Amplification of the magnetic field by r-mode instability: Role of the back-reaction

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CFS instability: Instability due to emission of GW waves

Uniform rotation => Minimal energy for given angular momentum *J* $E_{\rm rot}(J); \quad \delta E_{\rm rot} = \Omega \delta J$

Excitation energy $$E_{\rm ex} = E - E_{\rm rot} > 0$$

Gravitational radiation: $$-\frac{\omega}{m} \dot{j}^{\rm GW} = \dot{E}^{\rm GW} < 0$$

[Thorne 1980]

$$\dot{E}^{\rm GW}_{\rm rot} = \Omega \dot{j}^{\rm GW}$$

$$\dot{E}^{\rm GW}_{\rm ex} = \dot{E}^{\rm GW} - \dot{E}^{\rm GW}_{\rm rot} = -\left(\frac{\omega}{m} + \Omega\right) \dot{j}^{\rm GW} = \frac{\omega + m\Omega}{\omega} \dot{E}^{\rm GW}$$

Increase of excitation energy $\dot{E}^{\rm GW}_{\rm ex} > 0$ => instability

CFS instability criteria: $$\omega + m\Omega > 0; \; \omega < 0$$

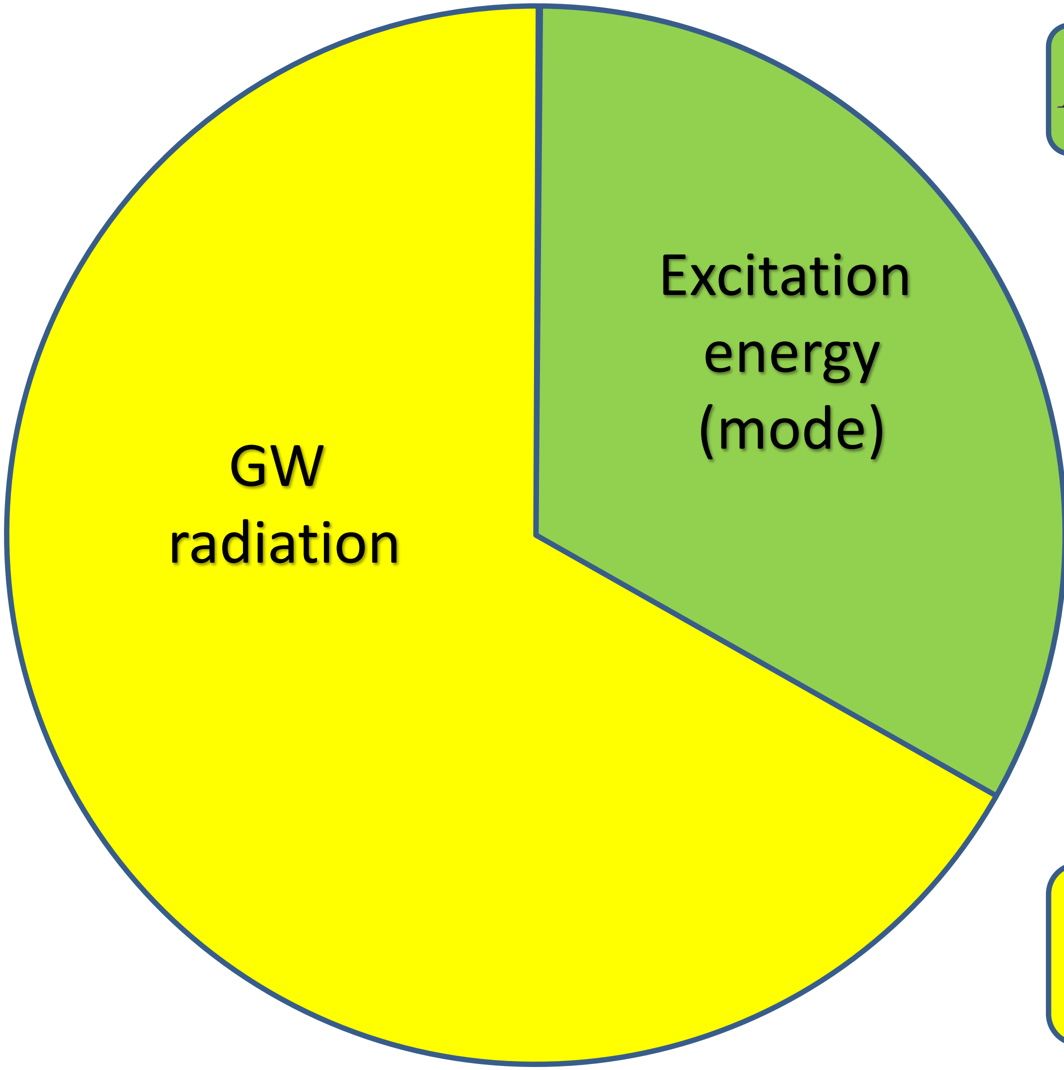
[Friedman&Schutz (1978a,b)]

The energy source is the rotation energy $\dot{E}^{\rm GW}_{\rm ex} + \left|\dot{E}^{\rm GW}\right| = \left|\dot{E}^{\rm GW}_{\rm rot}\right|$

CFS-instability energy budget

Income: decrease of rotational energy, $\left|\dot{E}^{\rm GW}_{\rm rot}\right|$

Expensies: emission of GW radiation $L^{\rm GW}$, enhancement of the mode $\dot{E}^{\rm GW}_{\rm ex}$



$$\dot{E}^{\rm GW}_{\rm ex} = \left(1 + \frac{\omega}{m\Omega}\right) \left|\dot{E}^{\rm GW}_{\rm rot}\right| > 0$$

$$\text{Pertubations} \propto \sin(m\phi + \omega t)$$

$$\omega + m\Omega > 0;\ \omega < 0$$

$$L^{\rm GW} + \dot{E}^{\rm GW}_{\rm ex} = \left|\dot{E}^{\rm GW}_{\rm rot}\right|$$

$$L^{\rm GW} \equiv -\dot{E}^{\rm GW} = -\frac{\omega}{m\Omega} \left|\dot{E}^{\rm GW}_{\rm rot}\right|$$

R-mode instability: key features

m=2 R-modes:
[Provost et al. 1981]

$$\omega = -\frac{4}{3}\Omega \qquad \delta^{(1)}\boldsymbol{v} = \alpha R \Omega \left(\frac{r}{R}\right)^2 \mathbf{Y}_{22}^{B} \exp^{i\omega t}$$

CFS instability criteria:
[Friedman&Schutz (1978a,b)]

$$\omega + m\Omega > 0; \ \omega < 0$$

Instability at any rotation rate (in absence of dissipation)
[Andresson 1998, Friedman&Morsink 1998]

Dissipation suppress r-mode instability at low spin frequency, but it can strongly affect evolution of rapidly rotating NSs (e.g., limit its spin frequency)
[Bildsten 1998, Andresson et al. 1999, Levin 2000…]

<u>Observations vs theory</u>

Observations put strong constraints to the suppression of the r-mode instability, but the mechanism of this suppression is still not well established (e.g., shear viscosity in npe-matter is not enough)
[Haskell et al. 2012, Haskell 2015, Schwenzer et al. 2017, AIC et al. 2017,….]

R-mode instability and magnetic windup

Magnetic damping of r-modes:
[Rezzolla et al. 2000-2001]

- r-modes are likely to be accompanied by differential drift of fluid elements
- Differential drift amplify magnetic field
- Enhancement of magnetic field removes energy from r-modes => r-mode instability might be prevented or suppressed

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Key question of this talk:

Does this mechanism actually work in neutron stars?

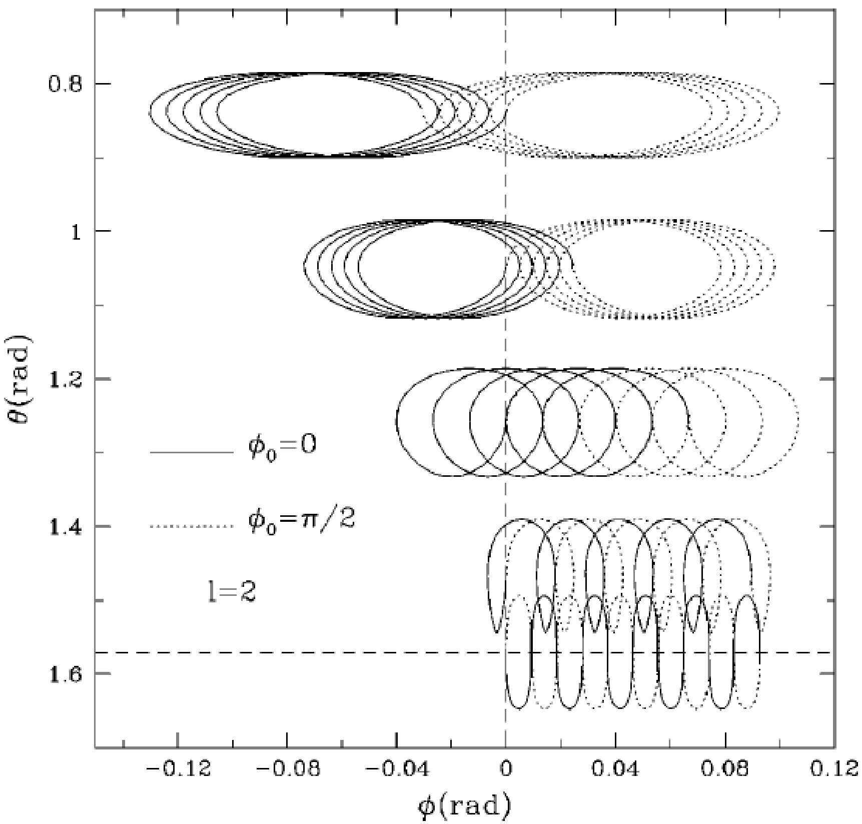
Yes

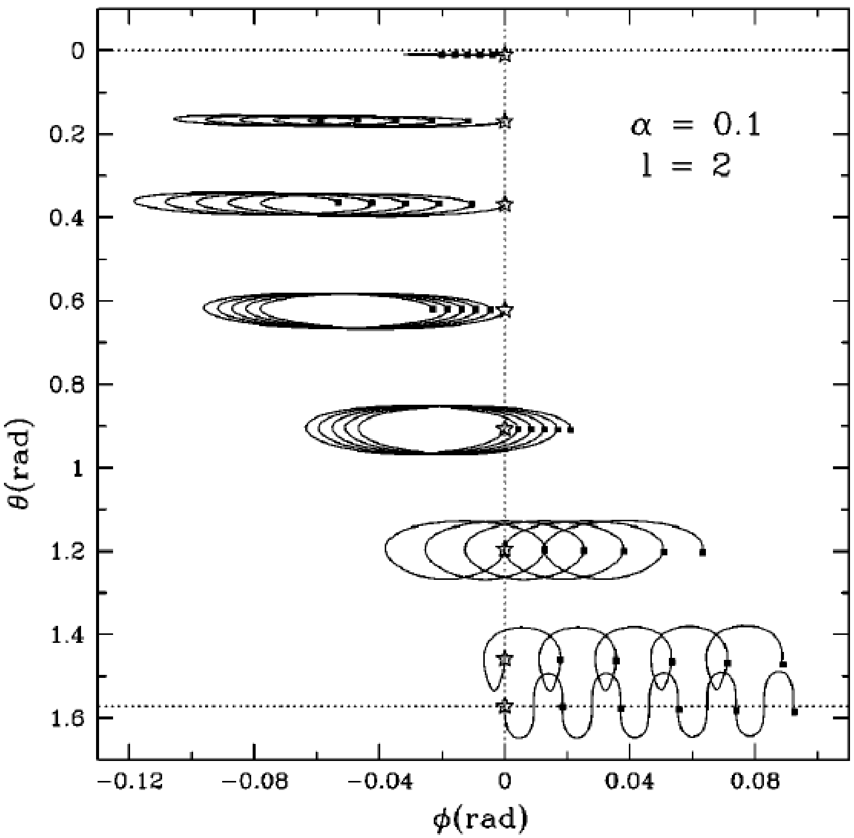
R-mode instability can generate magnetic field, but after that NS should be r-mode stable.
No problems with r-modes, but **no detailed information on NSs**

No

Other mechanism of suppression of the r-mode instability is required, leading thus to **constraints to the (micro)physics of NSs**

Rezzolla et al. (2001a):
Differential drift of fluid elements enhance magnetic field





$$\frac{\partial B}{\partial t} = [\nabla \times [\boldsymbol{v} \times \boldsymbol{B}]]$$

Magnetic field is winded up and enhanced

$$\boldsymbol{v} = \delta^{(1)} \boldsymbol{v}$$

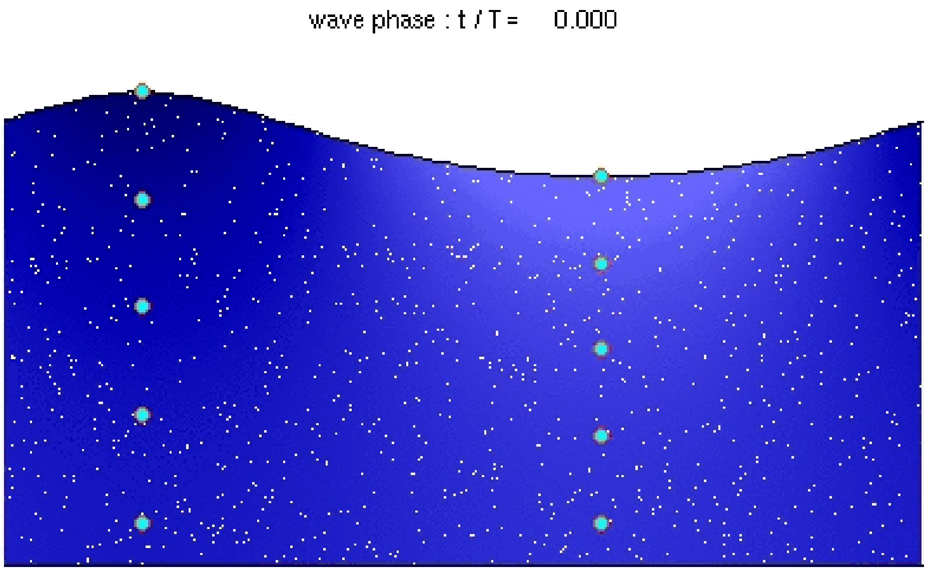
⇒ Secular increase of magnetic field

$$\Delta B \propto \alpha^2 t$$

Stokes drift

*G.G. Stokes (1847). "On the theory of oscillatory waves". Transactions of the Cambridge Philosophical Society **8**: 441–455*

Wikipedia:
For a pure wave motion in fluid dynamics, the **Stokes drift velocity** is the average velocity when following a specific fluid parcel as it travels with the fluid flow.



(from wikipedia)

Main features of Stokes drift:

- Second order in amplitude
- Even if time averaged Eulerian velocity (even at every point) is (exactly) zero, macroscopic drift of fluid elements can take place
- This drift is not correction to the Eulerian velocity field

Stokes drift and Eulerian velocity profile

Second order perturbation theory

Eulerian velocity: $v = \epsilon v_1 + \epsilon^2 v_2 + \ldots; \quad \epsilon \ll 1$

Velocity of the fluid element, which was at coordinate x_0 at the moment t=0:

$$V(x_0, t) = v\left(x_0 + \int_0^t V\mathrm{d}t, t\right) \approx v(x_0, t) + \int_0^t V\mathrm{d}t \cdot \nabla v(x_0, t)$$

$$V = \epsilon V_1 + \epsilon^2 V_2 + \ldots$$

$$\overline{V}_1 = \overline{v}_1; \quad \overline{V}_2 = \overline{v}_2 + \overline{\int_0^t v_1 \mathrm{d}t \cdot \nabla v_1}$$

Stokes drift is NOT the only contribution to secular drift

Longuet-Higgins M. S., Phil. Trans. R. Soc. Long. A, **245** (1953), 535

R-mode instability and (differential) second order rotation

Rotation energy acts as energy source for instability ➡ Physical angular momentum should decrease $\delta J < 0$

$$\boldsymbol{J} = \int \rho[\boldsymbol{r} \times \boldsymbol{v}]\mathrm{d}V \qquad \delta\boldsymbol{J} = \boldsymbol{J} - \boldsymbol{J}_0$$

$$\delta\rho = O(\Omega^2) = o(\Omega) \quad \Longrightarrow \quad \delta^{(i)}\boldsymbol{J} = \int \rho[\boldsymbol{r} \times \delta^{(i)}\boldsymbol{v}]\mathrm{d}V$$

$$\delta^{(1)}v \propto \exp(im\phi) \quad \Longrightarrow \quad \delta^{(1)}\boldsymbol{J} = 0; \quad \delta^{(2)}\boldsymbol{J} = \int \rho[\boldsymbol{r} \times \delta^{(2)}_{sym}\boldsymbol{v}]\mathrm{d}V$$

$\delta^{(2)}_{sym}\boldsymbol{v} \neq 0$ is essential for instability, but should ***drift*** be differential?
(fixed δ*J* does not fix velocity perturbation profile and thus drift profile)

Is differential drift essential for Newtonian r-modes (no GW radiation)?

Newtonian r-modes at the second order in amplitude

[Sa, Phys. Rev. D **69** (2004), 084001]

$$\begin{aligned}
\partial_t \delta^{(1)} v_i + \delta^{(1)} v^k \nabla_k v_i + v^k \nabla_k \delta^{(1)} v_i &= -\nabla_i \delta^{(1)} U, \\
\partial_t \delta^{(1)} \rho + v^i \nabla_i \delta^{(1)} \rho + \nabla_i (\rho \delta^{(1)} v^i) &= 0, \\
\triangle \delta^{(1)} \Phi &= 4\pi G \delta^{(1)} \rho,
\end{aligned}$$

$$\delta^{(i)} U = \delta^{(i)} \left(p/\rho\right) + \delta^{(i)} \Phi$$

$$\begin{aligned}
\partial_t \delta^{(2)} v_i &+ \delta^{(2)} v^k \nabla_k v_i + v^k \nabla_k \delta^{(2)} v_i + \delta^{(1)} v^k \nabla_k \delta^{(1)} v_i \\
&= -\nabla_i \delta^{(2)} U + \frac{\delta^{(1)} \rho}{\rho} \nabla_i \left(\frac{\delta^{(1)} p}{p} \right), \\
\partial_t \delta^{(2)} \rho &+ v^i \nabla_i \delta^{(2)} \rho + \nabla_i (\rho \delta^{(2)} v^i) \\
&+ \nabla_i (\delta^{(1)} \rho \delta^{(1)} v^i) = 0, \\
\triangle \delta^{(2)} \Phi &= 4\pi G \delta^{(2)} \rho.
\end{aligned}$$

Induced by the first order solution

Newtonian r-modes at the second order in amplitude

[Sa, Phys. Rev. D **69** (2004), 084001]

$$
\begin{aligned}
\partial_t \delta^{(2)} v_i &+ \delta^{(2)} v^k \nabla_k v_i + v^k \nabla_k \delta^{(2)} v_i + \delta^{(1)} v^k \nabla_k \delta^{(1)} v_i \\
&= -\nabla_i \delta^{(2)} U + \frac{\delta^{(1)}\rho}{\rho} \nabla_i \left(\frac{\delta^{(1)} p}{p} \right), \\
\partial_t \delta^{(2)} \rho &+ v^i \nabla_i \delta^{(2)} \rho + \nabla_i (\rho \delta^{(2)} v^i) \\
&+ \nabla_i (\delta^{(1)} \rho \delta^{(1)} v^i) = 0, \\
\triangle \delta^{(2)} \Phi &= 4\pi G \delta^{(2)} \rho.
\end{aligned}
$$

Crucial properties

- General second order solution = partial solution + general solution of homogeneous equations (i.e., it is not unique)
- Homogeneous equations are the same as equation for linear perturbations in nonoscillating star

Newtonian r-modes: drift velocity

[Chugunov, MNRAS **451,** 2772–2779 (2015)]

$$(v^{(\mathrm{d})})^{\phi} = \alpha^2 \Omega f(\varpi)$$

$$\varpi = r \sin(\theta)$$

Drift velocity is a solution of linearized equations:
arbitrary (differential) rotation, stratified on cylinders.
Cylindrical layers are decoupled and moves independently.
Differential drift can be absent

Newtonian NS: Drift and oscillations are decoupled

[Chugunov, MNRAS **451,** 2772–2779 (2015)]

Drift and oscillations are independent degrees of freedom:
General second order r-mode solution: superposition of
(a) oscillating solution with *vanishing drift*
(b) drift
Secular evolution of magnetic field is coupled only with drift

$$B = 0$$

Oscillating solution with vanishing drift exists

Drift: Arbitrary stationary motion of cylindrical layers

$$B \neq 0$$

Oscillating solution with vanishing drift is unaffected by *B*, if $B \ll 10^{16}$ G

Drift & magnetic field evolves as in nonoscillating star (Alfven waves + uniform rotation).

R-mode energy is conserved. Magnetic damping is absent

Effect of gravitational radiation-reaction force

In absence of magnetic field: exponentially increasing of mode amplitude and displacement of fluid elements [Friedman et al., 2016]

$$\alpha = \alpha_0 \exp(\beta t)$$

R-mode energy density: $\mathcal{E}_{mode} \sim \rho \, [\alpha \, \Omega \, R]^2$

Drift:

$$\langle \dot{\phi}(t) \rangle \approx -\frac{3}{2}\alpha^2(t)\Omega \left[\frac{\varpi^2}{4} + \Upsilon(\varpi) \right]; \quad \varpi = r \sin(\theta)$$

Drift is cylindrically stratified, but differential.
As far as cylindrical layers are decoupled it is not surprising

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Magnetic field: Couples cylindrical layers at the Alfven timescale

$$\tau_{\rm A} \equiv \omega_{\rm A}^{-1} = R/\sqrt{B^2/(4\pi\rho)}$$

Effect of gravitational radiation-reaction force

Magnetic field: Couples cylindrical layers at the Alfven timescale

$$\omega_{\mathrm{A}}^2 = \frac{B^2}{4\pi\rho\, R^2}$$

Heuristic approach: oscillator under action of external force

$$\ddot{x} + \omega_{\mathrm{A}}^2 x = f, \qquad x = R\xi^{\phi}$$

$$\xi^{\phi}_{\mathrm{max}} = \phi(t) - \phi(0) \sim \frac{|\boldsymbol{f}_{\mathrm{max}}|}{R\,\omega_{\mathrm{A}}^2}$$

External force (per unit mass): axisymmetric part of gravitational radiation-reaction force [Friedman et al., PRD 93 (2016), 024023] + second order magnetic terms [Friedman et al., to be submitted]. It can be estimated as

$$f_{\mathrm{max}} \lesssim \alpha^2 \omega_{\mathrm{A}} \Omega R \quad \Rightarrow \quad \xi^{\phi}_{\mathrm{max}} \lesssim \alpha^2 \frac{\Omega}{\omega_{\mathrm{A}}}$$

Effect of gravitational radiation-reaction force

Magnetic field: Couples cylindrical layers at the Alfven timescale

$$\omega_{\mathrm{A}}^2 = \frac{B^2}{4\pi\rho\, R^2}$$

Displacement: oscillator under action of external force

$$\xi^{\phi}_{\max} \lesssim \alpha^2 \frac{\Omega}{\omega_{\mathrm{A}}}$$

Magnetic field, generated by the drift:

$$\delta B \sim B_0 \xi^{\phi} \lesssim \alpha^2_{\mathrm{sat}} B_0 \frac{\Omega}{\omega_{\mathrm{A}}}$$

Strict mathematical derivation via second order Lagrangian perturbation theory:
Friedman et al. (to be submitted)

Effect of gravitational radiation-reaction force

Magnetic field: Couples cylindrical layers at the Alfven timescale

$$\omega_{\rm A}^2 = \frac{B^2}{4\pi\rho\, R^2}$$

Displacement: oscillator under action of external force

$$\xi^{\phi}_{\rm max} \lesssim \alpha^2 \frac{\Omega}{\omega_A}$$

Energy density of generated magnetic field (=energy of the oscillator)

$$\mathcal{E}_m \sim \rho\omega_{\rm A}^2 (R\xi^{\phi})^2 \sim \alpha^2 \mathcal{E}_{mode} \ll \mathcal{E}_{mode}$$

Mode energy density $\mathcal{E}_{mode} \sim \rho\,[\alpha\,\Omega\,R]^2$

Strict mathematical derivation via second order Lagrangian perturbation theory:
Friedman et al. (to be submitted)

Summary

- The generated magnetic field is strongly limited

$$\delta B \lesssim \alpha_{\rm sat}^2 B_0 \Omega/\omega_A$$

For example, 10^{11} G can be generated from 10^9 G if $\alpha_{sat} \sim 10^{-3}$

- Increase of magnetic energy can not suppress r-mode instability

$$\mathcal{E}_m \sim \alpha^2 \mathcal{E}_{mode} \ll \mathcal{E}_{mode}$$

Take away messages:

- The magnetic damping can not suppress r-mode instability in case of low saturation amplitude
- Mechanism of r-mode instability suppression is required by observations, but it is still not well established. Solution of this problem can put tight constraints to physics of NSs [not limited by mass-radius curve, see, e.g., poster 20 by Kantor et al.].