An approximate method to study oscillations of superfluid hyperon stars

V. A. Dommes[1,2], M. E. Gusakov[1]

[1]Ioffe Institute, St. Petersburg

[2]St. Petersburg Academic University

Abstract

We analyze hydrodynamic equations governing oscillations of superfluid neutron stars with hyperon cores (hyperon stars). With this aim we extend the approximate method presented in [1] in application to npe–matter and split the oscillation equations into two weakly coupled systems of equations, describing normal and superfluid modes. The proposed scheme allows us to significantly simplify calculations of these modes. An efficiency of this scheme is illustrated by calculation of sound waves in hyperon stars at arbitrary temperature.

1. Introduction

Theory of neutron-star (NS) oscillations is complicated by the fact that at a temperature $T < 10^8 - 10^{10} K$ baryons in the internal layers of NSs become superfluid. Thus, to model oscillations one has to employ superfluid hydrodynamics which is much more complex than the ordinary one, describing "normal" (nonsuperfluid) matter.

Lindblom and Mendell in 1994 [2] numerically found two distinct classes of pulsation modes for a simple model of a superfluid Newtonian star: (i) normal modes which practically coincide with the corresponding modes of a normal star; and (ii) superfluid modes in which the matter pulsates in such a way that the mass current density approximately vanishes. General explanation of this result has been proposed by Gusakov and Kantor in 2011 [1]. These authors also presented an approximate scheme which allows one to decouple equations describing normal and superfluid modes and greatly simplifies calculations of pulsating superfluid NSs (consisting of npe–matter).

We extend this approximate method tomassive neutron stars, whose cores are composed of neutrons (n), protons (p), electrons (e), muons (μ), as well as Λ–, Ξ^-–, Ξ^0– and Σ^-–hyperons. We illustrate this scheme by calculating sound waves and r–modes in hyperon stars for various equations of state [3] in wide range of temperatures T and baryon number densities n_b.

2. Relativistic superfluid hydrodynamics

2.1 Definitions

$i, k = n, p, \Lambda, \Xi^-, \Xi^0, \Sigma^-$	indices for baryons
$l = e, \mu$	indices for leptons
q_i, q_l	charge of given particle
$\alpha, \beta, \gamma = 0, 1, 2, 3$	spacetime indicies
u^α	4-velocity of normal fluid
$v^\alpha_{sfl(i)}$	4-velocity of superfluid particles i
$w^\alpha_{(i)} = \mu_i(v^\alpha_{sfl(i)} - u^\alpha)$	superfluid 4-velocity
μ_i, μ_l	relativistic chemical potential
Y_{ik}	Symmetric relativistic entrainment matrix
$j^\alpha_{(i)} = n_i u^\alpha + Y_{ik} w^\alpha_{(k)}$	4-current of baryons i
$j^\alpha_{(e)} = n_e u^\alpha$, $j^\alpha_{(\mu)} = n_\mu u^\alpha$	4-currents of electrons and muons
n_i, n_l	number density of particles i, l
$n_b = \sum_i n_i$	baryon number density
$j^\alpha_{(b)} = \sum_i j^\alpha_{(i)} = n_b U^\alpha_b = n_b(u^\alpha + W^\alpha)$	baryon 4-current
s_i	strangeness of particle i
$n_s = -\sum_i s_i n_i = n_\Lambda + 2n_{\Xi^-} + 2n_{\Xi^0} + n_{\Sigma^-}$	(minus) strangeness concentration
$j^\alpha_{(s)} = -\sum_i s_i j^\alpha_{(i)} = n_s U^\alpha_s$	"strange" 4-current
$T^{\alpha\beta}$	energy-momentum tensor
$X_{;\alpha}$	covariant derivative of quantity X

2.2 Main processes of particle transformations

Strong processes in nucleon-hyperon matter:

$$\Lambda + \Lambda \leftrightarrow p + \Xi^-, \quad \Lambda + \Lambda \leftrightarrow n + \Xi^0, \quad n + \Lambda \leftrightarrow p + \Sigma^-. \tag{1}$$

We assume that the perturbed matter is always in chemical equilibrium with respect to these reactions:

$$2\mu_\Lambda = \mu_p + \mu_{\Xi^-}, \quad \mu_n + \mu_\Lambda = \mu_p + \mu_{\Sigma^-}, \quad 2\mu_\Lambda = \mu_n + \mu_{\Xi^0}. \tag{2}$$

2.3 Hydrodynamic equations

Hereafter we consider small perturbations in non-rotating spherically symmetric neutron star. We give a brief overview of superfluid relativistic hydrodynamics (see, e.g., [4] for details).

Density currents for baryons and leptons are expressed in terms of the "normal" four-velocity u^α and additional four-vectors $w^\alpha_{(i)} = \mu_i(v^\alpha_{sfl(i)} - u^\alpha)$, responsible for superfluid degrees of freedom.

$$j^\alpha_{(i)} = n_i u^\alpha + Y_{ik} w^\alpha_{(k)}, \quad j^\alpha_{(l)} = n_l u^\alpha, \quad u_\alpha j^\alpha_{(i)} = n_i. \tag{3}$$

Note that in equilibrium $u^\alpha = (u^0, 0, 0, 0)$, $w^\alpha_{(i)} = 0$. The system of hydrodynamic equations is formulated below.

The baryon current and continuity equation for baryons are:

$$j^\alpha_{(b)} = \sum_i j^\alpha_{(i)} = n_b u^\alpha + \sum_i Y_{ik} w^\alpha_{(k)} = n_b\left(u^\alpha + W^\alpha\right), \quad j^\alpha_{(b);\alpha} = 0. \tag{4}$$

We also assume that weak processes of particle transformations are slow on typical hydrodynamic timescale. This assumption leads to the following continuity equations for electrons, muons and strangeness:

$$j^\alpha_{(e);\alpha} = j^\alpha_{(\mu);\alpha} = j^\alpha_{(s);\alpha} = 0. \tag{5}$$

Energy-momentum conservation law takes the form:

$$T^{\alpha\beta}_{\ ;\beta} = 0, \text{ with } T^{\alpha\beta} = (P + \varepsilon) u^\alpha u^\beta + P g^{\alpha\beta} + Y_{ik}\left[w^\alpha_{(i)} w^\beta_{(k)} + \mu_i w^\alpha_{(k)} u^\beta + \mu_k w^\beta_{(i)} u^\alpha\right]. \tag{6}$$

In case of small perturbations $T^{\alpha\beta}$ can be simplified:

$$T^{\alpha\beta} = (P + \varepsilon) u^\alpha u^\beta + P g^{\alpha\beta} + \mu_n n_b \left(W^\alpha u^\beta + W^\beta u^\alpha\right) + \text{(quadratically small terms)}. \tag{7}$$

Einstein equations read:

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = 8\pi T^{\alpha\beta}. \tag{8}$$

Thermodynamic relations in the low temperature limit:

$$P + \varepsilon = \mu_i n_i + \mu_l n_l = \mu_n n_b - \delta\mu_\Lambda n_s - \delta\mu_e n_e - \delta\mu_\mu n_\mu, \tag{9}$$
$$dP = n_b d\mu_n - n_e d\delta\mu_e - n_\mu d\delta\mu_\mu - n_s d\delta\mu_\Lambda, \tag{10}$$
$$\delta\mu_e \equiv \mu_n - \mu_p - \mu_e, \quad \delta\mu_\mu \equiv \mu_n - \mu_p - \mu_\mu, \quad \delta\mu_\Lambda \equiv \mu_n - \mu_\Lambda. \tag{11}$$

In equilibrium $\delta\mu_e = \delta\mu_\mu = \delta\mu_\Lambda = 0$

Quasineutrality condition, $q_i j^\alpha_{(i)} + q_l j^\alpha_{(l)} = 0$, implies 2 equations:

$$q_i n_i + q_l n_l = q_p(n_p - n_e - n_\mu - n_{\Xi^-} - n_{\Sigma^-}) = 0, \tag{12}$$
$$q_i Y_{ik} w^\alpha_{(k)} = q_p(Y_{pk} - Y_{\Xi^- k} - Y_{\Sigma^- k}) w^\alpha_{(k)} = 0. \tag{13}$$

Potentiality of superfluid motion (A_α is four-potential of electromagnetic field):

$$\left(w_{(i)\alpha} + \mu_i u_\alpha + q_i A_\alpha\right)_{;\beta} - \left(w_{(i)\beta} + \mu_i u_\beta + q_i A_\beta\right)_{;\alpha} = 0. \tag{14}$$

In superfluid $npe\mu\Lambda\Xi^-\Xi^0\Sigma^-$–matter there are 6 superfluid four–velocities: ($w^\alpha_{(n)}, w^\alpha_{(p)}, w^\alpha_{(\Lambda)}, w^\alpha_{(\Xi^-)}, w^\alpha_{(\Xi^0)}, w^\alpha_{(\Sigma^-)}$) which are subject to 4 constraints (quasineutrality condition (13) and equilibrium conditions for fast reactions (1)).

Using these constraints one can write the expression for baryon current in terms of just 2 additional variables, e.g $w^\alpha_{(n)}$ and $w^\alpha_{(\Lambda)}$:

$$j^\alpha_{(b)} = n_b u^\alpha + \sum_i Y_{ik} w^\alpha_{(k)} = n_b u^\alpha + \tilde{Y}_n w^\alpha_{(n)} + \tilde{Y}_\Lambda w^\alpha_{(\Lambda)}. \tag{15}$$

Here $\tilde{Y}_n$ and $\tilde{Y}_\Lambda$ can be expressed through the entrainment matrix Y_{ik}.

3. Decoupling of superfluid and normal equations

3.1 Superfluid equations

We follow derivation of ref. [1] applied to npe–matter.

Using the energy-momentum conservation law (6) and the potentiality equation for neutrons (14) we compose a vanishing combination

$$\begin{aligned} &T_{\alpha\ ;\beta}^{\ \beta} + u_\alpha u_\gamma T^{\gamma\beta}_{\ ;\beta} - n_b u^\beta\left[\left(w_{(n)\alpha} + \mu_n u_\alpha\right)_{;\beta} - \left(w_{(n)\beta} + \mu_n u_\beta\right)_{;\alpha}\right] = \\ &= (P + \varepsilon - \mu_n n_b) u^\beta u_{\alpha;\beta} + (P_{;\beta} - n_b \mu_{n;\beta}) u_\alpha u^\beta + (P_{;\alpha} - n_b \mu_{n;\alpha}) + \\ &+ (g_{\alpha\gamma} + u_\alpha u_\gamma) u^\beta (\mu_n n_b W^\gamma)_{;\beta} + \mu_n n_b \left(u^\beta_{\ ;\beta} W_\alpha + u_{\alpha;\beta} W^\beta\right) - n_b u^\beta\left[w_{(n)\alpha;\beta} - w_{(n)\beta;\alpha}\right] = 0. \end{aligned} \tag{16}$$

Employing thermodynamic relations (9) and (10) in equation (16) it is easily verified that each term in equation (16) depends on one of the small (and vanishing in equilibrium) quantities $\delta\mu_e, \delta\mu_\mu, \delta\mu_\Lambda, w^\alpha_{(n)}, w^\alpha_{(\Lambda)}$. Thus, in linear approximation, one can replace all other quantities in this equation with their equilibrium values.

Now let us consider nonrotating star with the Schwarzschild metric, $ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, and assume that all perturbations depend on time as $e^{i\omega t}$. In this case spatial components ($\alpha = 1, 2, 3$) of superfluid equation take a simple form:

$$i\omega n_b(\mu_n W_\alpha - w_{(n)\alpha}) = n_e \frac{\partial}{\partial x^\alpha}\left(\delta\mu_e e^{\nu/2}\right) + n_\mu \frac{\partial}{\partial x^\alpha}\left(\delta\mu_\mu e^{\nu/2}\right) + n_s \frac{\partial}{\partial x^\alpha}\left(\delta\mu_\Lambda e^{\nu/2}\right). \tag{17}$$

The second superfluid equation can be derived by subtracting the potentiality equation (14) for neutrons from the potentiality equation for Λ–hyperons,

$$i\omega\left(w_{(\Lambda)\alpha} - w_{(n)\alpha}\right) = \frac{\partial}{\partial x^\alpha}\left(\delta\mu_\Lambda e^{\nu/2}\right), \quad \alpha = 1, 2, 3. \tag{18}$$

Note that disbalances of chemical potentials play the role of "driving force" for superfluid oscillations.

3.2 Normal equation and coupling parameters

In the linear approximation a perturbation δ of energy-momentum tensor (7) $T^{\alpha\beta}$ can be written as

$$\delta T^{\alpha\beta} = (\delta P + \delta\varepsilon) U^\alpha_b U^\beta_b + (P + \varepsilon)\left(U^\alpha_b \delta U^\beta_b + U^\beta_b \delta U^\alpha_b\right) + \delta P g^{\alpha\beta} + P \delta g^{\alpha\beta}. \tag{19}$$

Here U^α_b, P, ε, $g^{\alpha\beta}$ are taken in equilibrium (note that in equilibrium $U^\alpha_b = u^\alpha$).

We describe perturbations in superfluid $npe\mu\Lambda\Xi^-\Xi^0\Sigma^-$–matter by choosing $\delta g^{\alpha\beta}$, δU^α_b, $w^\alpha_{(n)}$, $w^\alpha_{(\Lambda)}$ as independent variables.

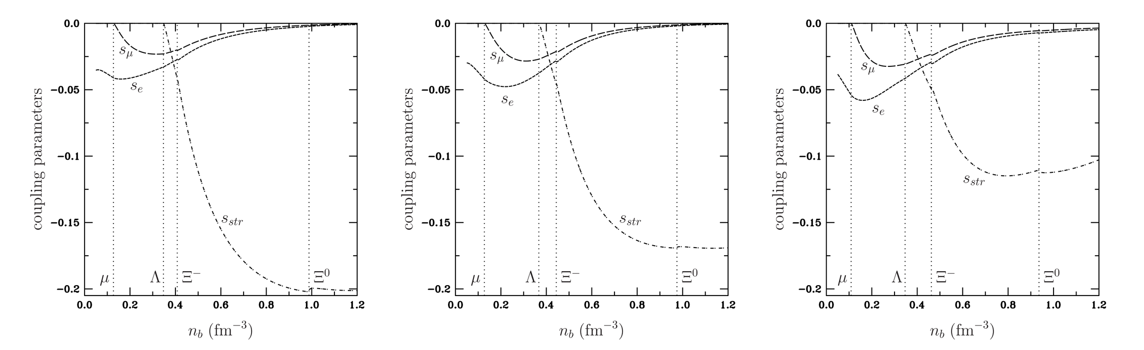
All thermodynamic quantities in degenerate matter are functions of (n_b, n_e, n_μ, n_s), hence their perturbations can be expressed in terms of $(\delta n_b, \delta n_e, \delta n_\mu, \delta n_s)$ or, using continuity equations, in terms of $(\delta U^\alpha_b, \delta g^{\alpha\beta}, w^\alpha_{(n)}, w^\alpha_{(\Lambda)})$.

Now let us express $\delta\varepsilon$ and δP through n_b, n_e, n_μ, n_s:

$$\delta\varepsilon = \mu_n \delta n_b, \quad \delta P = n_b \frac{\partial P(n_b, n_e, n_\mu, n_s)}{\partial n_b}\left(\frac{\delta n_b}{n_b} + s_e \frac{\delta n_e}{n_e} + s_\mu \frac{\delta n_\mu}{n_\mu} + s_{str} \frac{\delta n_s}{n_s}\right), \tag{20}$$

$$s_e = \frac{\partial \ln P/\partial \ln n_e}{\partial \ln P/\partial \ln n_b}, \quad s_\mu = \frac{\partial \ln P/\partial \ln n_\mu}{\partial \ln P/\partial \ln n_b}, \quad s_{str} = \frac{\partial \ln P/\partial \ln n_s}{\partial \ln P/\partial \ln n_b}, \tag{21}$$

where we introduced the electron, muon and strange coupling parameters s_e, s_μ, s_{str}, respectively.



(GM1A) (GM1'B) (TM1C)

Figure 1: *Coupling parameters s_e, s_μ, s_{str} vs baryon number density n_b for EOS GM1A, GM1'B, TM1C [3]. Vertical lines: thresholds of appearance for muons and Λ–, Ξ^-–, Ξ^0–hyperons.*

Note that δn_b depends only on $(\delta U^\alpha_b, \delta g^{\alpha\beta})$. Thus, if we set $s_e = s_\mu = s_{str} = 0$, then $\delta T^{\alpha\beta} = \delta T^{\alpha\beta}(\delta n_b) = \delta T^{\alpha\beta}(\delta U^\alpha_b, \delta g^{\alpha\beta})$ does not depend on superfluid velocities and has exactly the same form as in the absence of superfluidity. This means that perturbed Einstein equations (8), $\delta\left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R\right) = 8\pi \delta T^{\alpha\beta}$, coincide with the corresponding equations for normal matter.

As a result, a solution to hydrodynamic equations in the approximation of vanishing coupling parameters has the following properties:

- Equations governing relativistic oscillations are split into two completely decoupled systems.
- The first system describes normal modes which depend on the variables $\delta U^\alpha_b, \delta g^{\alpha\beta}$. Frequencies of normal modes can be calculated within nonsuperfluid hydrodynamics.
- The second system describes superfluid modes which depend only on superfluid variables $w^\alpha_{(n)}, w^\alpha_{(n)}$. Frequencies of superfluid modes can be calculated by using only two "superfluid" equations (17), (18) instead of full system containing Einstein equations.
- For decoupled superfluid modes perturbations of baryon velocity δU^α_b, baryon number density δn_b, metric $\delta g^{\alpha\beta}$ and pressure δP vanish.

4. Sound waves

Let us consider small harmonic perturbations ($\sim e^{i(\omega t - kx)}$) in homogeneous superfluid matter in Minkowski spacetime: $g^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. Equations describing sound waves follow from the general equations described above,

$$\omega(P + \varepsilon)\delta U_b = k\delta P, \tag{22}$$
$$\omega(\mu_n \tilde{Y}_n w_{(n)} + \mu_n \tilde{Y}_\Lambda w_{(\Lambda)} - n_b w_{(n)}) = -k\left(n_e \delta\mu_e + n_\mu \delta\mu_\mu + n_s \delta\mu_\Lambda\right), \tag{23}$$
$$\omega\left(w_{(\Lambda)} - w_{(n)}\right) = -k\delta\mu_\Lambda. \tag{24}$$

After expressing $\delta P, \delta\mu_e, \delta\mu_\mu, \delta\mu_\Lambda$ in terms of $\delta U_b, w_{(n)}, w_{(\Lambda)}$ one can see that this system reduces to the cubic equation on squared speed of sound $c_S^2 = \omega^2/k^2$.

We calculate sound waves for EOS GM1A, GM1'B, TM1C [3]. We adopt the following values for baryon critical temperatures:

$T_{cn} = 5 \times 10^8\ K, \quad T_{cp} = T_{c\Xi^-} = T_{c\Xi^0} = 5 \times 10^9\ K$

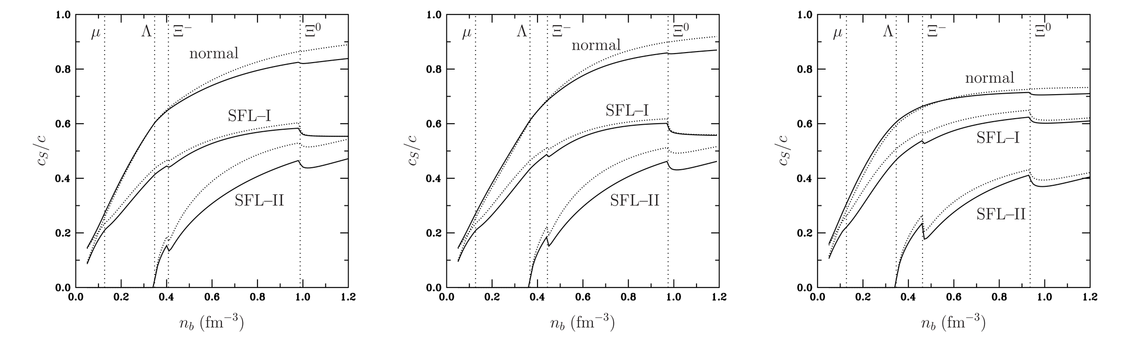
As for Λ–hyperons, we consider 2 different cases: (i) Λ–hyperons are superfluid, $T_{c\Lambda} = 10^9 K$; (ii) Λ–hyperons are normal at $T > 10^7 K$.

At low temperatures, in the first case, the additional degree of freedom and, as a consequence, second superfluid mode (SFL-II) arises after the appearance of Λ–hyperons (see fig. 2). If Λ–hyperons are normal, second superfluid mode arises after the appearance of Ξ^-–hyperons (see fig. 5).

Note that, at high densities, a difference between the exact and decoupled solutions is determined mainly by the strange coupling parameter s_{str} (see fig. 1).

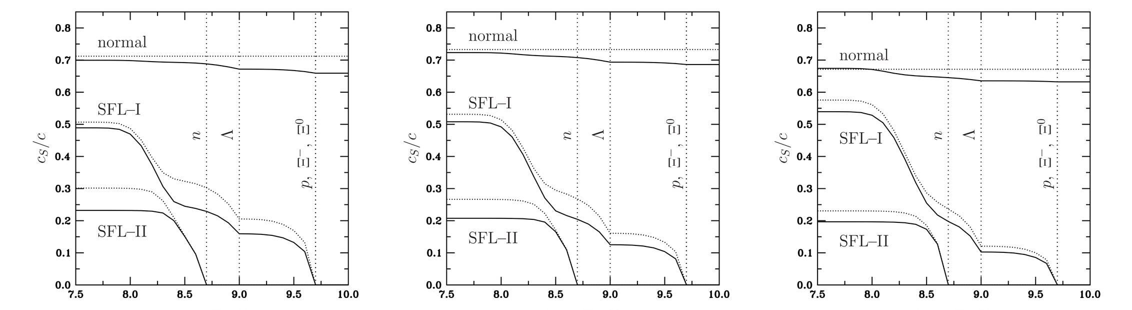
The temperature dependence of sound modes is determined by temperature dependence of entrainment matrix Y_{ik}.

4.1 Results for the case of superfluid Λ–hyperons, $T_{c\Lambda} = 10^9 K$



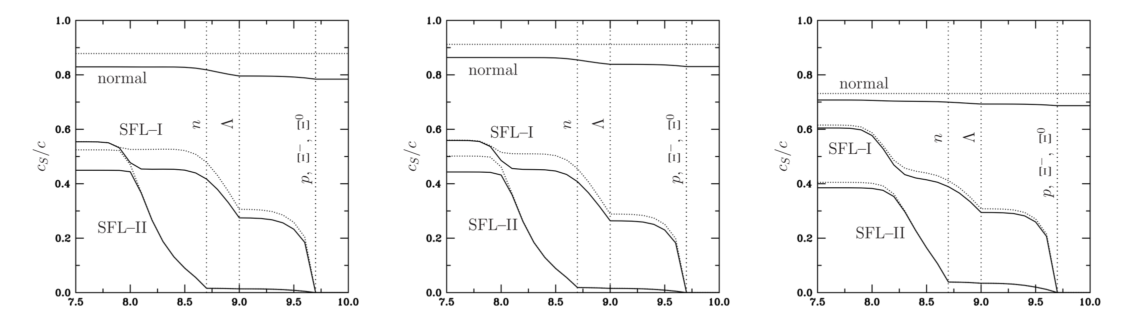
(GM1A) (GM1'B) (TM1C)

Figure 2: *Speed of sound c_S (in units of c) vs baryon number density n_b for EOS GM1A, GM1'B, TM1C [3] at $\log_{10} T = 7.5$. Solid lines: exact solution. Dotted lines: decoupled solution. Vertical lines: thresholds of appearance for muons and Λ–, Ξ^-–, Ξ^0–hyperons.*



(GM1A) (GM1'B) (TM1C)

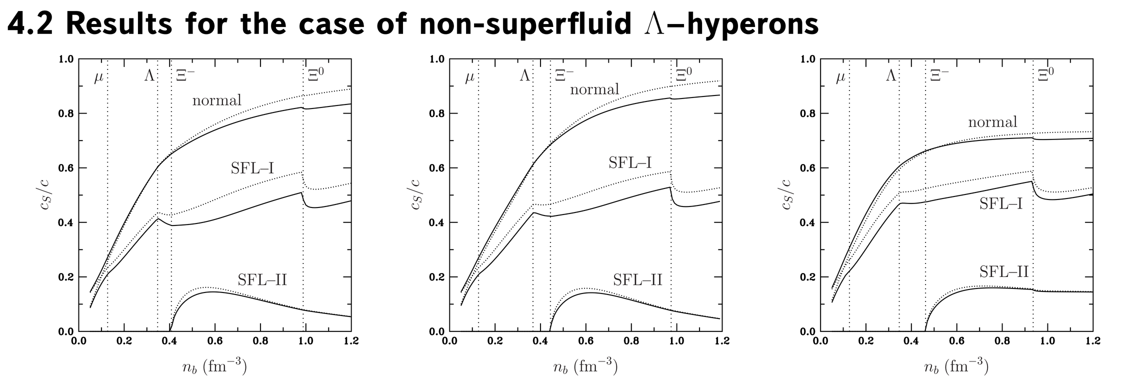
Figure 3: *Speed of sound c_S (in units of c) vs temperature $\log_{10} T$, K for EOS GM1A, GM1'B, TM1C [3] at $n_b = 0.5\ \mathrm{fm}^{-3}$. Solid lines: exact solution. Dotted lines: decoupled solution. Vertical lines: critical temperatures for baryons.*



(GM1A) (GM1'B) (TM1C)

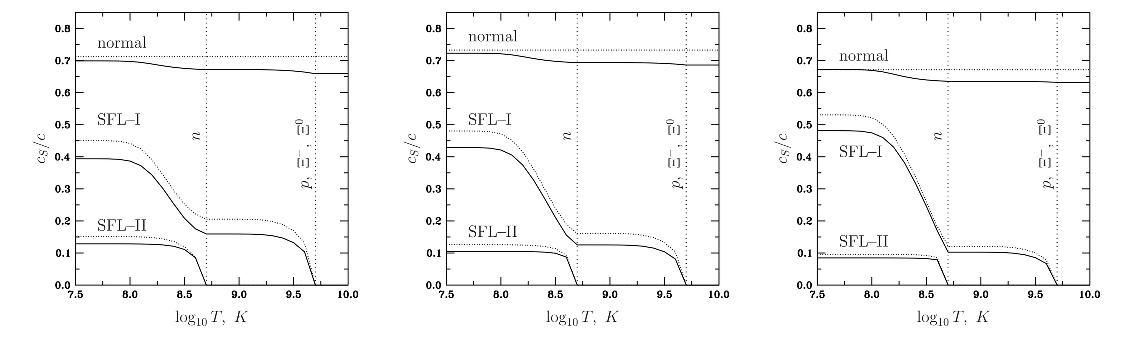
Figure 4: *Speed of sound c_S (in units of c) vs temperature $\log_{10} T$, K for EOS GM1A, GM1'B, TM1C [3] at $n_b = 1.1\ \mathrm{fm}^{-3}$. Solid lines: exact solution. Dotted lines: decoupled solution. Vertical lines: critical temperatures for baryons.*

4.2 Results for the case of non-superfluid Λ–hyperons



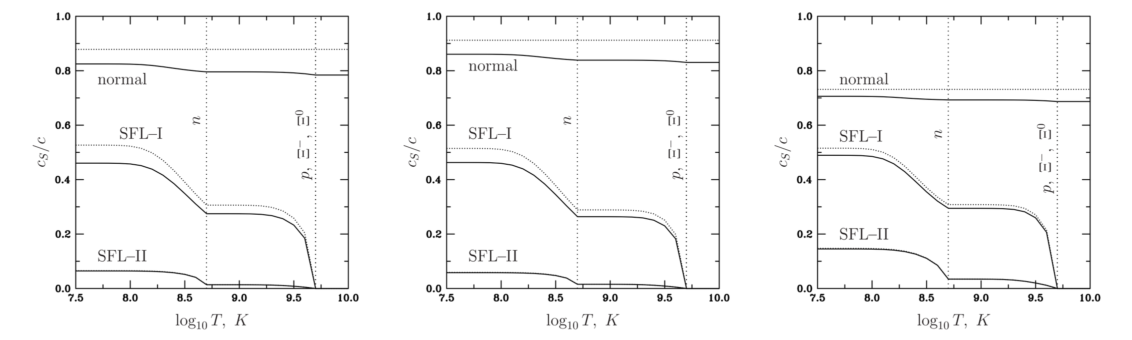
(GM1A) (GM1'B) (TM1C)

Figure 5: *Speed of sound c_S (in units of c) vs baryon number density n_b for EOS GM1A, GM1'B, TM1C [3] at $\log_{10} T = 7.5$. Solid lines: exact solution. Dotted lines: decoupled solution. Vertical lines: thresholds of appearance for muons and Λ–, Ξ^-–, Ξ^0–hyperons.*



(GM1A) (GM1'B) (TM1C)

Figure 6: *Speed of sound c_S (in units of c) vs temperature $\log_{10} T$, K for EOS GM1A, GM1'B, TM1C [3] at $n_b = 0.5\ \mathrm{fm}^{-3}$. Solid lines: exact solution. Dotted lines: decoupled solution. Vertical lines: critical temperatures for baryons.*



(GM1A) (GM1'B) (TM1C)

Figure 7: *Speed of sound c_S (in units of c) vs temperature $\log_{10} T$, K for EOS GM1A, GM1'B, TM1C [3] at $n_b = 1.1\ \mathrm{fm}^{-3}$. Solid lines: exact solution. Dotted lines: decoupled solution. Vertical lines: critical temperatures for baryons.*

5. Conclusion

- Equations governing relativistic oscillations of hyperon stars can be split into two systems of weakly coupled equations with the coupling parameters s_e, s_μ, and s_{str} given by eq. (21). These two systems describe normal and superfluid oscillation modes.
- Neglecting this small coupling ($s_e = s_\mu = s_{str} = 0$), the normal modes coincide with the ordinary modes of nonsuperfluid star.
- In this approximation superfluid modes can be calculated by using only two "superfluid" equations (17) and (18) instead of the full system of hydrodynamic equations. These modes do not perturb metric, pressure, baryon current density and are localized in superfluid region of a star.
- An efficiency of this decoupling scheme is illustrated by calculation of the sound waves in superfluid hyperon matter at arbitrary temperature, which gives qualitatively correct results.
- We also calculated the frequencies of r–modes in hyperon stars in both coupled and fully decoupled cases. We found that 2 classes of r-modes exist: superfluid and normal r–modes. We showed that their frequencies are both equal to $\omega_r = \frac{2m\Omega}{l(l+1)}$ up to the terms $\sim (\Omega/\Omega_K)^3$, where Ω is the rotation stellar frequency, Ω_K is the Kepler frequency, and m and l are the spherical harmonic indices.

References

[1] M. E. Gusakov and E. M. Kantor. Decoupling of superfluid and normal modes in pulsating neutron stars. Phys. Rev. D, 83(8):081304, apr 2011.

[2] L. Lindblom and G. Mendell. The oscillations of superfluid neutron stars. ApJ, 421:689–704, February 1994.

[3] M. E. Gusakov, P. Haensel, and E. M. Kantor. Physics input for modelling superfluid neutron stars with hyperon cores. MNRAS, 439:318–333, March 2014.

[4] M. E. Gusakov and N. Andersson. Temperature-dependent pulsations of superfluid neutron stars. MNRAS, 372:1776–1790, nov 2006.

This work was partially supported by RFBR (grants 14-02-00868-a and 14-02-31616-mol-a), by RF president programme (grants MK-506.2014.2 and NSh-294.2014.2) and by Dynasty Foundation.