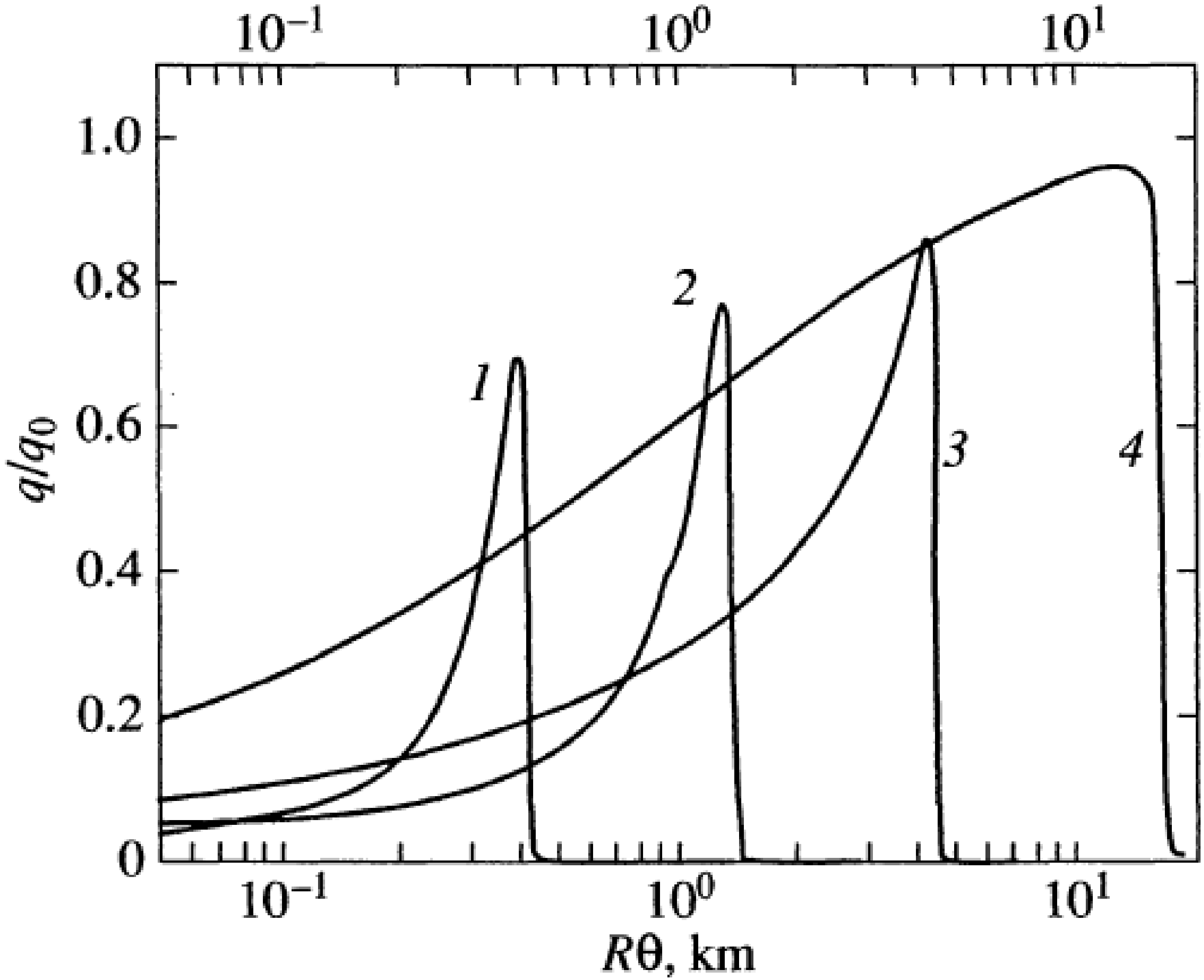


Meriodinal advection of radiation energy from equatorial belt

$$g_{\text{eff}} \approx g_0(1 - W^2), \text{ where } W = v_\varphi / v_\varphi^k$$



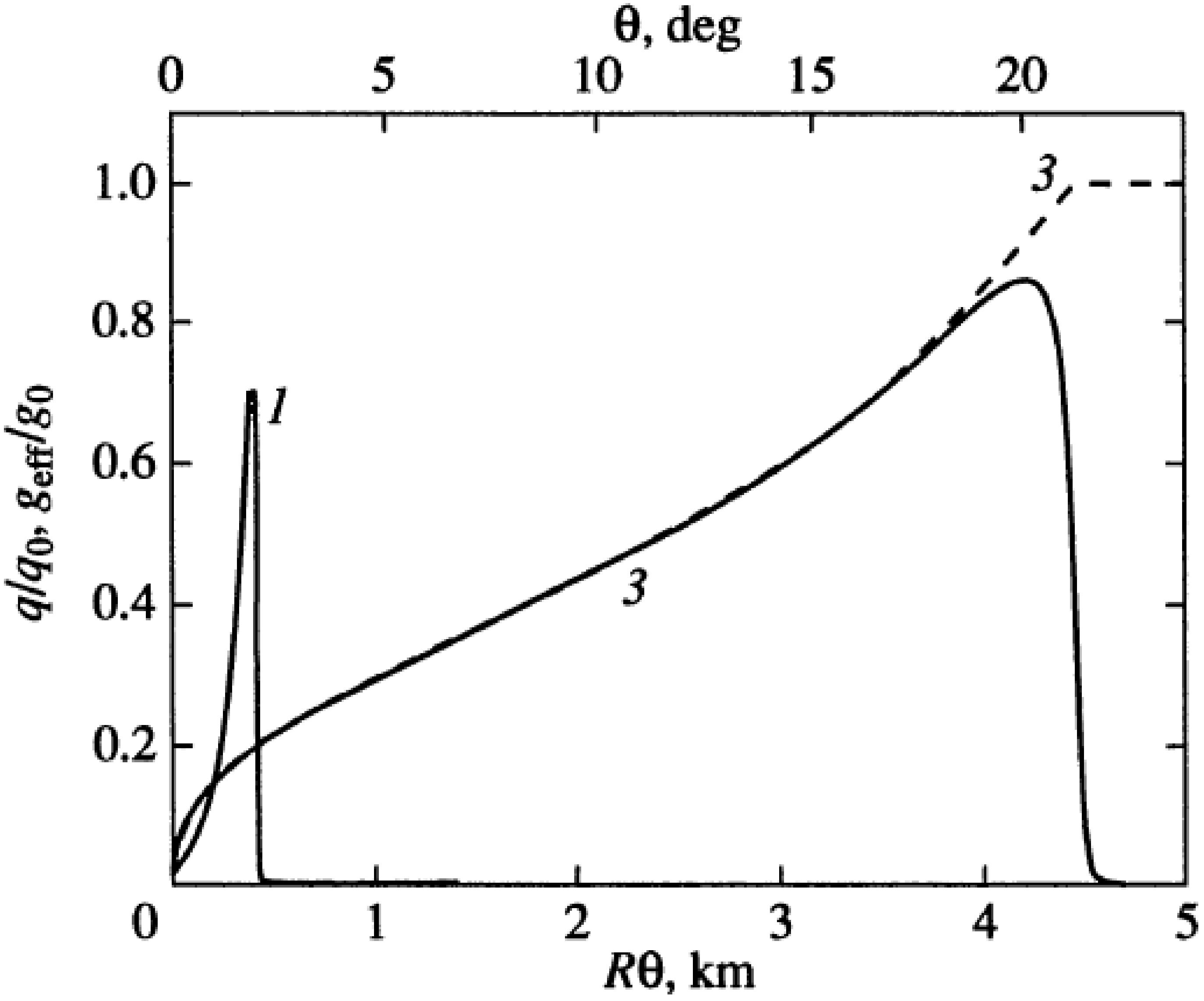
Advection of energy



two bright rings equidistant from equator

this distance Increases with luminosity or accretion rate

Fig. 8. $q(\theta)$ profiles in a wide range of $\dot{M}$. Labels *1*, *2*, *3*, and *4* refer to $L_{SL}/L_{edd} = 0.01$, 0.04, 0.2, and 0.8, respectively.



Levitating layer

1 km thick

Low luminosity

High luminosity

$$g_{\text{eff}} \approx g_0(1 - W^2),\ \text{where } W = v_\varphi / v_\varphi^k$$

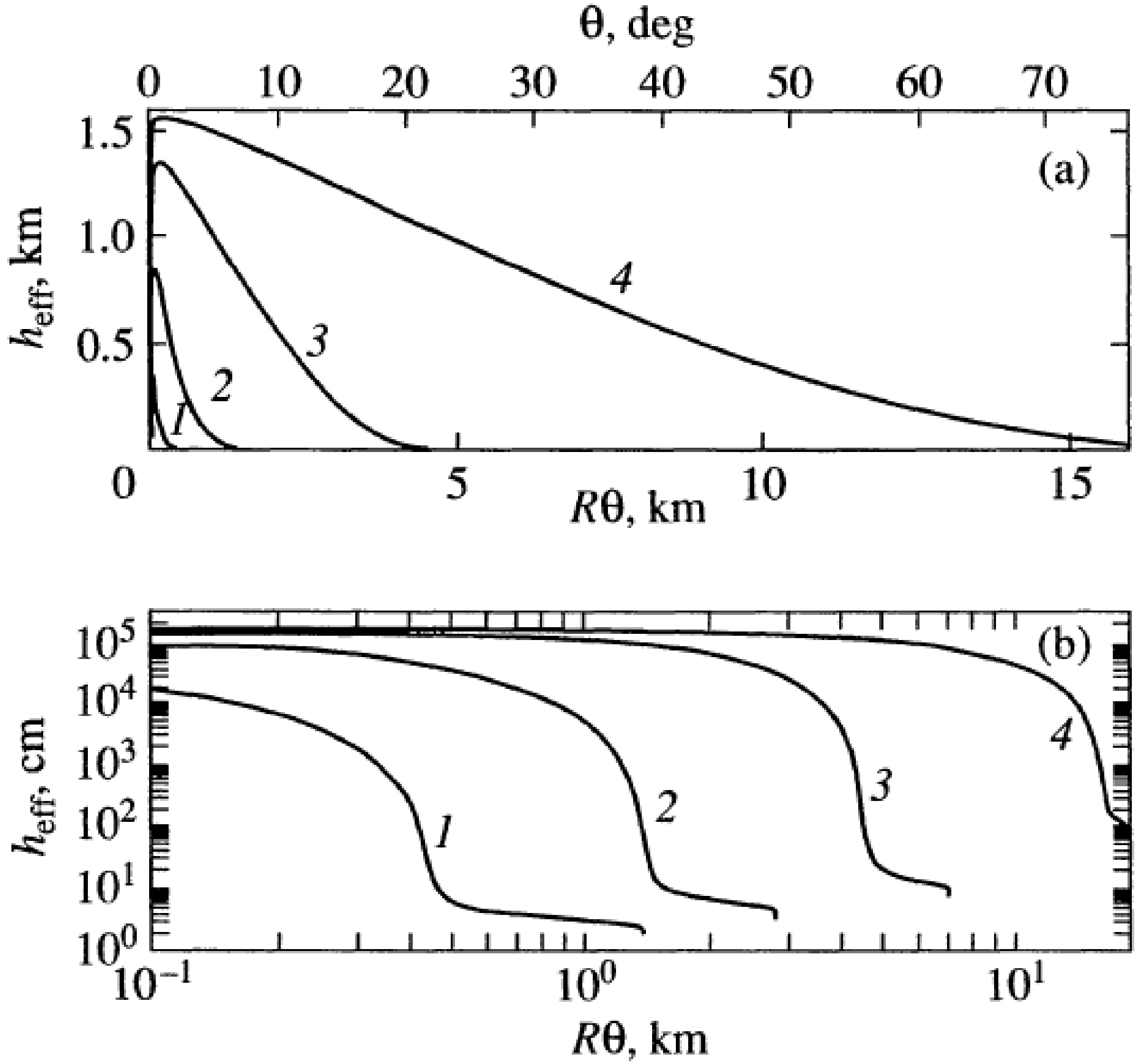
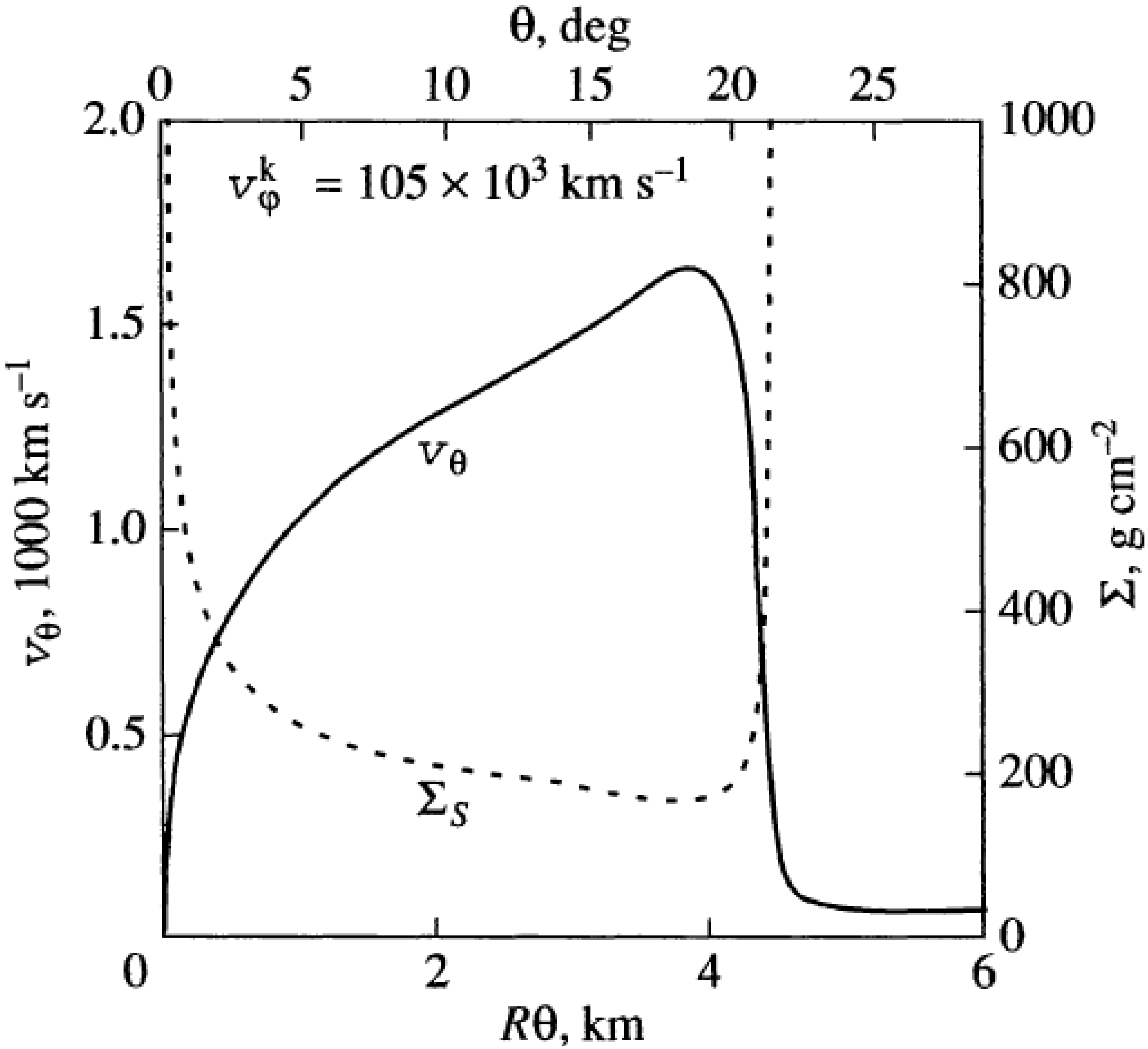


Fig. 5. Profiles *1*, *2*, *3*, and *4* of the spread-layer (S.L.) effective thickness $h_{\mathrm{eff}}(\theta)$ for various accretion rates $\dot{M}$. The relative luminosities $L_{\mathrm{SL}}/L_{\mathrm{edd}}$ that correspond to labels *1*, *2*, *3*, and *4* are 0.01, 0.04, 0.2, and 0.8, respectively. The stellar



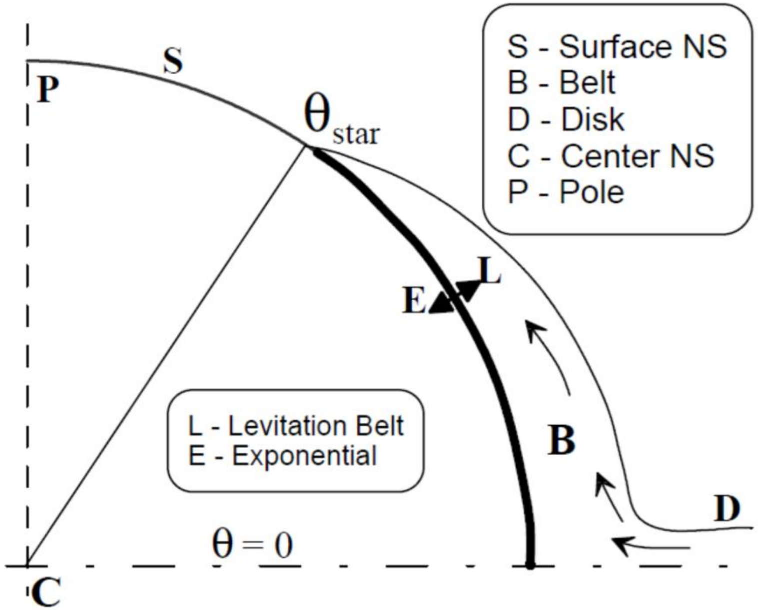
θ, deg
0
5
10
15
20
25
2.0
1.5
1.0
0.5
$v_\varphi^k = 105 \times 10^3$ km s^{-1}
v_θ
Σ_S
v_θ, 1000 km s^{-1}
1000
800
600
400
200
0
Σ, g cm^{-2}
0
2
4
6
$R\theta$, km

New paper: Inogamov and Sunyaev, 2010

a little more detailed physics (but still no full analogue in the Lab or on the Earth).

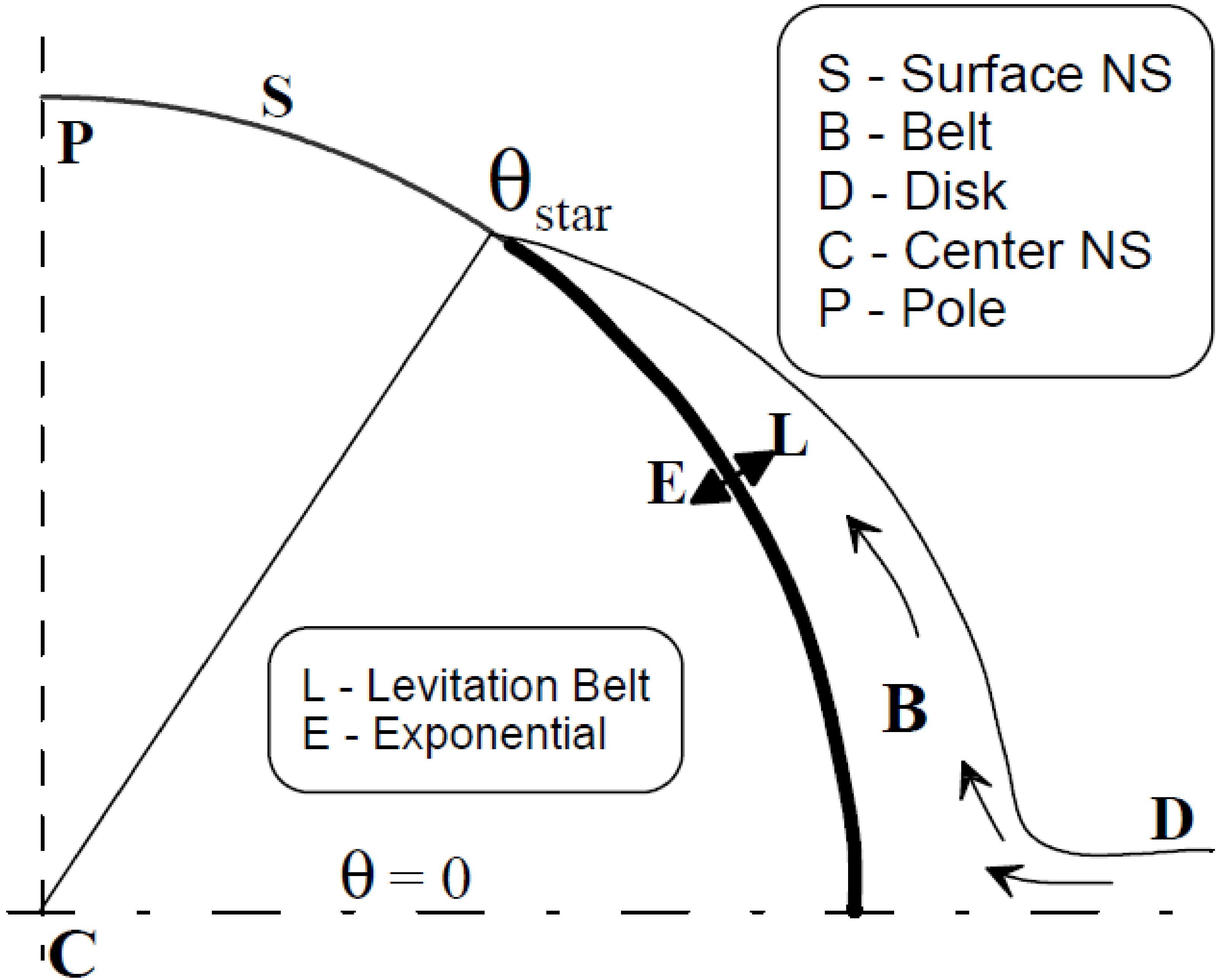
No smooth solid wall !!!

There are levitating speading layer and boundary layer below it
formed by the matter
which accreted hundreds of microseconds before



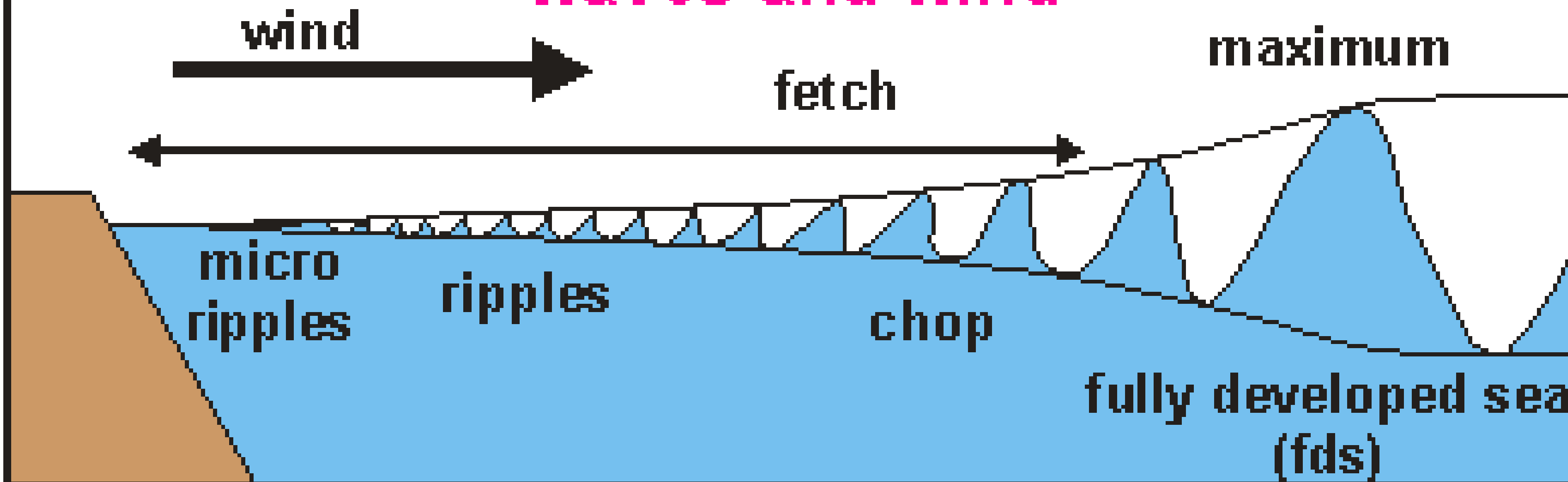
levitating speading layer
H ~ 1 km

boundary layer below it
H ~ 1 – 10 m



S - Surface NS
B - Belt
D - Disk
C - Center NS
P - Pole
S
P
θ_{star}
L
E
B
D
L - Levitation Belt
E - Exponential
$\theta = 0$
C

waves and wind



As waves develop, they offer more surface area for the wind to press against (wind stress). Depending on both fetch and time, the size of the waves increases quadratically to a maximum. The energy imparted to the sea increases with the fourth power of the wind speed! As waves develop, they become more rounded and longer and they travel faster. Their maximum size is reached when they travel almost as fast as the wind. A 60 knot storm lasting for 10 hours makes 15 m high waves in open water.

Oversimplification: **strong wind (levitating layer) over more dense boudary layer**

surface and internal gravity waves

We **separate problem of levitating layer** assuming that
~10-15% of keplerian velocity dissipates there due to
logarithmic law effects at large distances from the boundary layer

Main problem now - to find how the rest of keplerian velocity
dissipates in the **quasiexponential stratified boundary layer**.

First attempt: to check the validity of **Prandtl-Karman model** of **turbulent boundary layer**

Main equations

We use equation of mass conservation

$$\rho v_r = -\dot{\Sigma}$$

and radial force balance

$$\frac{d}{dr}(p+\rho v_r^2) = -\rho\left(g_{grav} - \frac{v_\varphi^2}{R}\right) = -\rho g_{eff}$$

to describe braking, cooling, and settling of accreting matter. Here

$$\dot{\Sigma} \sim \Sigma_T\, c/R,$$

is accretion rate per unit of NS surface, $\Sigma_T = m_p/\sigma_T$, σ_T is Thomson cross-section, R is radius of neutron star, v_φ and v_r are rotational and radial components of velocity. The last corresponds to settling of matter.

Equation of state is

$$p = p_{pl} + p_{rad}, \quad p_{pl} = \rho\frac{2T}{m_p}, \quad p_{rad} = \frac{aT^4}{3},$$

where p_{pl} and p_{rad} are plasma and radiative pressures.

Conservation of angular momentum gives

$$w = J - \dot{\Sigma}\, v_\varphi.$$

This conservation law follows from equation

$$\frac{\partial(\rho\, v_\varphi)}{\partial t} = -\frac{\partial(\rho\, v_\varphi\, v_r)}{\partial r} + \frac{\partial w}{\partial r}$$

in the steady-state case when $\partial_t = 0$. The constant

$$J \sim \dot{\Sigma}\, v_K$$

is an integral of this equation; here v_K is Keplerian velocity.

Newton friction law for turbulent viscous stress is

$$w = \rho\,\nu\, dv_\varphi/dr,$$

where ν is total kinematic viscosity

$$\nu = \nu_i + \nu_{rad} + \nu_t.$$

Partial viscosities are

$$\nu_{rad} = \frac{4}{15}\,\frac{a\,T^4\,\Sigma_T}{c\,\rho^2} \simeq \frac{\rho_{rad}}{\rho}\, l_T\, c$$

(Weinberg, 1972), here $\rho_{rad} = aT^4/c^2$ is “density” of photons, $l_T = 1/n\sigma_T$ is photon mean free path and $n = \rho/m_p$. Ion viscosity is

$$\nu_i = 10^3\,\frac{T^{2.5}}{\rho\,\ln\Lambda},$$

where Λ is Coulomb logarithm (Spitzer, 1962), temperature is in keV, while density and kinematic viscosity are in g/cc and cm^2/ s. Under the physical conditions in the spreading layer the radiative viscosity often exceeds the ionic one, but the turbulent viscosity dominates.

Energy balance

$$-\dot{\Sigma}\left(\frac{v_r^2+v_\varphi^2}{2}+5\,\frac{T}{m_p}+\frac{4}{3}\,\frac{aT^4}{\rho}+\psi\right)-wv_\varphi+q=Q$$

presents advection of rotational and radiation energies $\dot{\Sigma}(v_\varphi^2/2+4\,a\,T^4/3\rho)$, viscous dissipation wv_φ, and radiative flux

$$q=-\frac{c\,\Sigma_T}{3\rho}\,\frac{d(a\,T^4)}{dr}.$$

In the Prandtl-Karman model turbulent viscosity ν_t is calculated as

$$\nu_t = \alpha_t l_t v_t =$$

$$= \alpha_t h_\rho (h_\rho \, dv_\varphi / dr) =$$

$$= \alpha_t h_\rho^2 \, dv_\varphi / dr,$$

Experiment:
alpha_t = 0.16

see equation (4.8) in the full paper. Here α_t is a non-dimensional coefficient, l_t and v_t - are respectively characteristic spatial scale (average size of vortices) and an amplitude of velocity fluctuations. Value $h_\rho(r) = [\,|d\ln(\rho)/dr|\,]^{-1}$ - is a spatial scale of density distribution $\rho(r)$. Kinematic viscosity ν_t allows the calculation of dynamic viscosity $\mu = \rho\nu$ and viscous stress w via the Newton friction law

$$w = \mu \frac{dv_\varphi}{dr}.$$

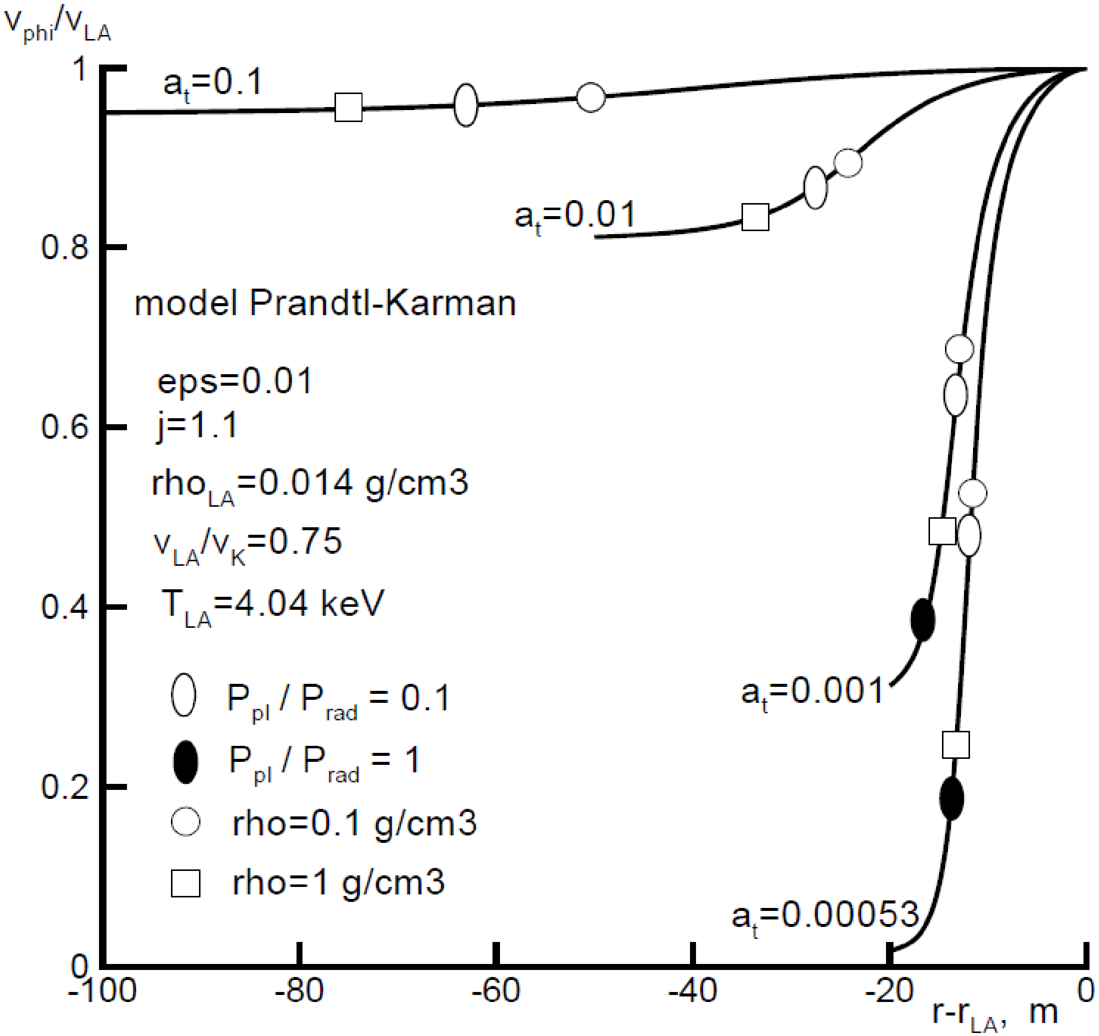


FIG. 5: Velocity profiles in the Prandtl-Karman model of friction (4.8). The influence of the coefficient α_t (the numbers near the curves) on the flow braking in the layer A is shown. The Prandtl-Karman model is traditionally used in calculations of stress in the turbulent shear flow.

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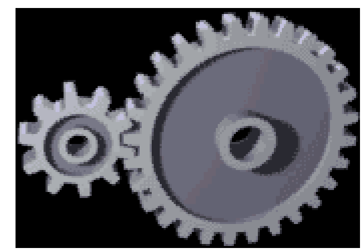


Gear !!!

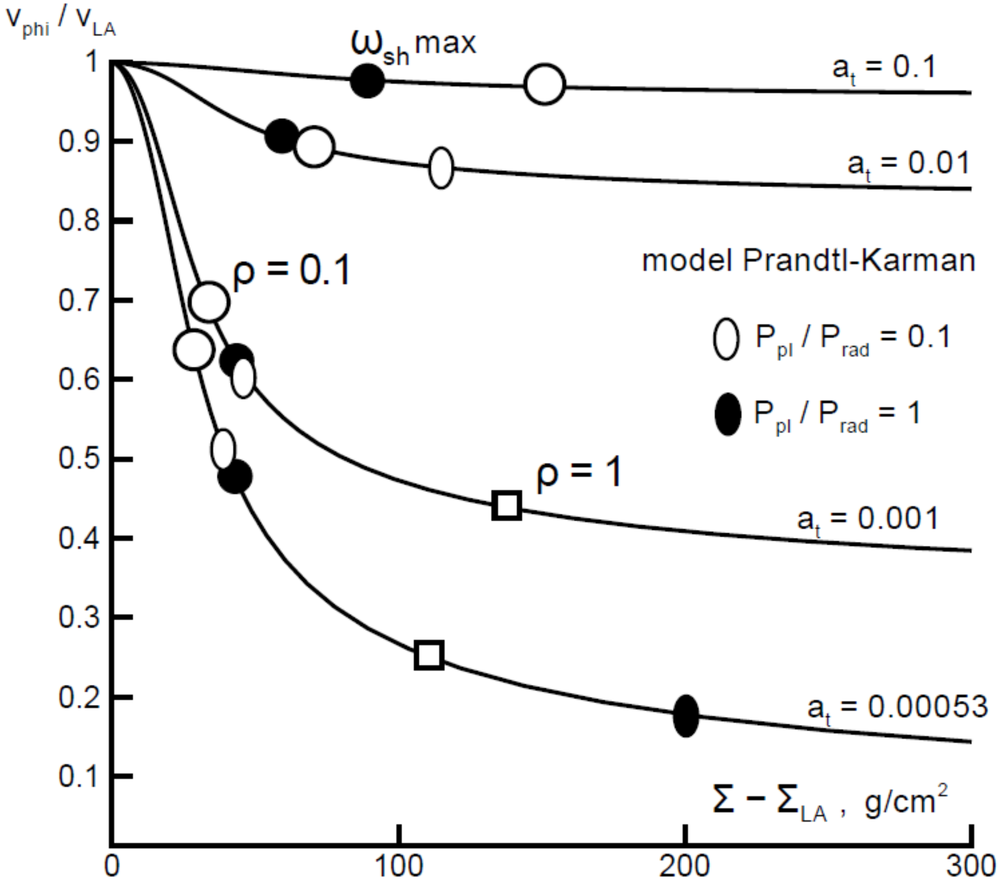


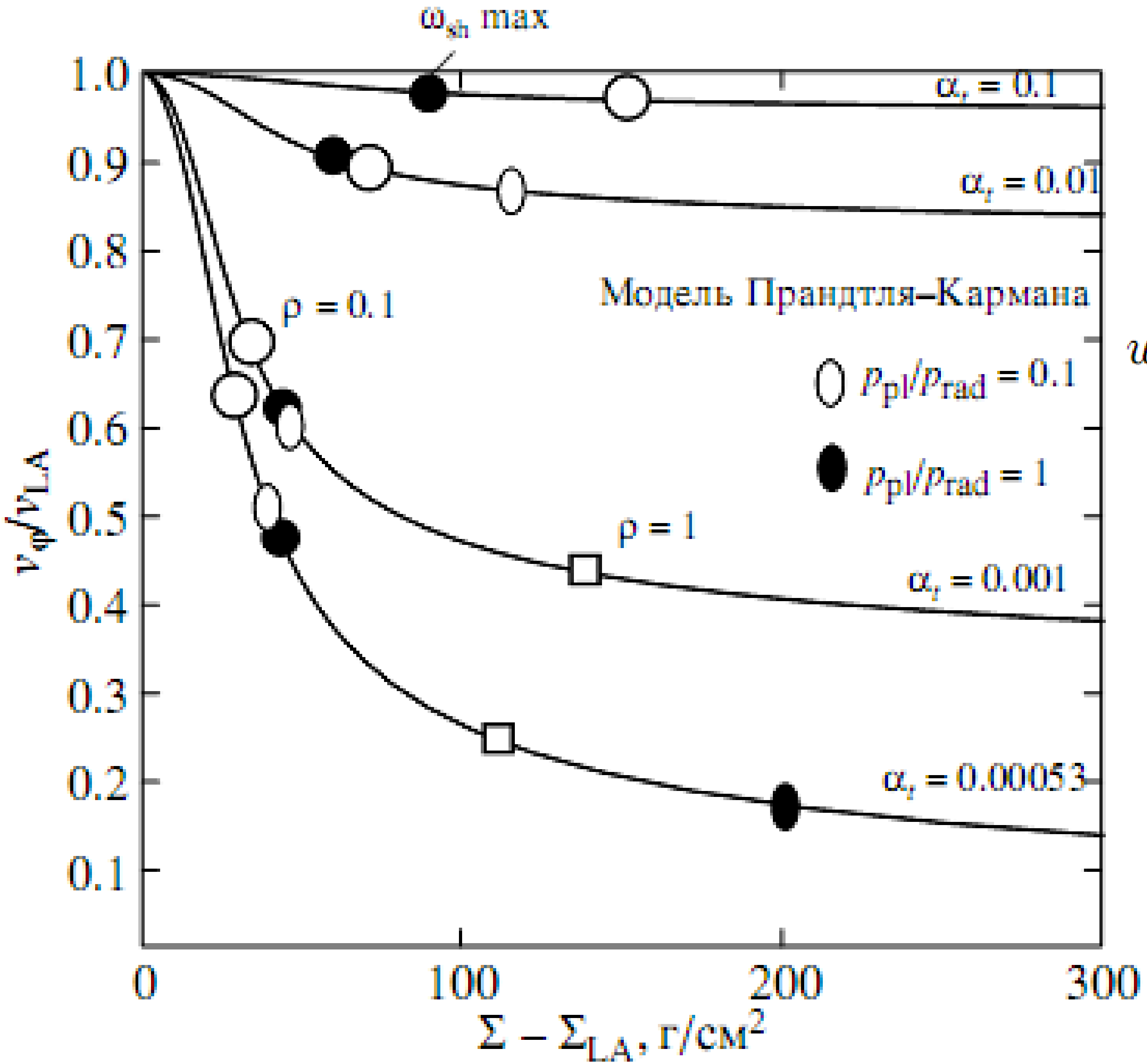
FIG. 4: Decrease of linear angular velocity v_φ and sharp accumulation of column mass Σ in the process of braking of velocity. The part of the zone A, where the velocity shear is dominated, is shown. This is marked by the thick curve in Fig. 3. The filled circles indicate the maxima of the shear frequency ω_{sh}; the open circles and squares represent respectively the depths where densities are: $\rho = 0.1$ and 1 g cm^{-3}. The elliptical symbols show the ratio of plasma and radiation pressures.

The stress between the rotating fast layers and the deeper interiors of NS decreases rotation frequency with depth.

The Prandtl-Karman model is traditionally used in analyzing shear turbulence. The coefficient α_t in this model is related to the Karman constant κ (Schlichting 1965) by the formula $\alpha_t = \kappa^2$. In the case of turbulence near a wall in a medium with neutral stratification, we have

$$\kappa \approx 0.4, \quad \alpha_t \approx 0.16.$$

paper). We arrive at a paradoxical result: *the universally accepted parameters of turbulent viscosity give rise to a massive rotating equatorial layer in which the rotation decays very slowly and the energy release takes place mainly at depths where the matter density exceeds considerably* 10^4 *g cm*$^{-3}$.



Experiment:
alpha_t = 0.16

$$w = \rho\nu\, dv_\varphi/dr$$

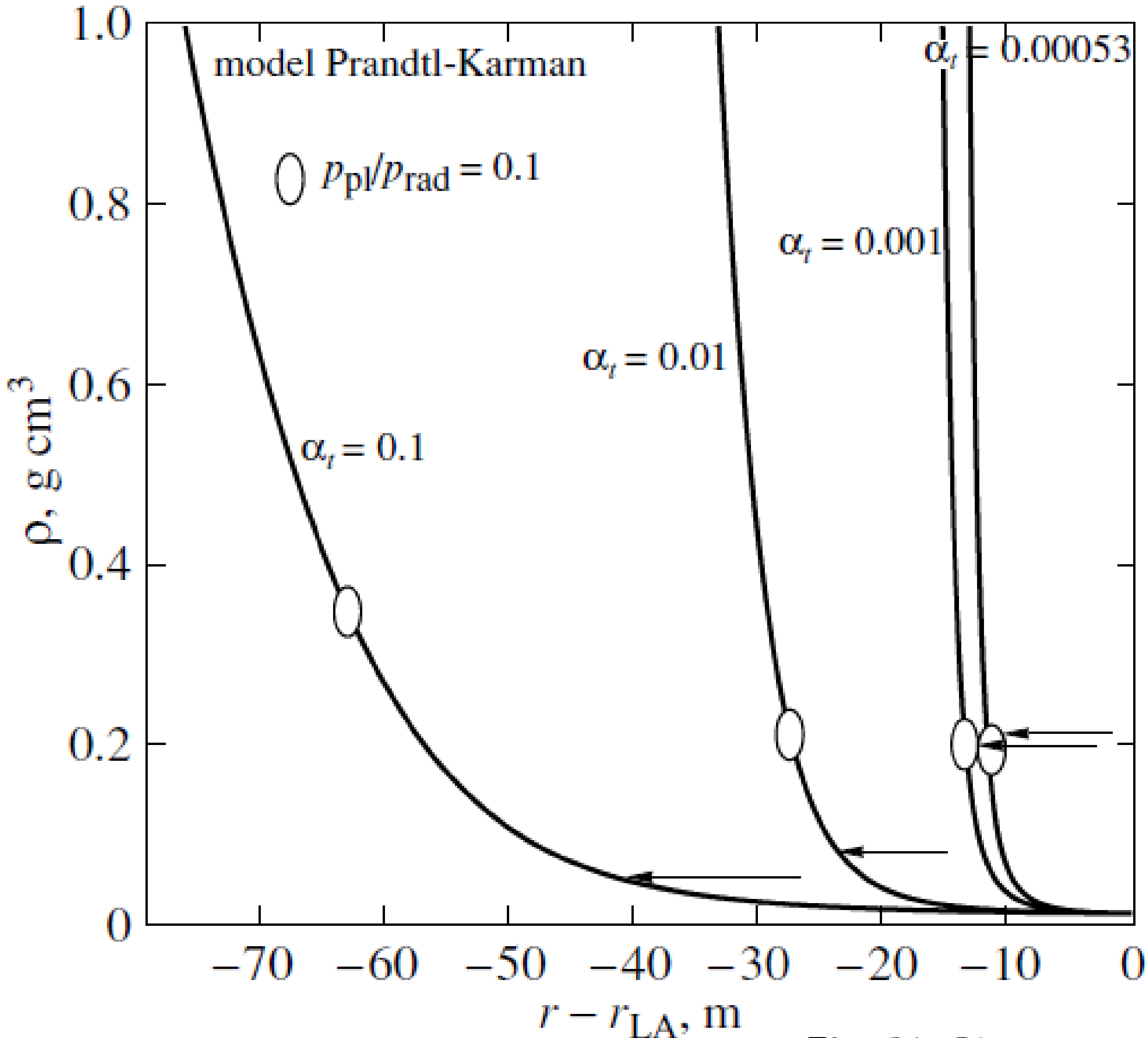
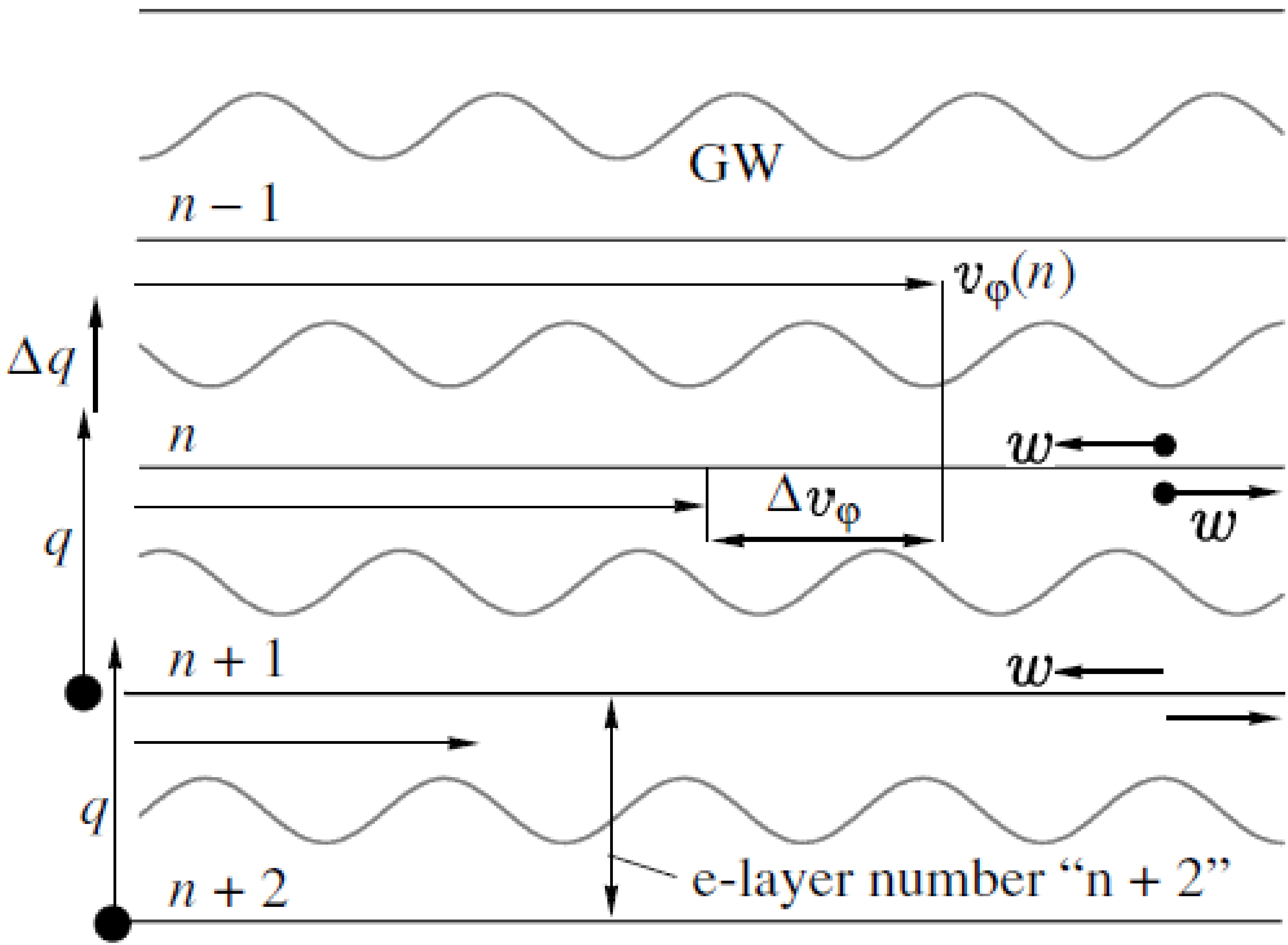
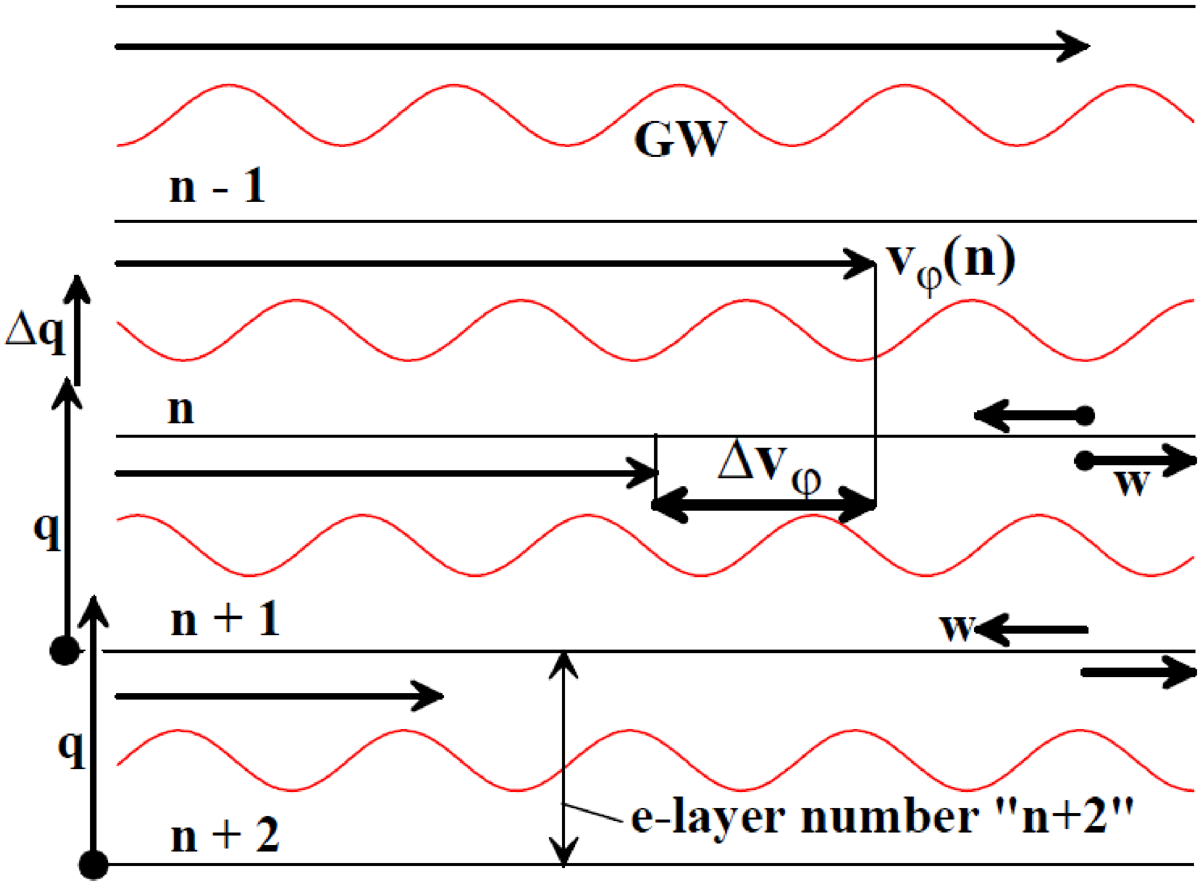


Fig. 21. Steep rise in density $\rho(r)$ with depth. For the parameters, see Fig. 20. The arrows mark the maxima of the shear frequency ω_{sh} with values of $\log(\omega_{sh}\ [s^{-1}]) = 5, 5.9, 7, 7.2$. The maximum is reached at the point of inflection of the curve $v_\varphi(r)$.

Second attempt:

internal gravity waves in the stratified atmosphere


GW
n − 1
$v_\varphi(n)$
Δq
n
w
Δv_φ
w
q
n + 1
w
q
e-layer number "n + 2"
n + 2



The gravity wave ladder

FIG. 7: Sequence or ladder of nonlinear gravity waves, each traveling in its layer with a thickness of $h_E = 2T/m_p g$ of an exponential atmosphere. The pairs of arrows w indicate the transferred shear stress through which the angular momentum is eventually transported to the bulk of the NS. The arrows $v_\varphi(n)$ indicate the average rotation velocity at the level of layer n. The arrows q and $q + \Delta q$ represent the radiation flux at the entrance to or exit from the e-layer (an increase in ρ by a factor of "e" takes place in the "e-layer"). The increment $\Delta q = w\Delta v_\varphi$ is related to wave breaking and the dissipation of kinetic energy of the gravity wave in this layer.

In the case of insignificant gravity stratification the braking of rotation as result of viscous deceleration is

$$\frac{dv_\varphi}{dr} = \frac{1}{\rho}\,\frac{w}{\nu} = \frac{1}{\rho}\,\frac{J - \dot{\Sigma}\,v_\varphi}{\nu_i + \nu_{rad} + \nu_t}.$$

In the alternative case of significant gravity stratification we use another description of the deceleration of rotation (decrease in shear frequency v'_φ). This another description is

$$\frac{dv_\varphi}{dr} = \frac{N}{\sqrt{\mathrm{Ri}}}.$$

Here N is the Brunt-Väisälä frequency and Ri is Richardson number. This frequency in the case of large contribution of radiation pressure p_{rad} in the layer transmitting radiative energy flux q is

$$N^2 = g_{eff}^2(c_T^{-2} - c_S^{-2})\left(1 - \frac{1}{2}\,\tilde{q}\,\frac{10 + 8\tilde{R} + 5/(4\tilde{R})}{1 + 4\tilde{R}}\right), \qquad \tilde{R} = \frac{p_{rad}}{p_{pl}}, \quad \tilde{q} = \frac{q}{g_{eff}\,c\Sigma_T}$$

The deceleration v'_φ changes from $v'_\varphi = w/\rho\nu$ (if $0 < \mathrm{Ri} < 0.25$) to $v'_\varphi = N/\sqrt{\mathrm{Ri}}$ (if $\mathrm{Ri} \approx 0.25$) depending on local value of the number Ri.

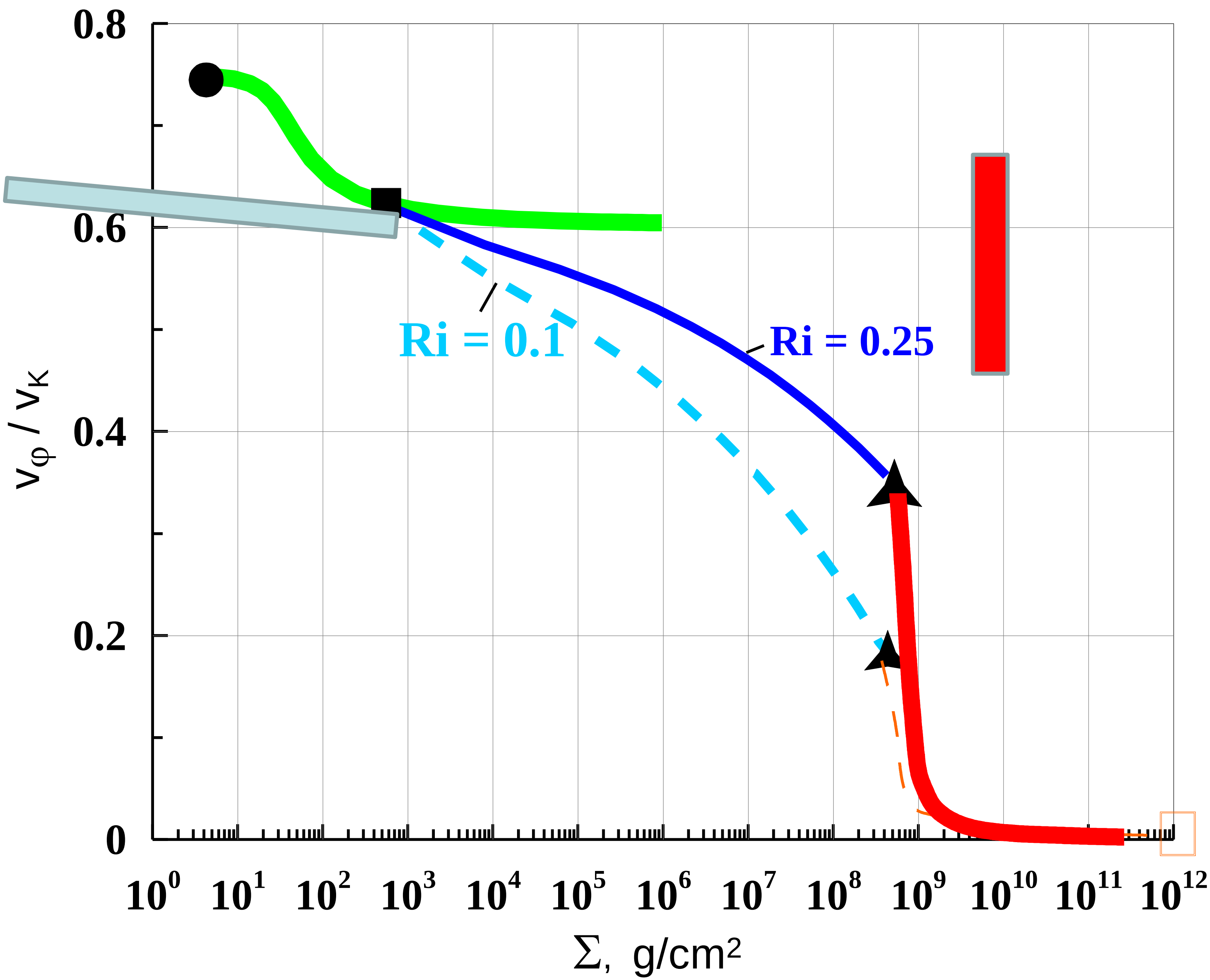
We constructed the gravity wave ladder by assuming that a small, compared to the Keplerian velocity, but noticeable decrease in rotation velocity (a decrease by a value of the order of the local speed of sound) occurs in each layer with a thickness equal to the pressure scale height h_E (see Fig. 7). Unfortunately, full flow braking at the commonly assumed Richardson number (see Sections 1.3 and 10) Ri = 0.25 requires a large number of ladder "steps" (here, a layer of thickness h_E is called the ladder step). Increasing the shear frequency dv_φ/dr by reducing the critical value of Ri from 0.25 to 0.1 does not save the situation in the sense of an overly high mass drawn into rapid rotation (see Fig. 3), al-

Richardson number

$$\mathrm{Ri} = N^2/\omega_{sh}^2$$

N and ω_{sh} - are the Brunt-Väisälä and shear frequencies

The profile of rotational velocity

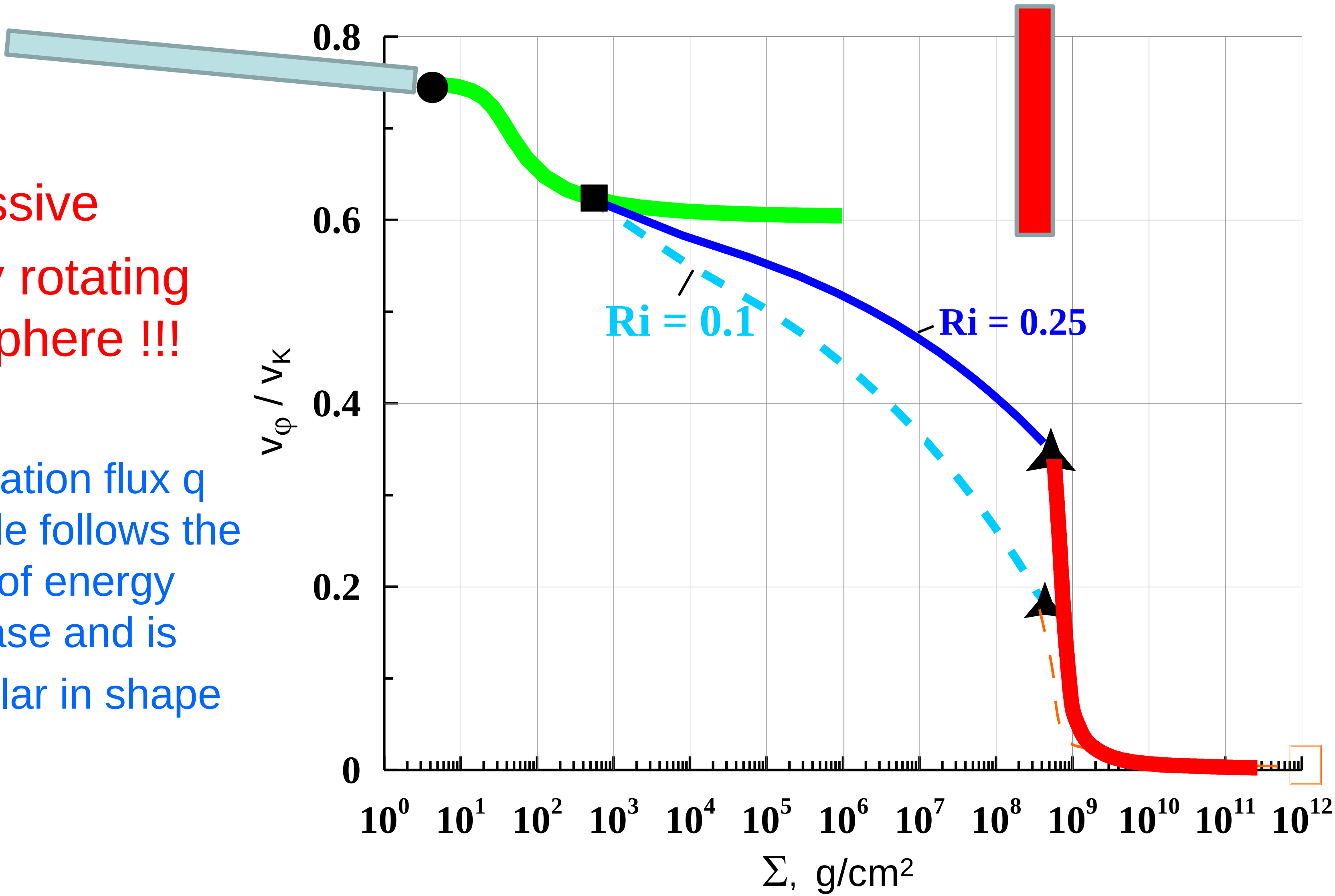


- Levitating layer, turbulent Prandtl-Karman deceleration, Richardson ladder deceleration, turbulent Prandtl-Karman deceleration above solid crust

- Massive

rapidly rotating atmosphere !!!

- Radiation flux q profile follows the rate of energy release and is similar in shape



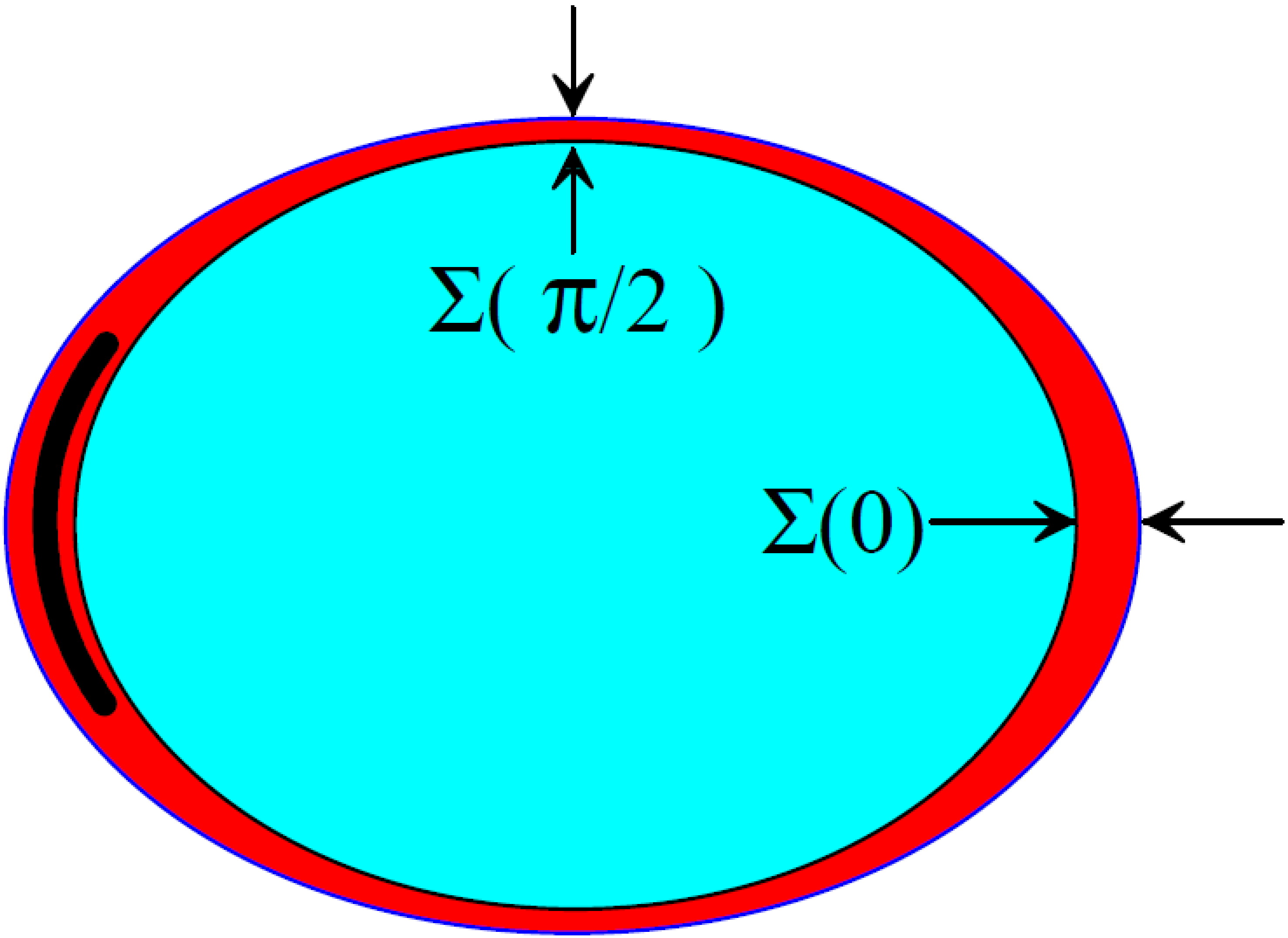
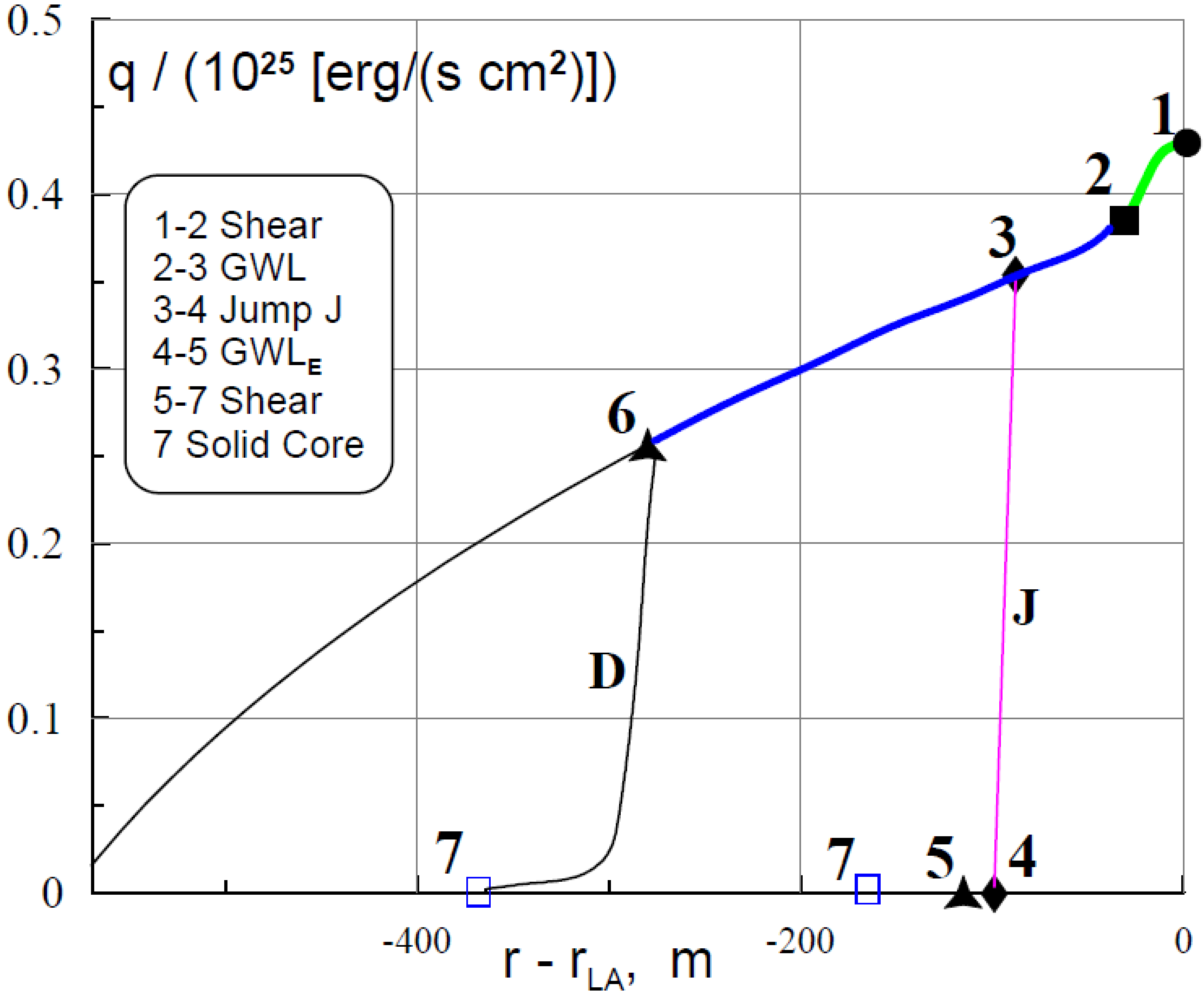
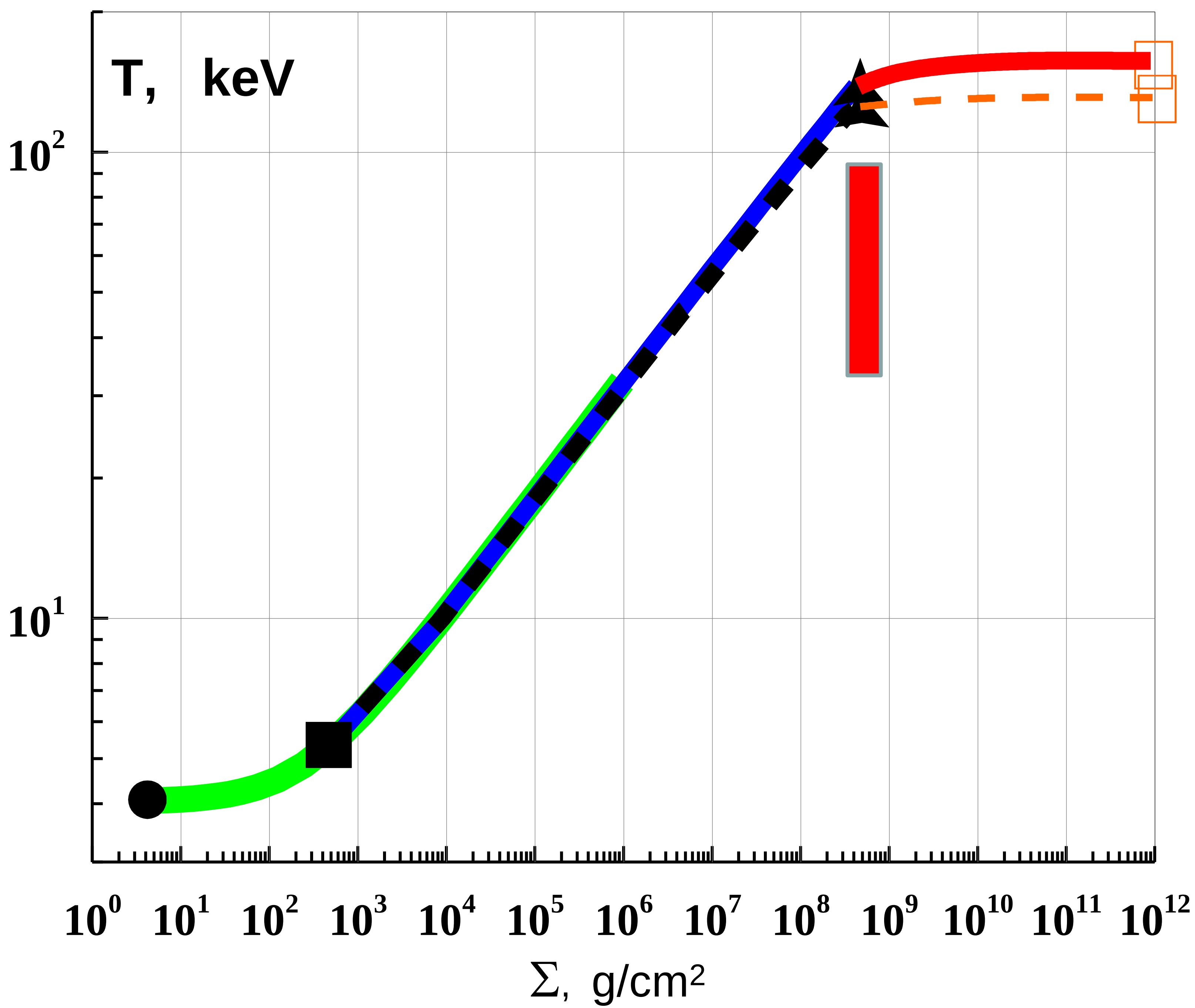


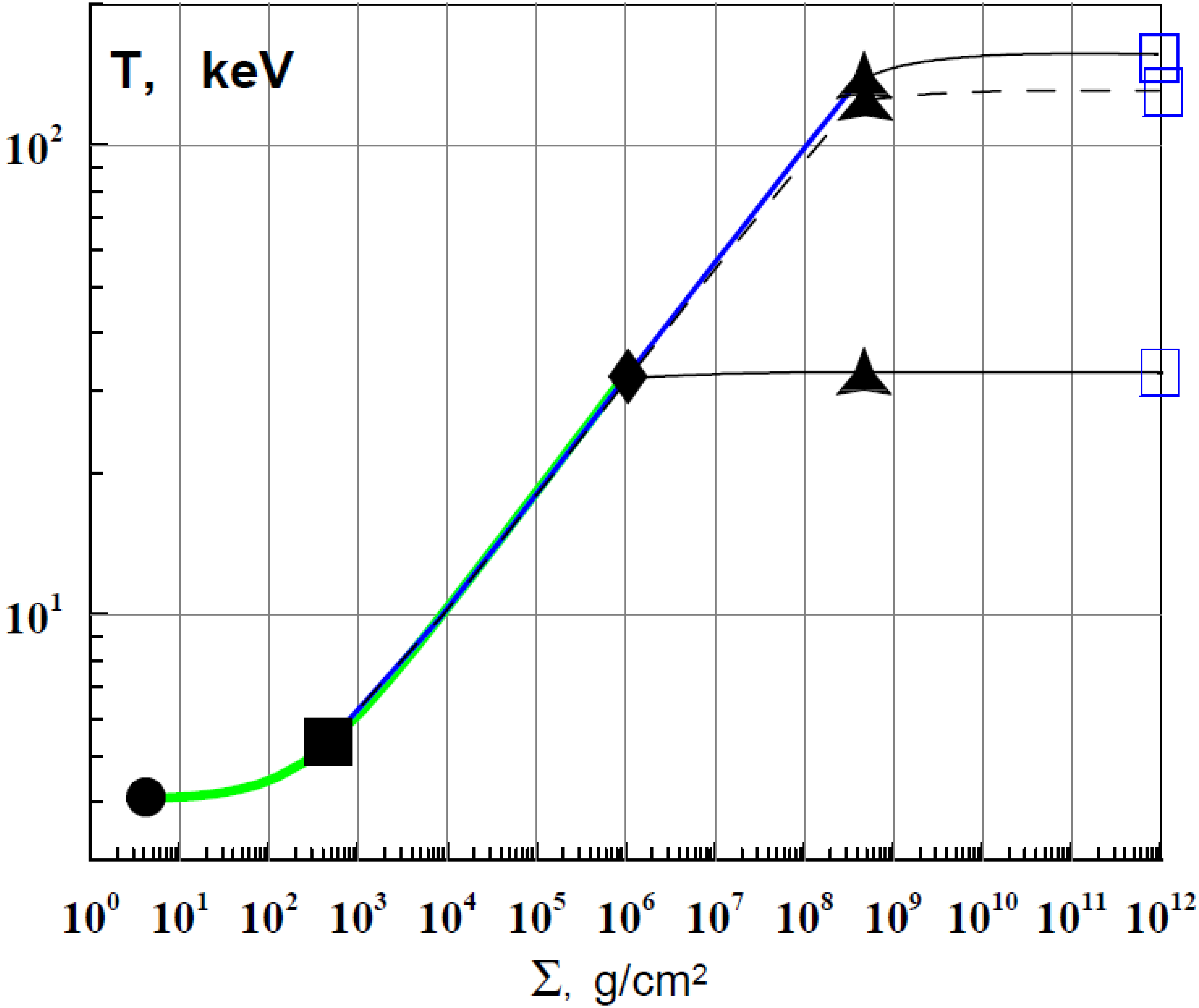
FIG. 12: The red layer is the layer of fresh accreted material (fuel) corotating together with neutron star. Fresh material is less dense than underlying ashes (blue) from previous bursts.

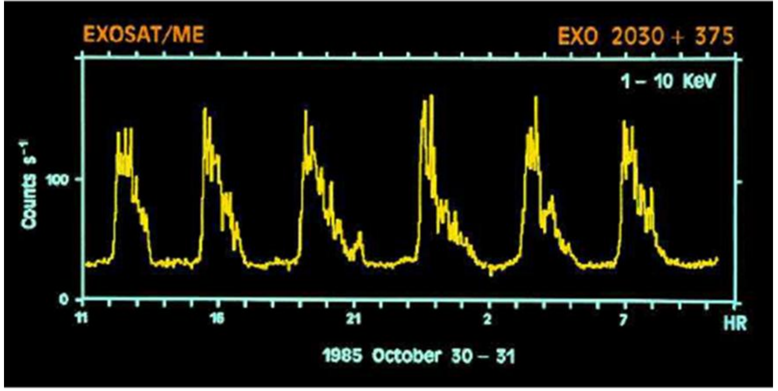


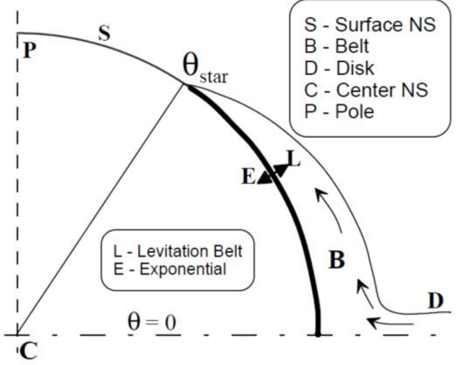
q / (10^25 [erg/(s cm^2)])
1-2 Shear
2-3 GWL
3-4 Jump J
4-5 GWL_E
5-7 Shear
7 Solid Core
1
2
3
4
5
6
7
7
J
D
0.5
0.4
0.3
0.2
0.1
0
-400
-200
0
r - r_LA, m



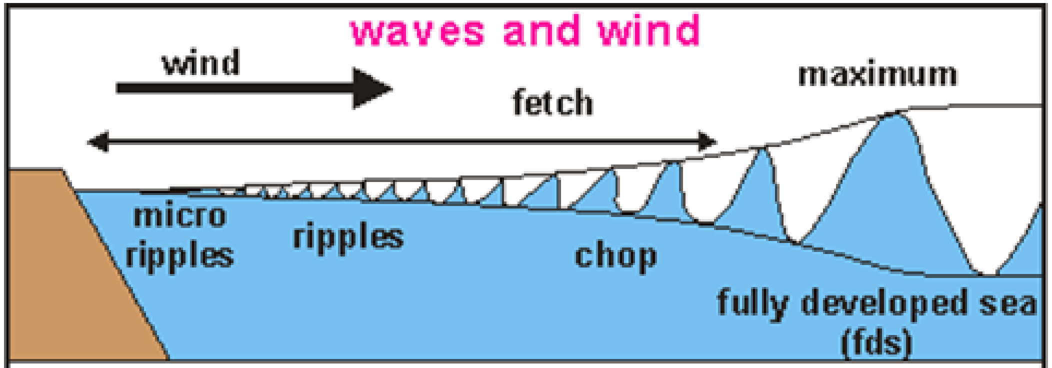
T, keV
10^2
10^1
10^0 10^1 10^2 10^3 10^4 10^5 10^6 10^7 10^8 10^9 10^{10} 10^{11} 10^{12}
Σ, g/cm2

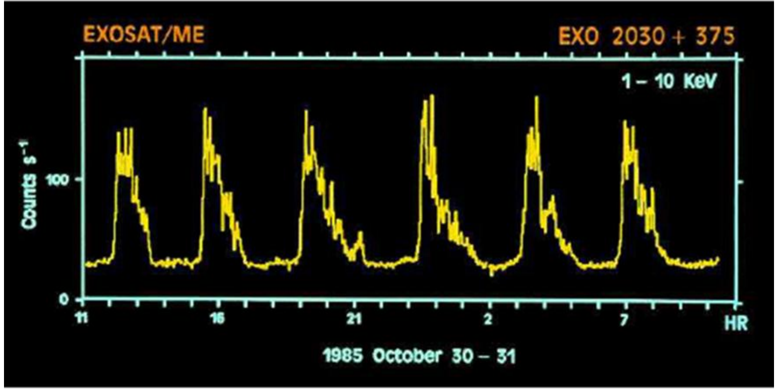

T, keV
10^2
10^1
10^0 10^1 10^2 10^3 10^4 10^5 10^6 10^7 10^8 10^9 10^10 10^11 10^12
Σ, g/cm²





There are many other very interesting consequences of this model for variability and spectral properties of such sources





There should be family of objects with thick rapidly rotating hot envelopes.

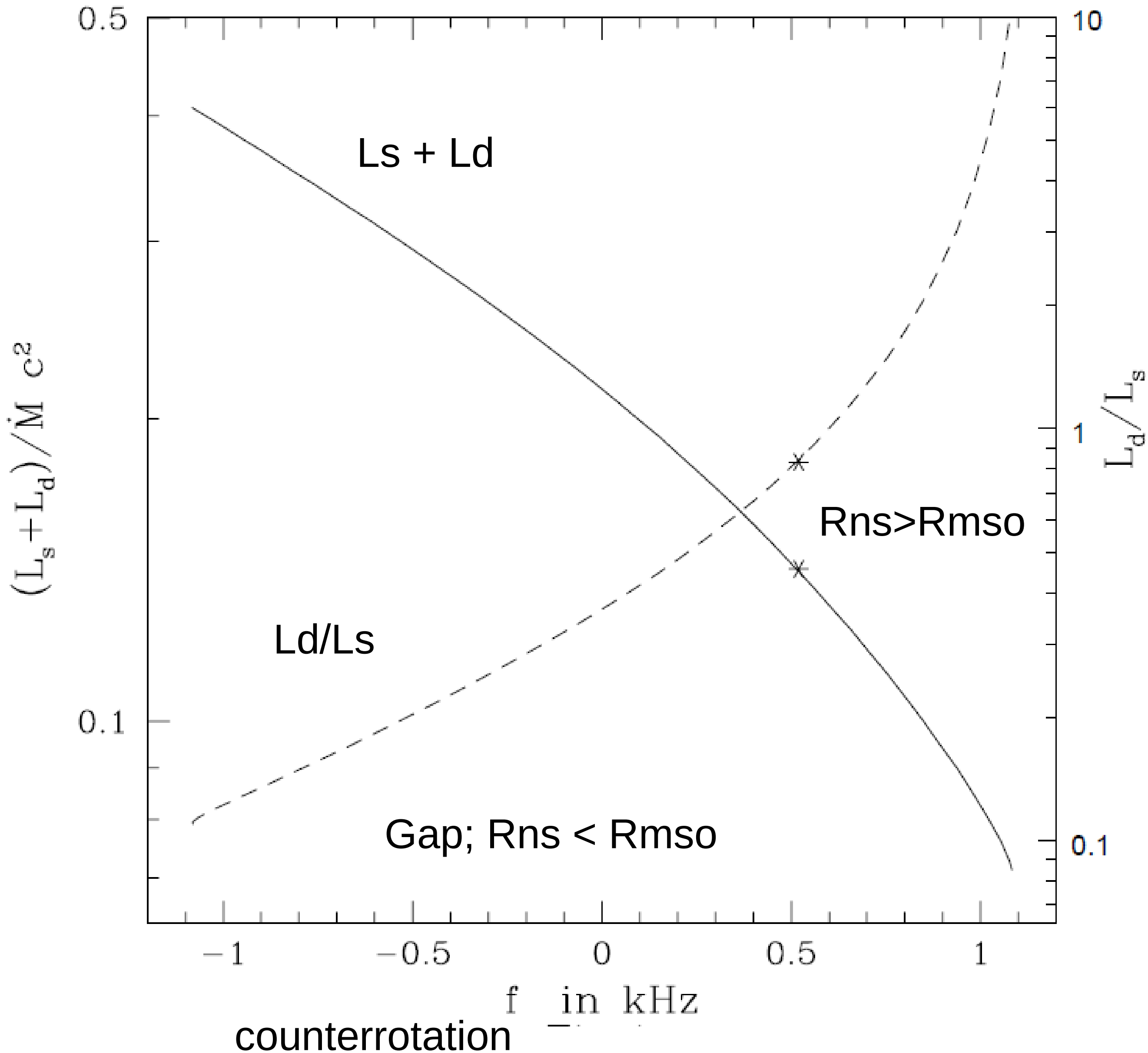
How to distinguish them from normal bursters ?

Persistent emission is coming from the deep layer and this leads
to softer spectra. Only 10-30% of energy is released in the levitating
Layer with relatively small optical depth.

It is important to remember that many sources demonstrate burst
activity only during few weeks and usually are in the quiet stationary state.

Thank you !!!

The energy release at NS surface and in accretion disc for EOS FPS $M = 1.4\ M_\odot$



Sibgatullin, Sunyaev, 2000

efficiency of accretion as function of frequenccy of rotation **“f”** **(solid line)**

ratio of disc luminosity to surface luminosity **(second curve)**

in the Newtonian mechanics **Ld=Ls** *(virial theorem)*

In Newtonian case (Sunyaev, Shakura, 1988 , Kluzniak, 1987
GR consideration: Sibgatullin, Sunyaev, 1998)

$$L_s = \frac{1}{2}\dot{M}GM(1 - f/f_K)^2/R, \quad L_d = \frac{1}{2}\dot{M}GM/R$$

Система уравнений

- Торможение, охлаждение за счет радиационного теплоотвода и оседание
- Сохранение массы при оседании $\rho v_r = -\dot{\Sigma}$
- Радиальный силовой баланс

$$\frac{d}{dr}(p + \rho v_r^2) = -\rho \left(g_{\rm grav} - \frac{v_\varphi^2}{R} \right) = -\rho g_{\rm eff}$$

$$p = p_{\rm pl} + p_{\rm rad}, \quad p_{\rm pl} = \rho \frac{2T}{m_p}, \quad p_{\rm rad} = \frac{aT^4}{3}$$

- Сохранение углового момента (w [dyn/cm^2])

$$J = w - \rho v_\varphi v_r = w + \dot{\Sigma} v_\varphi$$

Система уравнений, стр. 2: трение

- Сохранен. вращательного момента J: $w = J - \dot{\Sigma} v_\varphi$
- Закон Ньютона для напряжения трения $(\mu = \rho\nu)$

$$w = \rho\nu\, dv_\varphi / dr$$

- Суммарная вязкость $\nu = \nu_i + \nu_{\mathrm{rad}} + \nu_t$
- Парциальные вязкости : ионная : $\nu_i = 10^3 T^{2.5} / \rho \ln\Lambda$
- Радиационная вязкость : $\nu_{\mathrm{rad}} = (4/15) a T^4 \Sigma_{\mathrm{T}} / c\rho^2$
- Турбулентная вязкость : next page

Турбулентное трение и тормож. вращен.

- Модель Прандтля-Кармана

$$\nu_t = \alpha_t\, l_t v_t = \alpha_t h_\rho (h_\rho dv_\varphi / dr) = \alpha_t h_\rho^2 dv_\varphi / dr$$

- Модель Шакуры и Сюняева $\nu_t = \alpha_t l_t v_t = \alpha_t h_\rho c_{\mathrm{T}}$

- Замедление вращения за счет вязкости

$$\frac{dv_\varphi}{dr} = \frac{\boxed{w}}{\rho \boxed{\nu}}$$

$$\boxed{w} = J - \dot{\Sigma} v_\varphi$$

$$\boxed{\nu} = \nu_i + \nu_{\mathrm{rad}} + \nu_t$$

Замедление вращения в устойчиво стратифицированном слое Е

- Критерий Ричардсона $\mathrm{Ri} = N^2/\omega_{\mathrm{sh}}^2, \quad \omega_{\mathrm{sh}} = dv_\varphi/dr$

$$\frac{dv_\varphi}{dr} = \frac{N}{\sqrt{\mathrm{Ri}}}$$

$$\frac{dv_\varphi}{dr} = \frac{w}{\rho\nu}$$

- Частота Брента-Вяйсяля в плазме с существенным вкладом радиационного давления и пронизанной потоком q

$$N^2 = g_{\mathrm{eff}}^2 (c_{\mathrm{T}}^{-2} - c_{\mathrm{S}}^{-2}) \left[1 - \frac{1}{2} \tilde{q} \frac{10 + 8\tilde{R} + 5/(4\tilde{R})}{1 + 4\tilde{R}} \right]$$

$$\tilde{R} = \frac{p_{\mathrm{rad}}}{p_{\mathrm{pl}}}, \quad \tilde{q} = \frac{q}{g_{\mathrm{eff}} c \Sigma_{\mathrm{T}}}$$

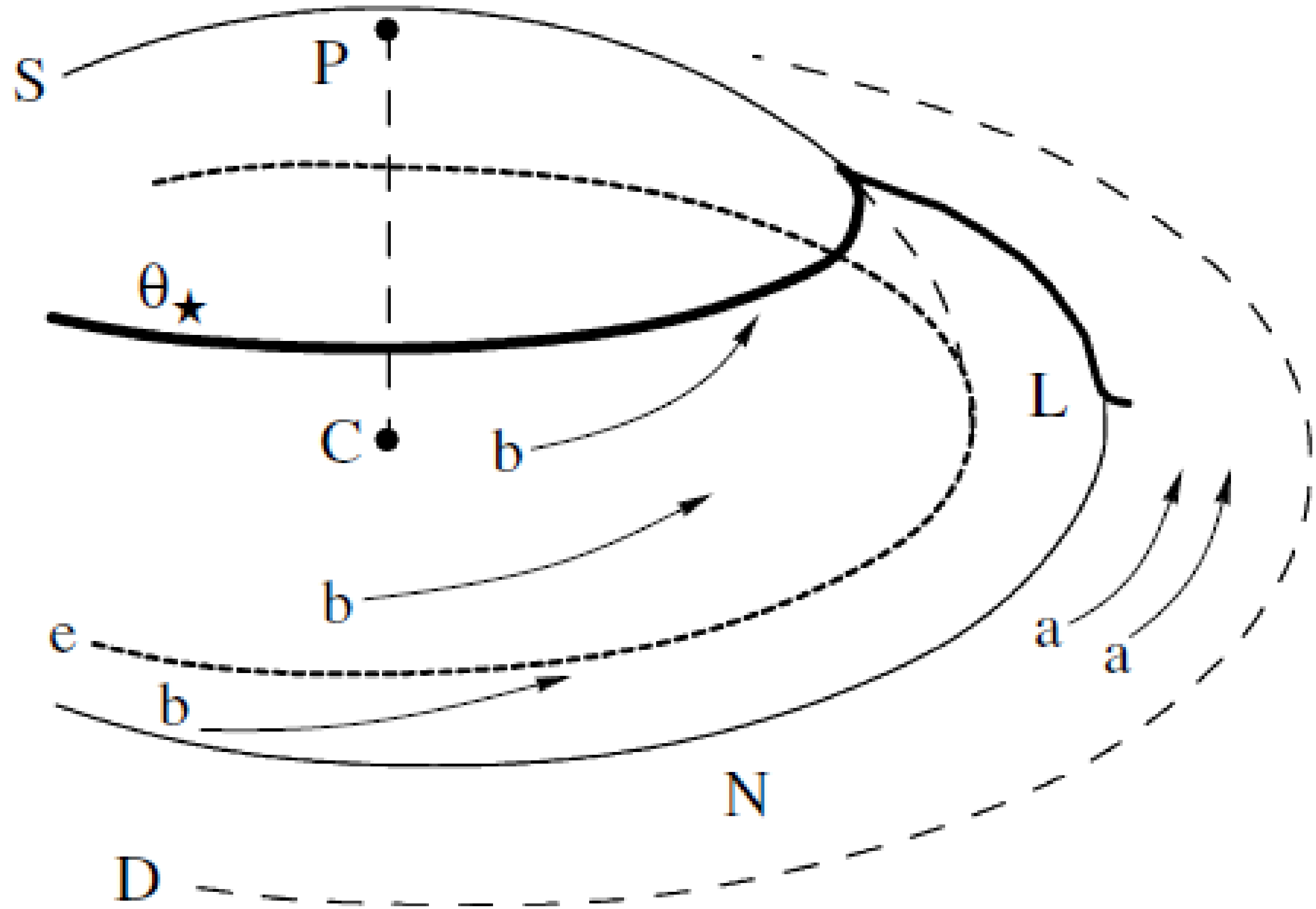


Fig. 10. Three-dimensional scheme of the disk D and the equatorial spreading belt L. Arrows a and b indicate the direction of rotation in the disk and the belt, e is the equatorial circumference on the NS surface.

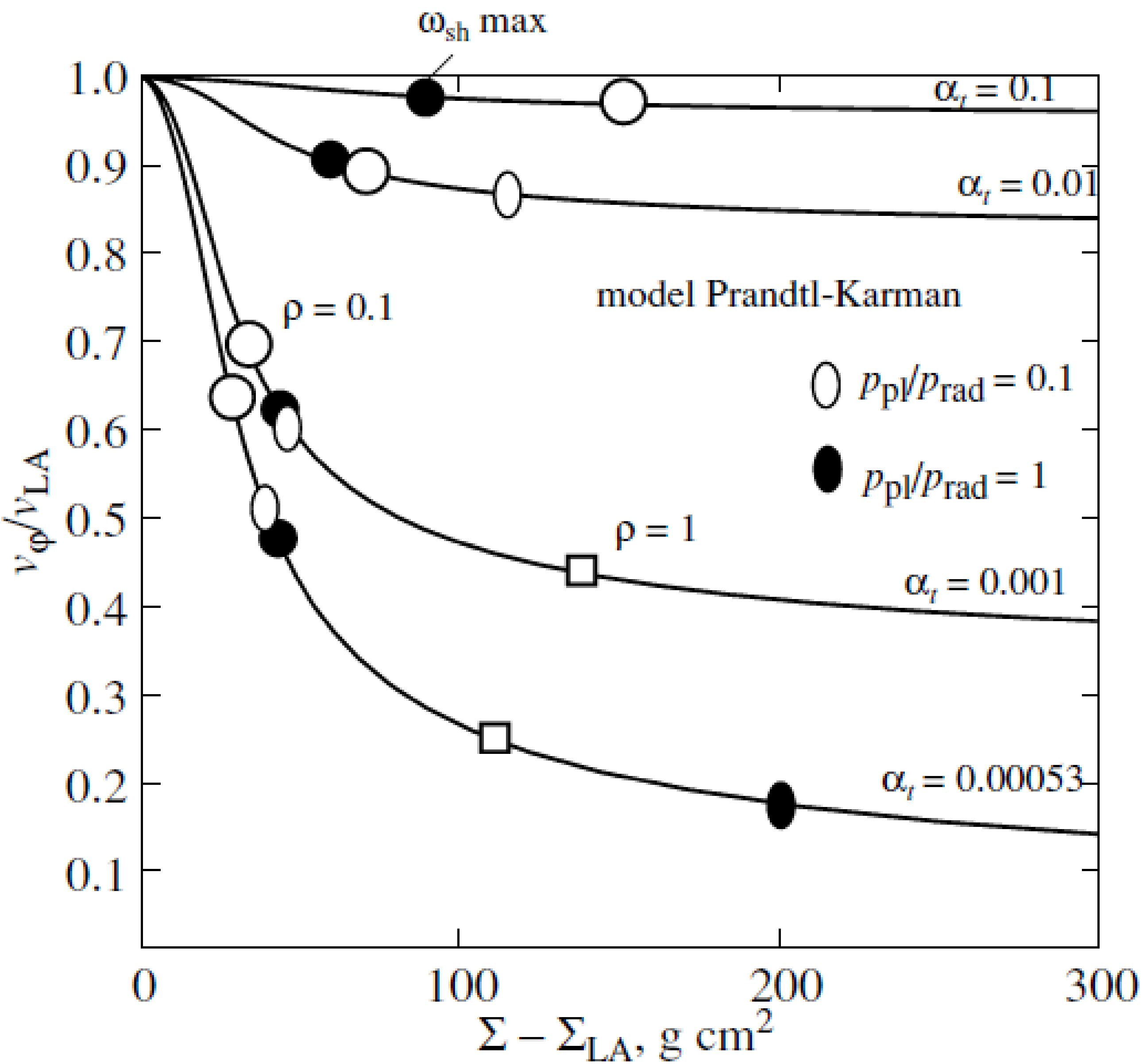


Fig. 23. Mass accumulation in the column in the braking process. The filled circles indicate the maxima of ω_{sh}, the open circles and squares represent $\rho = 0.1$ and 1 g cm^{-3}, respectively.