We consider a strongly coupled Coulomb plasma (SCCP) of ions immersed in a nearly uniform charge-compensating electron gas. The ions may be disordered (liquid phase) or arranged in a crystalline lattice. The energetically favorable body-centered cubic (bcc) lattice appears at \( \Gamma > \Gamma_m = 172 \) [1], where \( \Gamma = (Ze)^2/aT \) is the ion-coupling parameter, \( T \) is the temperature, \( a = (4\pi n_i/3)^{-1/3} \), and \( n_i \) is the ion number density.

Many astrophysical objects (interiors of white dwarfs, massive stars, and giant planets; envelopes of neutron stars) are made of such a plasma. Its kinetic properties required for various applications are determined mainly by electron-ion (ei) scattering. A general framework for calculation of these transport properties has been described in [2]. Numerous calculations (e.g., [3–7]), done under additional assumption of strong electron degeneracy, predict large (a factor of 3–4) discontinuities of the electric and thermal conductivities at the melting point.

In contrast, the thermodynamic quantities in the liquid and solid phases, determined solely by ions, are very similar near \( \Gamma = \Gamma_m \) (e.g., [1,8]). This suggests that properties of the ion system serving as a main scatterer for electrons should vary smoothly through the melting transition. In this Letter, we propose a modification of the transport theory which removes large jumps of the transport coefficients.

The differential \( ei \) scattering rate in a SCCP averaged over initial and summed over final electron spin states \( \sigma \) and \( \sigma' \) is

\[
\Gamma(p \to p') = \frac{2\pi N}{\hbar^2} \sum_{\sigma, \sigma'} |U_{q,\sigma'\sigma}|^2 S(q, \omega),
\]

where \( N \) is the total number of ions, \( p \) and \( p' \) are the electron momenta before and after scattering, respectively, \( \hbar q = p' - p \), \( \hbar \omega = \epsilon' - \epsilon \) is the difference between final and initial electron energies, and \( U_{q,\sigma'\sigma} \) is the matrix element of the operator of elementary \( ei \) interaction. \( S(q, \omega) \) is the dynamical structure factor of the plasma, the most important quantity of the theory. In the liquid regime, \( \hat{\rho}(x, t) \) is the operator of the charge density in units of \( Z[|e|] \): \( \hat{\rho}(x, t) = \hat{n}_I(x, t) - n_i \), where \( \hat{n}_I(x, t) \) is the ion-density operator and \( n_i = n_i/Z \) takes account of the compensating electron background with the electron density \( n_e \). In the solid regime, \( \hat{\rho}(x, t) = \hat{n}_I(x, t) - \sum_i \delta(x - \mathbf{R}_i) \) (where \( \mathbf{R}_i \) is a lattice vector), i.e., the operator of fluctuations of the charge density.

Integrating over \( x \) and \( x' \) we obtain the structure factor of the ion-density fluctuations in the solid phase in the form

\[
NS_{sol}(q, t) = \left\langle \sum_{i,j} e^{i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)} [e^{i\mathbf{q} \cdot \mathbf{u}(t)} - 1] \right\rangle_T \times \left[ e^{-i\mathbf{q} \cdot \mathbf{u}(0)} - 1 \right],
\]

(3)

where \( \mathbf{u}_i \) is an ion displacement from \( \mathbf{R}_i \). Expanding \( \mathbf{u}_i \) in the phonon normal coordinates and using the Weyl operator identity \( e^{Ae^B} = e^{A+e^{[A,B]/2}} \), we can decompose \( S_{sol}(q, t) \) into the elastic (Bragg) and inelastic parts \( S_{sol}(q, t) = S'_{sol}(q) + S''_{sol}(q, t) \). The elastic part is easily calculated (cf. [9]):

\[
S'_{sol}(q) = (1 - e^{-W})(2\pi)^3 n_i \sum_G \delta(q - G),
\]

(4)

where \( G \) is a reciprocal lattice vector, and \( W = W(q) \) is the Debye-Waller factor, \( \exp(-W) = \langle \exp(i\mathbf{q} \cdot \mathbf{u}) \rangle_T \).

\[
W = \frac{\hbar}{2MN} \sum_{\nu} \frac{(\mathbf{q} \cdot \mathbf{e}_\nu)^2}{\omega_\nu} \left( \tilde{n}_\nu + \frac{1}{2} \right).
\]

(5)

In this case, \( M \) is the ion mass, \( \nu \equiv (\mathbf{k}, s), s = 1, 2, 3 \) enumerates phonon modes, \( \mathbf{k} \) is a phonon wave vector,
The polarization vector, $\omega$, the frequency, and $\tilde{n}_v = (e^{\omega t} - 1)^{-1}$ is the mean number of phonons, $z_v = h\omega_v / T$. For the lattice types of interest (e.g., bcc), $W = r_F^2 q^2 / 6$, where $r_F^2 = (u^2) / T$ is the mean-squared ion displacement (cf. [6,9]).

The Bragg scattering of electrons results in the energy band structure of the electron Bloch states but does not contribute to the $ei$ collision integral in the kinetic equation [2]. Indeed, this scattering occurs at the boundaries of the Brillouin zones and translates an electron from one zone to another. The transition requires change of the electron energy by the value of the interband gap; thus another particle must be involved to carry the excess energy.

Therefore only the inelastic part of the structure factor contributes to the collision integral. The inelastic part can be found by the same technique [9]:

$$\begin{align*}
NS_{\text{sol}}(q, t) = e^{-2W} & \sum_{ij} e^{i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)} \sum_{n=1}^{\infty} \frac{1}{n!} \\
& \times \left[ \frac{\hbar}{2MN} \sum_v \frac{(q \cdot e_v)^2}{\omega_v} \\
& \times [\alpha_{ijv}(\tilde{n}_v + 1) + \alpha_{ijv}^* \tilde{n}_v] \right]^n ,
\end{align*}$$

(6)

where $\alpha_{ijv} = \exp[\mathbf{i} \mathbf{k} \cdot (\mathbf{R}_i - \mathbf{R}_j) - i \omega_v t]$. The summation over $i, j$ yields the delta function which removes one summation over $k$ (included in the sums over $v$). Thus we have $n$ sums over $s$ and $n - 1$ sums over $k$ in each nth term of Eq. (6).

Retaining the first term $n = 1$, we recover the one-phonon approximation employed in previous works (e.g., [2–6]). Our point is that this approximation fails near the melting point. In fact, the contribution of the $n$-phonon processes ($nth$ term) at $T$ above the Debye temperature can be estimated as $(q T)^{2n+1} / n! < (k_F r_F)^{2n+1} / n!$, where $k_F^2 = (3\pi^2 n_p)^{1/3}$ is the electron Fermi wave vector, $r_F^2 = u_{-2} a^2 / \Gamma$, $u_{-2} = (\omega_p^2 / \omega_0^2)^{ph} \approx 13$ is a frequency moment for a bcc lattice [8], $\omega_p$ is the ion plasma frequency, and $\langle \cdots \rangle_{\text{ph}}$ denotes averaging over phonon spectrum (e.g., [6]) in the harmonic-lattice approximation.

For instance, for the Fe plasma at the melting point. Indeed, this scattering occurs at the boundaries of the Brillouin zone ($q_B$) in the harmonic-lattice approximation. We have checked that the result is almost the same for face-centered cubic (fcc) and hexagonal close-packed (hcp) lattices.

At this stage we need to specify the matrix element of the elementary $ei$ interaction $U_{q,q',\sigma}$. Assuming the Coulomb potential screened by the static polarization of ideal, relativistic, strongly degenerate ($p = p' = \hbar k_F$) electrons, we obtain

$$\begin{align*}
\frac{2\pi N}{\hbar^2} \sum_{q \sigma} |U_{q,q',\sigma}|^2 &= 2\pi N \frac{16\pi^2 Z^2 e^4}{q^4 |\epsilon(q)|^2} \\
& \times \left( 1 - \frac{\hbar^2 c^2 g^2}{4\epsilon_F^2} \right) ,
\end{align*}$$

(9)

where $\epsilon_F = [m_e^* e^4 + p_F c^2] / 2$ is the electron Fermi energy, $V$ is the normalization volume, and $\epsilon(q)$ is the electron longitudinal static dielectric function [15].

The electric ($\sigma$) and thermal ($\kappa$) conductivities, and shear viscosity ($\eta$) can be written in the form [2]

$$\begin{align*}
\sigma &= \frac{n_e e^2}{m_e^* \nu_\sigma} , & \kappa &= \frac{\pi^2 T n_e}{3m_e^* \nu_\kappa} , & \eta &= \frac{n_e m_e^* v_F^2}{5\nu_\eta} ,
\end{align*}$$

(10)

where $m^*_e = \epsilon_F / c^2$, and

$$\begin{align*}
\nu_{\sigma,\kappa,\eta} &= 4\pi m_e^* Z^2 e^4 L_{\sigma,\kappa,\eta} (n_i / (\hbar k_F))^3
\end{align*}$$

(11)
are the effective collision frequencies. Here, the effective Coulomb logarithms are

\[
L_{\sigma,\kappa} = \int_{q_0}^{2k_F} \frac{dq}{q^4} |e(q)|^2 \left[ 1 - \frac{1}{4} \left( \frac{\hbar q}{m_e c} \right)^2 \right]^2 S_{\sigma,\kappa}(q),
\]

(12)

\[
L_\eta = 3 \int_{q_0}^{2k_F} \frac{dq}{q^4} |e(q)|^2 \left( 1 - \frac{q^2}{4k_F^2} \right) \times \left[ 1 - \frac{1}{4} \left( \frac{\hbar q}{m_e c} \right)^2 \right]^2 S_\eta(q),
\]

(13)

\[ q_0 = 0 \] for the liquid phase and \[ q_0 = q_B \] for the solid phase [6], and \( S_{\sigma,\kappa}(q) \) are the effective static structure factors. In the liquid regime, we approximate \( S_{\sigma,\kappa}(q) \) by \( S_{\text{liq}}(q) \) as described above. In the solid regime, we have

\[
S_\sigma(q) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} dt \, e^{-i\omega t} \frac{z}{1 - e^{-z}} S^\eta_{\text{sol}}(q,t),
\]

(14)

\[
S_\kappa(q) = S_\sigma(q) + \left( \frac{3k_F^2}{q^2} - \frac{1}{2} \right) \delta S_\kappa(q),
\]

(15)

\[
\delta S_\kappa(q) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} dt \, e^{-i\omega t} \times \frac{z^3}{1 - e^{-z}} S^\eta_{\text{sol}}(q,t),
\]

(16)

with \( z = \hbar \omega / T \). The integration over \( \omega \) can be performed analytically. The remaining numerical integration over \( t \) is then facilitated by shifting the integration path in complex plane: \( t = t' - \frac{i\hbar}{2T} t' \), where \( t' \) is real. The final result reads

\[
S_\sigma(q) = \frac{1}{2} \int_{-\infty}^{+\infty} dx \frac{1}{\cosh^2 x} e^{-2W} K(q,T,t'),
\]

(17)

\[
\delta S_\kappa(q) = \int_{-\infty}^{+\infty} dx \frac{2 - 2 \sinh^2 x}{\cosh^4 x} e^{-2W} K(q,T,t'),
\]

(18)

where \( x = \pi t' \hbar / T \) and

\[
K(q,T,t') = \exp \left[ \frac{\hbar q^2}{2M} \cos(\omega_v t') \left( \omega_v \sinh(z_v/2) \right) \right] - 1.
\]

(19)

Retaining the term \( \propto q^2 \) in the expansion of \( K(q,T,t) \), we recover the standard one-phonon approximation [3–6].

Figures 1 and 2 show temperature dependence of the electric and thermal conductivities for carbon plasma at density \( 10^9 \) g cm\(^{-3} \) and for iron plasma at \( 10^9 \) g cm\(^{-3} \), respectively, calculated in the Born approximation. In spite of large differences in densities and chemical elements, the figures are fairly similar. Dashes show the traditional results calculated with the full structure factor \( S_{\text{liq}}(q) \) in ion liquid [3,7] and in the one-phonon approximation [4,6] for bcc crystal (notice that the results of Ref. [6] for fcc crystals are in error; actually, they are very similar to those for bcc). One can see strong jumps of \( \kappa \) and \( \sigma \) at the melting point \( \Gamma_m = 172 \). Solid lines are the present results in the solid phase (including multiphonon processes), while dots show the present results in the liquid obtained using the analytic fits for the static structure factor \( S_{\text{liq}}(q) \) at \( \Gamma = 225 \) [14] by subtracting the long-range correlations (see above). For illustration (as suggested by H. DeWitt), we have extended the improved results in liquid by shifting artificially the melting point to lower \( T \) (to \( \Gamma = 225 \), considering thus supercooled liquid) and the improved results in solid by shifting the melting to higher \( T \) (lower \( \Gamma \), superheated crystal). The curves for liquid and solid ions match one another quite well, and the jumps at the melting point actually disappear. In a wide temperature range the improved curves for liquid and solid almost coincide. We have verified that the same is true for all transport coefficients (including shear viscosity) in a wide range of densities for a number of chemical elements. Thus one can observe that electron transport properties in SCCP of ions appear to be fairly insensitive to the state of SCCP (liquid or solid, bcc or other crystals). This should be taken into account in various astrophysical applications, for instance, in calculating the temperature growth from the surface into interior of the isolated neutron stars important for theories of neutron
FIG. 2. Same as in Fig. 1 but for iron matter at density $10^8 \text{ g cm}^{-3}$.

star cooling [7,16]. The thermal conductivity of neutron-star envelopes to be reconsidered lies exactly in the “sensitivity strip” [16] which strongly affects the temperature profiles and neutron-star cooling.

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