Differential rotation of neutron star polar caps

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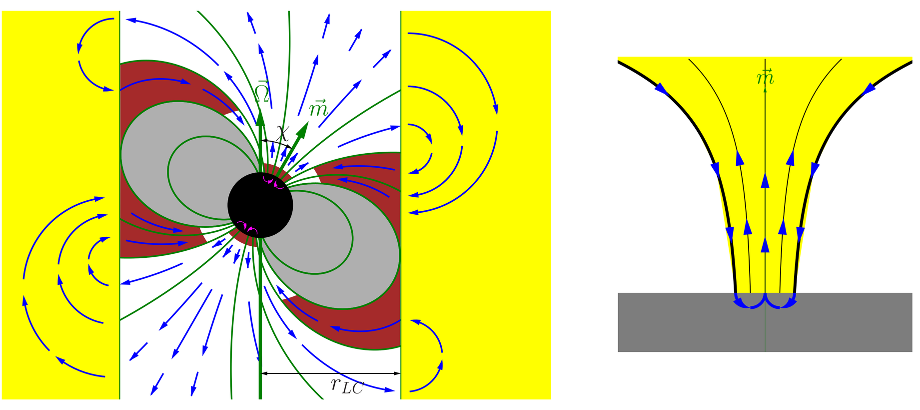
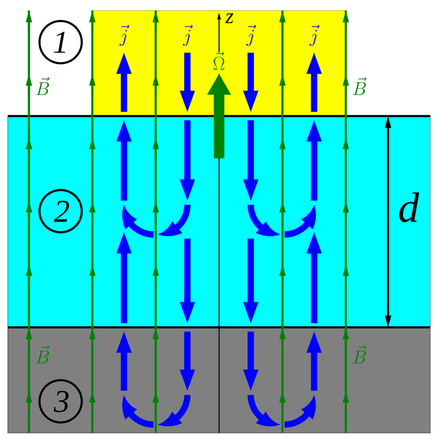
Rotating neutron stars with electric current flowing from magnetosphere into the liquid polar magnetic caps is considered. In the case of simple axisymmetric configuration the current has the one direction in central part of the polar cap and oposite direction in outer ring. It is shown that the closure of electric current under the surface leads to the differential rotation of the polar caps. Velocity structure is determined by the current distribution on the star surface. There are two rings rotating in opposite directions. The inner ring rotates always slower than main rotation of the star. The velocity value is proportional to the gradient of the electric current with radius. The velocity is $10^{-2} - 10^{-4}$ cm/sec (i.e. the period of rotation is 0.1 -10 years) for typical current changing scale 1 - 10^2 cm, the main rotation period $P = 1$ sec and magnetic field $B = 10^{12}$ Gauss.

Electric currents in pulsar magnetosphere



Electric currents nearby the polar cap



1. Magnetosphere $z > 0$
   electric current $\vec{j}$ flows along field lines inside the pulsar tube, shown by yellow area
2. Liquid ocean $-d < z < 0$
3. Rigid crust $z < -d$
   rigdly rotates with angular velocity
   $\vec{\Omega} = \Omega\vec{e}_z$

Magnetohydrodynamics equations

The flow is assumed to be stationary and axisymmetric

$$(\vec{U}\cdot\nabla)\vec{U} = -\frac{1}{\rho}\nabla P + \nu\Delta\vec{U} - \frac{1}{4\pi\rho}\left(\vec{B}\times \mathrm{curl}\vec{B}\right) + \vec{g},$$
$$\mathrm{curl}\left(\vec{U}\times\vec{B}\right) + \eta\Delta\vec{B} = 0,$$
$$\mathrm{div}\vec{U} = 0, \qquad \mathrm{div}\vec{B} = 0,$$

where $\vec{U}$ is the velocity of liquid, P is the pressure, $\vec{B}$ is the magnetic field strenght, $\eta = c^2/(4\pi\sigma)$ is the magnetic diffusion coefficient, σ is the electric conductivity, $\rho\nu$ is the shear viscosity coefficient, ρ is the mass density, $\vec{g}$ is the gravitional force per unit mass.

$$\rho = const, \qquad \eta = const, \qquad \nu = const$$

Linearization

1. zeroth order approximation

$$\vec{U} = \Omega r\vec{e}_\phi, \qquad \vec{B} = B\vec{e}_z, \qquad \vec{j} = 0, \qquad P = P(r,z) = \rho\left(-gz + \frac{\Omega^2 r^2}{2}\right)$$

$$\Omega = const \qquad \text{and} \qquad B = const$$

2. small perturbation caused by electric current $\vec{j}$

$$\vec{U} = \Omega r\vec{e}_\phi + \vec{u}, \qquad \vec{B} = B\vec{e}_z + \vec{b}, \qquad P = P(r,z) + p,$$

Dimensionless values

$$R \to dR,\ z \to dz,\ \vec{b} \to B\vec{b},\ \vec{u} \to \frac{\eta}{d}\vec{u},\ p \to \frac{\rho\nu\eta}{d^2}p$$

$$\mathrm{Re} = \frac{\Omega d^2}{\nu}, \qquad \mathrm{Ha} = \frac{Bd}{(4\pi\rho\nu\eta)^{1/2}},$$

If we assume
$\Omega \sim 10 rad/s$, $B \sim 10^{12}G$, $\rho \sim 10^4 g/cm^3$, $T \sim 10^7 K$,
$d \sim 3\cdot 10^3 cm$ [1], $\sigma \sim 10^{18}s^{-1}$ [2], $\nu \sim 10^{-2}cm^2/s$ [3]
then

$$\mathrm{Re} \sim 10^{10} \quad \text{and} \quad \mathrm{Ha} \sim 10^{13}$$

Dimensionless equations

$$-2\mathrm{Re}u_\phi = -\frac{\partial p}{\partial r} + L_1(u_r) + \frac{\partial^2 u_r}{\partial z^2} - \mathrm{Ha}^2\left(\frac{\partial b_z}{\partial r} - \frac{\partial b_r}{\partial z}\right),$$
$$2\mathrm{Re}u_r = L_1(u_\phi) + \frac{\partial^2 u_\phi}{\partial z^2} + \mathrm{Ha}^2\frac{\partial b_\phi}{\partial z},$$
$$\frac{\partial p}{\partial z} = L_0(u_z) + \frac{\partial^2 u_z}{\partial z^2}, \qquad -\frac{\partial u_z}{\partial z} = L_0(b_z) + \frac{\partial^2 b_z}{\partial z^2},$$
$$-\frac{\partial u_r}{\partial z} = L_1(b_r) + \frac{\partial^2 b_r}{\partial z^2}, \qquad -\frac{\partial u_\phi}{\partial z} = L_1(b_\phi) + \frac{\partial^2 b_\phi}{\partial z^2},$$
$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0, \qquad \frac{1}{r}\frac{\partial}{\partial r}(rb_r) + \frac{\partial b_z}{\partial z} = 0,$$

where $L_0 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$, $L_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}$

Hankel transforms

u_z, b_z and p can be represented as

$$f(r,z) = \int_0^{+\infty} kf^0(k,z)J_0(kr)dk, \quad f^0(k,z) = \int_0^{+\infty} rf(r,z)J_0(kr)dr,$$

u_r, u_ϕ and b_ϕ can be represented as

$$f(r,z) = \int_0^{+\infty} kf^1(k,z)J_1(kr)dk, \quad f^1(k,z) = \int_0^{+\infty} rf(r,z)J_1(kr)dr.$$

Transformed equations

$$-2\mathrm{Re}u_\phi^1 = kp^0 + \frac{\partial^2 u_r^1}{\partial z^2} - k^2u_r^1 + \mathrm{Ha}^2\left(kb_z^0 + \frac{\partial b_r^1}{\partial z}\right),$$
$$2\mathrm{Re}u_r^1 = \frac{\partial^2 u_\phi^1}{\partial z^2} - k^2u_\phi^1 + \mathrm{Ha}^2\frac{\partial b_\phi^1}{\partial z},$$
$$\frac{\partial p^0}{\partial z} = \frac{\partial^2 u_z^0}{\partial z^2} - k^2u_z^0, \qquad -\frac{\partial u_z^0}{\partial z} = \frac{\partial^2 b_z^0}{\partial z^2} - k^2b_z^0,$$
$$-\frac{\partial u_r^1}{\partial z} = \frac{\partial^2 b_r^1}{\partial z^2} - k^2b_r^1, \qquad -\frac{\partial u_\phi^1}{\partial z} = \frac{\partial^2 b_\phi^1}{\partial z^2} - k^2b_\phi^1,$$
$$ku_r^1 + \frac{\partial u_z^0}{\partial z} = 0, \qquad kb_r^1 + \frac{\partial b_z^0}{\partial z} = 0.$$

Characteristic equation

The solution is sought in the following form

$$f(z,k) = f(k)\exp(-sz)$$

where s is the root of equation

$$(s^2-1)[(s^2-k^2)^2 - \mathrm{Ha}^2s^2)^2 + 4\mathrm{Re}^2s^2(s^2-k^2)] = 0.$$

This equation is of the 5-th order with respect to s^2

Boundary conditions

Boudary conditions at $z = 0$

$$\frac{\partial u_r^1}{\partial z} = 0, \quad \frac{\partial u_\phi^1}{\partial z} = 0 \text{ and } u_z^0 = 0$$

$$b_\phi^1 = b_0(k) \quad \text{and} \quad \frac{\partial b_z^0}{\partial z} + kb_z^0 = 0,$$

where $b_0(k)$ is determined by electric current flowing along pulsar tube at $z > 0$

$$j_z(r,z) = \frac{1}{r}\frac{\partial}{\partial r}(rb_\phi) \Rightarrow b_\phi^1(k,z) = \frac{j_z^0(k,z)}{k} \quad \text{and} \quad b_0(k) = b_\phi^1(k,0) = \frac{j_z^0(k,0)}{k}.$$

Boudary conditions at $z = -1$

$$u_r^1 = u_\phi^1 = u_z^0 = 0$$

$$\frac{\partial b_z^0}{\partial z} - kb_z^0 = 0 \quad \text{and} \quad \frac{\partial b_\phi^1}{\partial z} - kb_\phi^1 = 0.$$

we assume that electric conductivity of the crust is constant and coincides with conductivity of the liquid.

Approximate solution

$$\mathrm{Ha} \gg 1 \text{ and } \frac{\mathrm{Re}^2}{\mathrm{Ha}^2} \ll 1$$

After neglecting $2\mathrm{Re}u_r^1$ term
the system of equations is reduced to

$$-\mathrm{Ha}^2\frac{\partial b_\phi^1}{\partial z} = \left(\frac{\partial^2 u_\phi^1}{\partial z^2} - k^2u_\phi^1\right) \quad \text{and} \quad -\frac{\partial u_\phi^1}{\partial z} = \left(\frac{\partial^2 b_\phi^1}{\partial z^2} - k^2b_\phi^1\right).$$

and characteristic equation may be rewritten as

$$(s^2-k^2)^2 - \mathrm{Ha}^2s^2 = 0$$

$$s_1 = -s_2, \quad s_3 = -s_4 = \frac{k^2}{s_1} \text{ where } s_1 = \sqrt{\frac{\mathrm{Ha}^2 + 2k^2 + \mathrm{Ha}(\mathrm{Ha}^2+4k^2)^{1/2}}{2}}$$

$$s_1 = -s_2 \approx \mathrm{Ha}\left(1 + \frac{k^2}{\mathrm{Ha}^2}\right) \text{ and } s_3 = -s_4 \approx \frac{k^2}{s_1} = \frac{k^2}{\mathrm{Ha}} \text{ at } \mathrm{Ha}^2 \gg k^2$$

$$b_\phi^1 = C_{b\phi}^{(1)}e^{s_1z} + C_{b\phi}^{(2)}e^{s_2z} + C_{b\phi}^{(3)}e^{s_3z} + C_{b\phi}^{(4)}e^{s_4z},$$
$$u_\phi^1 = C_{u\phi}^{(1)}e^{s_1z} + C_{u\phi}^{(2)}e^{s_2z} + C_{u\phi}^{(3)}e^{s_3z} + C_{u\phi}^{(4)}e^{s_4z},$$

$$C_{u\phi}^{(i)} = -\frac{s_i\mathrm{Ha}}{s_i^2-k^2}C_{b\phi}^{(i)} \text{ and } C_{u\phi}^{(i)} = \frac{k^2-s_i^2}{s_i}C_{b\phi}^{(i)}.$$

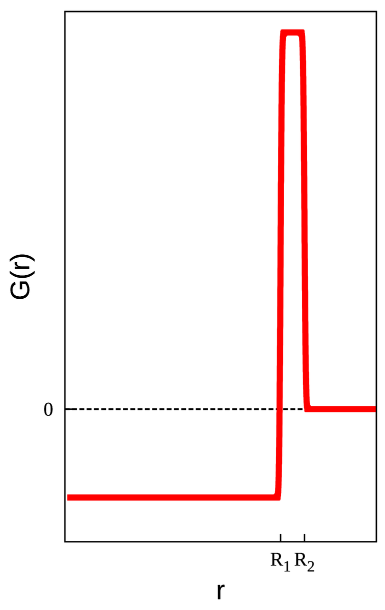
$$C_{u\phi}^{(i)} = -\mathrm{Ha}C_{b\phi}^{(i)} \text{ at } i = 1 \text{ and } i = 4$$
$$C_{u\phi}^{(i)} = \mathrm{Ha}C_{b\phi}^{(i)}, \text{ at } i = 2 \text{ and } i = 3$$

$$u_\phi^1 = \mathrm{Ha}b_0(k)\left\{\frac{k}{\mathrm{Ha}}\left[1 + k(1+z) - \frac{k}{\mathrm{Ha}}\right] - \frac{k^2}{\mathrm{Ha}^2}\left(1 + \frac{k^2}{\mathrm{Ha}}z\right)e^{\mathrm{Ha}z}\right.$$
$$\left. - \frac{k}{\mathrm{Ha}}\left[1 - \frac{k}{\mathrm{Ha}}(1 + (1+z)k) - \frac{k^2}{\mathrm{Ha}^2}\left(2 - k(1+k)z - \frac{k^2z^2}{2}\right)\right]e^{-\mathrm{Ha}(1+z)}\right\},$$
$$b_\phi^1 = b_0(k)\left\{1 - \frac{k^2}{\mathrm{Ha}^2}\left[1 - k(1+k)z - \frac{k^2z^2}{2}\right] + \frac{k^2}{\mathrm{Ha}^2}\left[1 + \frac{k^2}{\mathrm{Ha}}z\right]e^{\mathrm{Ha}z}\right.$$
$$\left. - \frac{k}{\mathrm{Ha}}\left[1 - \frac{k}{\mathrm{Ha}}(1 + k(1+z)) - \frac{k^2}{\mathrm{Ha}^2}\left(2 - k(1+k)z - \frac{k^2z^2}{2}\right)\right]e^{-\mathrm{Ha}(1+z)}\right\}.$$

$$u_\phi^1(0) = b_0(k)k(1+k) \text{ at } z = 0$$

$$u_r^1(k,z) = \mathrm{Re}b_0(k)\frac{k}{\mathrm{Ha}}\left\{(1+2z)\frac{k}{\mathrm{Ha}} - \left[1 + z - \frac{k}{\mathrm{Ha}} - (1+z)^2\frac{k^2}{\mathrm{Ha}}\right]e^{-(z+1)\mathrm{Ha}} + \right.$$
$$\left. + \frac{k}{\mathrm{Ha}}\left[z - \frac{3}{\mathrm{Ha}} + \frac{k^2z^2}{\mathrm{Ha}}\right]e^{\mathrm{Ha}z}\right\}$$

Tube current profile model



$j_z(r,0) = j_0G(r)$ and $j_z^0(k,0) = j_0G^0(k)$

where $G(r) = \int_0^\infty kG^0(k)J_0(kr)dk$

$$G^0(k) = \frac{R_1^2R_2^2}{R_2^2-R_1^2}\frac{1}{k}\left[\frac{1}{R_2}J_1(kR_2) - \frac{1}{R_1}J_1(kR_1)\right]e^{-\beta k^2}$$

$$G(r) = \frac{1}{R_2^2-R_1^2}\left[R_1^2\theta(R_2-r) - R_2^2\theta(R_1-r)\right] \text{ at } \beta \to 0$$

$u_\phi(r,0) \approx j_0F(r)$ and $u_\phi^1(r,0) \approx j_0F^1(k)$

where $F^1(k) = k(1+k)G^0(k)$

$$F(r) = \int_0^\infty k(1+k)G^0(k)J_1(kr)dk$$

This equations depend on three dimensionless parameters

1. $R_1 = \frac{R_{pc}}{d}$ is the ratio of polar cap radius to the ocean depth.
   We take $R_1 = 1, 3, 10$.
2. $\delta = \frac{R_2-R_1}{R_1}$ is relative width of returning current layer.
   We assume that this layer is very thin $\delta = 10^{-1}$, 10^{-2}, 10^{-3}.
3. β is the current profile sharpness.
   $1/\sqrt{\beta}$ is the scale at which the current changes.
   We assume that $1/\sqrt{\beta} \sim (R_2 - R_1)$.

Current and velocity profiles

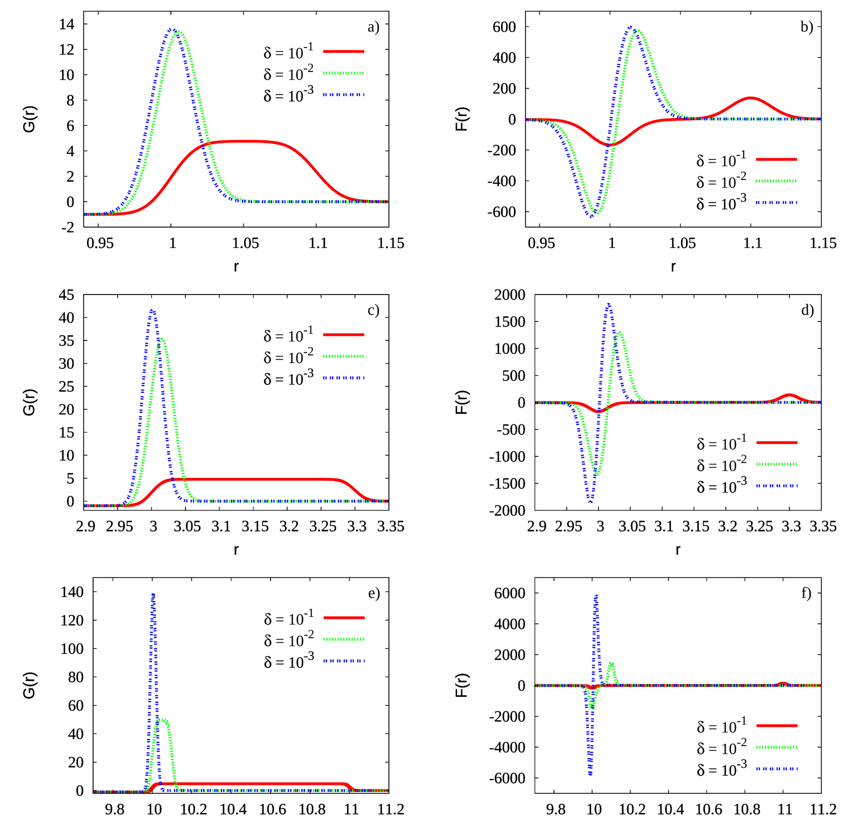


Fig.1 Functions $G(r)$ and $F(r)$ with $\beta = 10^{-4}$ and $R_1 = 1$ (a and b graphs), $R_1 = 3$ (c and d graphs), $R_1 = 10$ (e and f graphs).

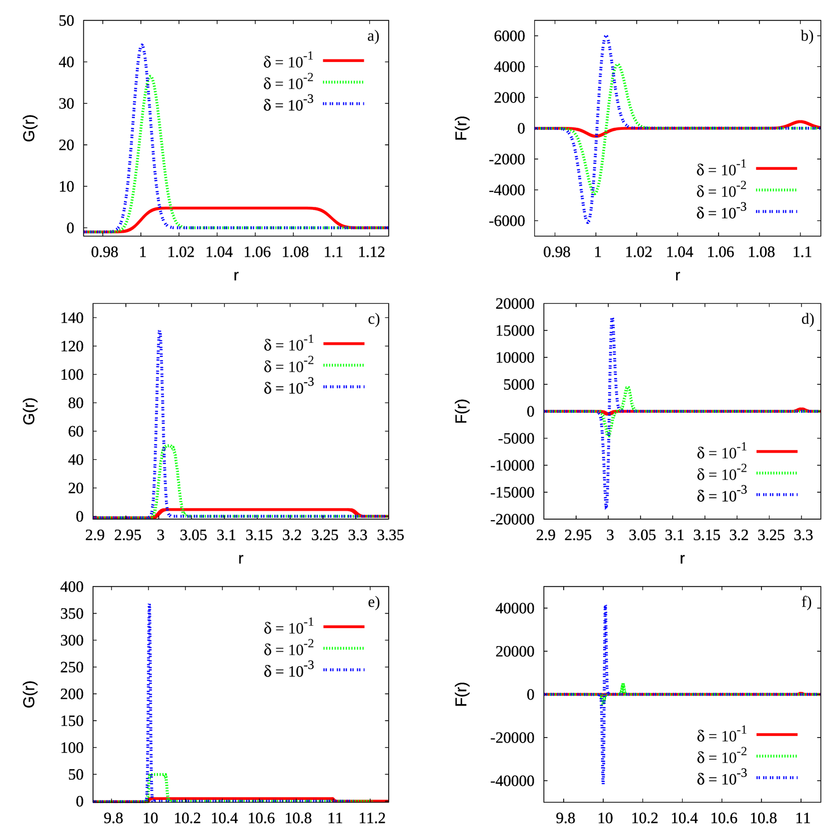
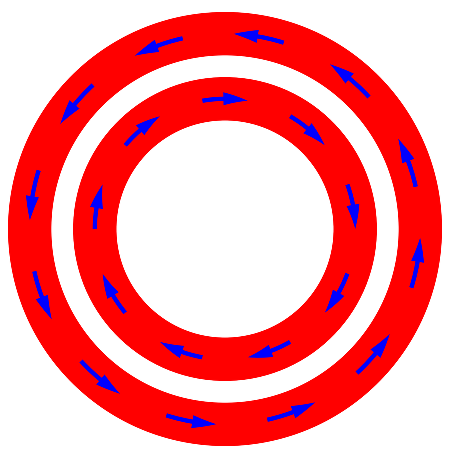


Fig.2 The same as in Fig.1, but with $\beta = 10^{-5}$

Let $j_0 = \frac{\Omega B}{2\pi}$ is Goldreich-Julian current density

$$u_\phi(r) = u_0\cdot F(r) = \frac{2\Omega\eta}{c}\cdot F(r).$$

where $u_0 = \frac{2\Omega\eta}{c} \sim 10^{-7}cm/s$.



$F(r)$ is large only inside two thin rings which correspond to rapid change of current density at $r = R_1$ and $r = R_2$. The sharper current profiles lead to larger function $F(r)$ values. So the decreasing of δ or β (if $\beta \lesssim \delta^2$) increases the $F(r)$ values. If pulsar tube electric current changes at scale $3-30$ cm then $F(r) \sim 10^5$. Hence, u_ϕ may achive $10^{-2}cm/sec$. It corresponds to one turn per 0.1-1 year.

Acknowledgments

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