On some properties of hybrid stars

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Physics of neutron stars - 2014

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Possible phase transitions in the matter at high density

Maxwellian type phase transition causes a density jump inside the star

Adapted from D.G. Yakovlev
Mass-radius diagram for hybrid stars

**Phase transition:**

**Nuclear Matter**

**Quark Matter**

EOS for pure nuclear matter is from Douchin & Haensel Astron. Astrophys., 380 151-167 (2001)

EOS for quark matter according to BAG model is linear: pressure $P_q$ is proportional to internal energy $E_q$. $B$ is a vacuum constant (model parameter)

$$P_q = \frac{1}{3} (E_q - 4B)$$

Density of phase transition $\rho_{pt}$ is connected with $B$: $\rho_{pt}/\rho_n \approx -3 + \ln [B - 91]$ $\rho_n = 2.6 \times 10^{14}$ g/ccm

Thick black line – Mass-Radius relation for pure nuclear matter EOS

Special “point” on Mass-Radius diagram!
The are other examples of special point on M-R diagram!

Schramm et al arXiv:1310.5804
What is the structure of the stars at special point?

The stars have different quark cores but very similar envelope.
The conditions for the envelope to remain unchanged at varying B:

**OVT-equations of equilibrium**

\[
\begin{align*}
\frac{dP}{dr} &= G(P + E) \left( m + \frac{4\pi r^3}{c^2} P \right) \\
\frac{dm}{dr} &= 4\pi r^2 \frac{E}{c^2}.
\end{align*}
\]

**Phase equilibrium**

\[
\begin{align*}
\mu_{pt} &= \frac{P_1 + E_1}{n_1} = \frac{P_2 + E_2}{n_2} \\
E_2 &= E_2(n_2, B)
\end{align*}
\]

\[
\Delta P = \frac{\delta B}{\lambda - 1} \left( \frac{\partial E}{\partial B} \right), \quad \lambda \equiv \frac{n_2}{n_1}
\]

\[
\Delta P = \frac{\partial P}{\partial r} \delta r + \frac{\partial P}{\partial P_c} \delta P_c + \frac{\partial P}{\partial B} \delta B,
\]

\[
\Delta m = \frac{\partial m}{\partial r} \delta r + \frac{\partial m}{\partial P_c} \delta P_c + \frac{\partial m}{\partial B} \delta B.
\]
The whole set of the equations:

\[
\frac{dP}{dr} \left[ \frac{\lambda - 1}{\lambda} \right] \delta r = \left( \frac{\partial E_2}{\partial B} \right) \delta B ,
\]

\[
- \frac{dP}{dr} \left[ \frac{\lambda - 1}{\lambda} \right] \delta r = \left( \frac{\partial P}{\partial P_c} \right)_{r,B} \delta P_c + \left( \frac{\partial P}{\partial B} \right)_{r,p_c} \delta B ,
\]

\[
- \frac{dm}{dr} \left[ \frac{P_2 + E_2}{E_2} \right] \left[ \frac{\lambda - 1}{\lambda} \right] \delta r = \left( \frac{\partial m}{\partial P_c} \right)_{r,B} \delta P_c + \left( \frac{\partial m}{\partial B} \right)_{r,p_c} \delta B .
\]

All the quantities in main equation refer to quark phase only!

Non-trivial solution
Determinant = 0!

The main equation

\[
\left( \frac{\partial P}{\partial P_c} \right)_{r,B} \left[ \frac{dm}{dr} \left( \frac{P_2 + E_2}{E_2} \right) \left( \frac{\partial E_2}{\partial B} \right) + \left( \frac{\partial m}{\partial B} \right)_{r,p_c} \right] = \left( \frac{\partial m}{\partial P_c} \right)_{r,B} \left[ \left( \frac{\partial E_2}{\partial B} \right) + \left( \frac{\partial P}{\partial B} \right)_{r,p_c} \right] .
\]

This is the main equation. If it fulfilled at some point, the whole mass M and radius R of the star will remain constant with small changes in B. Pay attention that \( \lambda \) dropped out from the main equation! Thus this property does not depend on the parameters of star’s envelope.
Dimensionless form of the Main Equation

For quark phase

\[ P = \alpha (E - E_0), \quad E_0 \equiv 4B, \quad \alpha_q = \frac{1}{3} \]

\[ \rho = \frac{E}{E_0}, \quad x = \frac{r}{r_{\text{dim}}}, \quad \mu = \frac{m}{m_{\text{dim}}} \]

\[ r_{\text{dim}} = \frac{c^2}{\sqrt{4\pi GE_0}}, \quad m_{\text{dim}} = \frac{c^4}{G\sqrt{4\pi GE_0}}. \]

Equilibrium equations:

\[ \begin{cases} 
\alpha \frac{d\rho}{dx} = -\left[ \rho + \alpha (\rho - 1) \right] \frac{\mu + \alpha x^3 (\rho - 1)}{x(x - 2\mu)}, \\
\frac{d\mu}{dx} = x^2 \rho. 
\end{cases} \]

This is the main equation in dimensionless variables:

\[ \left( \frac{\partial \mu}{\partial \rho_c} \right)_x \left[ \rho - \frac{\alpha}{1 + \alpha} + \frac{x}{2} \frac{d\rho}{dx} \right] \frac{d\rho}{dx} = x^2 \left( \frac{\partial \rho}{\partial \rho_c} \right)_x \left[ \rho - \frac{\alpha}{1 + \alpha} + \frac{x}{2} \frac{d\rho}{dx} \left( \rho - \frac{\mu}{x^3} \right) \right] \]
The way to solve main equation:

**Homology-invariant variables**

For some specific EOSes **TWO** TOV-equations of equilibrium can be combined to **ONE** differential equation with homological variables.

\[
\begin{align*}
\frac{dx}{du} &= f_u(u,v), \\
\frac{dx}{dv} &= f_v(u,v),
\end{align*}
\Rightarrow \quad \frac{dv}{du} = \frac{f_v(u,v)}{f_u(u,v)} = f(u,v).
\]

**Examples:**

**Newtonian limit, polytropes:**

\[
\begin{align*}
\frac{d\ln m}{d\ln r} &= 4\pi r^2 \rho, \\
\frac{d\ln P}{d\ln r} &= \frac{Gm\rho}{r^2},
\end{align*}
\Rightarrow \quad \frac{u}{\nu} \frac{dv}{du} = -\frac{u + \frac{\nu}{1+n} - 1}{u + \frac{\nu}{1+n} - 3}.
\]

**General Relativity, P=qE**

\[
\begin{align*}
\frac{u}{\nu} \frac{dv}{du} &= -1 - \frac{2q}{1+q} (1+3q)u + \frac{q(3+5q)}{1+q} \frac{uv}{1+q} + \frac{2q^2(1-q)}{(1+q)^2} \frac{wv^2}{1+q} , \\
\frac{v}{\nu} \frac{du}{dv} &= \frac{1+q}{3} - \frac{\nu}{1+q} (1+3q) - \frac{q(3+q)}{1+q} uv - \frac{4q^2}{(1+q)^2}uv^2.
\end{align*}
\]

“Almost” homological variables \( u \) and \( v \)

\[
\begin{aligned}
v &= \frac{\mu + \alpha x^3 (\rho - 1)}{x - 2 \mu}, \\
u &= \frac{x^3 [\rho + \alpha (\rho - 1)]}{3 \mu + \alpha x^3 (\rho - 1)}.
\end{aligned}
\]

The structure of real quark stars in homologous variables is indicated by the lines of various colors, almost coincident with the spiral ones.

The arrows indicate the points corresponding to the surface \( (\rho = 1) \) of these stars. The number at the arrow indicates the corresponding dimensionless central density \( \rho_c \).

This is the solution of main equation in variables \((u,v)\)

\[
u_* = \left[ v_*^2 (3 + \alpha) + v_* (3 - \alpha) - 6 \right] \frac{1 + (1 + \alpha)(\rho_* - 1)}{4\alpha^2 (\rho_* - 1)(1 - v_*)(3 - v_*)}
\]
The condition that a small shift from a fixed point bring us to another fixed point

\[ \lambda = \frac{u(3-v)[(1+v)^2(3-v)+8(1-v^2)\alpha-(3-v)^2\alpha^2]}{(1+v)(7v^2-6v+3)+8(1-v^2)(3-v)\alpha-(3-v)^3\alpha^2} \]

\[ \lambda \equiv \frac{n_2}{n_1} \]

Large scale

For a given EOS the point on M-R diagram where the main equation holds we call fixed point. This is a local property.

The reasons for the existence of a special point on the M-R diagram of hybrid stars:

1. The linearity of quark EOS
   \[ P = \alpha(E - E_0) \]
2. The “phase diagram” of quark matter

Global structure of fixed points
Open questions:
Other (non-linear) EOS as the solution of the main equation?
Other topology of fixed-points because different envelope?

Calculations with non-linear quark EOS:

The existence of special point means that the hybrid stars with the same Mass and Radius can have nevertheless absolutely different inner structure depending on EOS.