Liquid Crystals in the Mantles of Neutron Stars

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Abstract

Recent calculations indicate that in the outer parts of neutron stars nuclei are rod-like or slab-like, rather than roughly spherical. We consider the elastic properties of these phases, and argue that they behave as liquid crystals, rather than rigid solids. We estimate elastic constants and discuss implications of our results for neutron star behavior.

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In the standard picture [1], the outer part of a neutron star is a solid from the bottom of the ocean of melted iron at a density $\rho \sim 10^6 - 10^8$ g cm^{-3} (depending on temperature) down to the boundary with the core of the star at a density of order the saturation density of nuclear matter, $\rho_s \sim 3 \times 10^{14}$ g cm^{-3}, and matter consists of roughly spherical atomic nuclei arranged on a regular bcc lattice. Charge neutrality is provided by a background of electrons, and, at densities in excess of about 4×10^{11} g cm^{-3}, there is a neutron liquid between the nuclei. However, recent work indicates that, in roughly half of the matter by mass in what is traditionally referred to as the "crust" of a neutron star, the nuclei are very aspherical, and have the form of essentially infinitely long rods or infinitely extended slabs (for a review, see Ref. [2]). Following Ref. [3], we shall refer to these phases as mesomorphic phases, since they have properties intermediate between liquids and solids.

The possibility of nuclei in matter at sub-nuclear densities adopting very non-spherical shapes was first pointed out in the context of stellar collapse by Ravenhall et al. [4], and confirmed by other authors using other models [5–8]. With increasing density, nuclei become first rods, then slabs. This is followed by two "inside-out" phases in which there are cylindrical or spherical "bubbles"

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in the nuclear matter. Subsequently the transition to the uniform liquid phase occurs. For neutron star matter with dripped neutrons, Lorenz et al. [9] found a similar sequence of phase transitions, and these conclusions were confirmed by other investigators [10,11].

Mechanical properties of the outer parts of neutron stars are important in attempting to understand many aspects of neutron star behavior, including glitches in the pulse repetition rate, magnetic field evolution, and neutron star models for γ-ray bursts [12–14]. Our purpose in this Letter is to discuss the elastic properties of the mesomorphic phases and estimate elastic constants.

We shall concentrate on the phases with rod-like and slab-like nuclei, since they are expected to be the dominant ones. According to Ref. [9], these "spaghetti" and "lasagna" phases constitute, respectively, roughly $\frac{1}{5}$ and $\frac{1}{2}$ of the mass lying between the core and the ordinary solid crust, where nuclei are roughly spherical. In Ref. [11] these proportions are found to be about $\frac{1}{2}$ and $\frac{1}{3}$. Thus the fraction of mass in the bubble phases is relatively small.

As a model, we shall assume uniform rods and slabs. For symmetric nuclear matter, Thomas–Fermi calculations [7] have confirmed that these simple configurations are the thermodynamically favorable ones. Clearly there is no increase in energy if rods or slabs are displaced in directions that lie in the plane of the slabs, or along the rods. Consequently there is no restoring force for certain sorts of distortion, and they thus have the elastic properties of liquid crystals. In the "lasagna" and "spaghetti" phases, microscopic calculations indicate that there is positional order in one and two directions, respectively, maintained by the Coulomb repulsion of rods and slabs [4–9]. Accordingly, they conform to the definitions of columnar phases and smectics A [3]. More complex positionally ordered phases (e.g., smectics C, cholesterics) are precluded by the symmetry of the equilibrium shapes of the nuclei. At the temperatures of neutron star interiors ($\sim 10^8$ K) positionally disordered (nematic) phases are unlikely, since one would expect the ordering temperature of rods and plates to be comparable with the melting temperature for matter with spherical nuclei, $\sim 10^{10}$ K. We emphasize, however, that the physical reasons for the spatial structure of the mesomorphic phases in the laboratory and in neutron stars are very different. For laboratory liquid crystals, the non-spherical shape of the molecules drives the tendency to form rod-like and slab-like structures, while in neutron stars, it is a spontaneous symmetry breaking brought about by the competition between the nuclear surface energy and the Coulomb energy. In the neutron star case, the basic objects, nucleons, from which structures are constructed are spherical, while in laboratory liquid crystals, the basic ingredients are non-spherical molecules.

To calculate energies of these phases we adopt a generalized liquid drop model, with bulk, surface and Coulomb energies. In the deformations that we study in this paper we assume that the total density remains constant. Distortion can lead to a redistribution of neutrons between nuclear matter and neutron matter, but this is small because bulk energy densities are large compared with those of surface and Coulomb energies. Therefore we may assume that the fractions of the total volume occupied by nuclear matter and neutron matter, and their local densities remain constant, and consequently only the Coulomb and surface energies are altered.

We begin by estimating the elastic constants for the layered phase. Since there is complete rotational symmetry about the axis perpendicular to the layers, which we denote by Oz, this phase is similar to a smectic A liquid crystal, and the energy density due to deformation may be written in the form [3]

$$E_d = \frac{B}{2}\left[\frac{\partial u}{\partial z} - \frac{1}{2}(\nabla_\perp u)^2\right]^2 + \frac{K_1}{2}(\nabla_\perp^2 u)^2, \tag{1}$$

where u is the displacement of layers in the z direction. The first term is associated with a change of interlayer distance (Fig. 1a). The shear shown in Fig. 1b is equivalent, due to rotational invariance, to reducing the interlayer spacing, an effect taken into account by the second term in the square brackets. The second-order elastic constant K_1 is associated with splay deformations. As a first case, we consider a distortion that maintains the planar character of the slabs but changes the layer spacing, which we denote by $2r_c$. The surface energy per unit volume, E_σ scales as r_c^{-1}, while the Coulomb energy per unit volume, E_C scales as r_c^2, and therefore in equilibrium these energies satisfy the condition $E_{\sigma 0} = 2E_{C0}$ [2,4], which determines r_c. Here the subscript "0" denotes equilibrium values. From the same scaling law we conclude that a small departure from equilibrium associated with a change δr_c of r_c results in an increase of energy

$$\delta E_{\sigma+C} = 3E_{C0}(\delta r_c/r_c)^2. \tag{2}$$

Applying Eqs. (1) to the same deformation, we obtain $E_d = \frac{1}{2}B(\delta r_c/r_c)^2$, whence it follows that

$$B = 6E_{C0}. \tag{3}$$

In order to estimate K_1, let us consider a harmonic perturbation $u(x) = u_0\cos(kx)$, assuming that its wavelength is large: $(kr_c)^2 \ll 1$ and $(ku_0)^2 \ll 1$. Then the deformation energy averaged over x is

$$\langle E_d\rangle = \frac{3}{64}B\,(ku_0)^4 + \frac{1}{4}K_1 k^4 u_0^2. \tag{4}$$

The first term in this expression dominates when $u_0 \gg \lambda$, where $\lambda = (K_1/B)^{1/2}$.

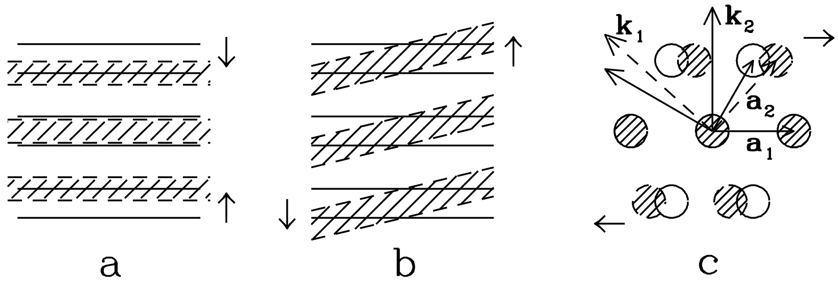

Fig. 1. Linear deformations. Cross sections of the slabs and rods in equilibrium are indicated by full lines, and, after deformation, by the hatched areas. a: compression perpendicular to the slabs (rods); b: shear; c: transverse shear in the columnar phase; the primitive translation vectors ($\mathbf{a}_1$, $\mathbf{a}_2$) and their counterparts in the reciprocal lattice ($\mathbf{k}_1$, $\mathbf{k}_2$) are also shown.

To order k^4, the relative change in the surface energy averaged over x is

$$\frac{\langle \Delta E_\sigma \rangle}{E_{\sigma 0}} = \frac{1}{4}\,(ku_0)^2 - \frac{3}{64}\,(ku_0)^4. \tag{5}$$

If one includes curvature contributions to the surface energy one finds that the first order terms average to zero, while the second order ones give an additional contribution $bk^4u_0^2/2$ to this equation, where b is the ratio of the second-order curvature coefficient to the surface tension. For symmetric nuclear matter, according to the estimate of Bennett and Ravenhall [15], $b \approx -(0.4\ \mathrm{fm})^2$.

It is convenient to express the charge density of nuclei as a Fourier series, and for the distortion given above one finds

$$\rho(x, z) = \sum_{mn} \rho_{mn} \exp(\mathrm{i}m\pi z/r_c + \mathrm{i}nkx). \tag{6}$$

Here $\rho_{mn} = \rho_p\,(m\pi)^{-1}\sin(m\pi r_N/r_c)(-\mathrm{i})^n J_n(m\pi u_0/r_c)$, ρ_p being the charge density of protons inside nuclei, r_N the halfwidth of the slab, and J_n the Bessel function of order n, and the summation is over all integers. The spatial average of the Coulomb energy is

$$\langle E_C \rangle = {\sum_{mn}}' \frac{2\pi\,|\rho_{mn}|^2}{k_{TF}^2 + (m\pi/r_c)^2 + (nk)^2}, \tag{7}$$

where k_{TF} is the Thomas–Fermi screening wavenumber, and the prime on the summation indicates that the term with $m = n = 0$ is excluded. In the absence of distortion ($u_0 = 0$), the screening correction can be neglected in evaluating the Coulomb energy, since $k_{TF}^2 \ll r_c^{-2}$, as argued in Refs. [7,16], and the summation over m in Eq. (7) yields $E_{C0} = 2\pi\,(\rho_p r_N)^2(1-w)^2/3$, in agreement with Ref. [4]. Here w is the fraction of space occupied by nuclear matter, and $w = r_N/r_c$ for the smectic case. From the equilibrium condition $E_{\sigma 0} = 2E_{C0}$ and the fact that the surface energy is given by $E_{\sigma 0} = w\sigma/r_N$, where σ is the surface tension of the interface between nuclear matter and neutron matter, it follows that in equilibrium $r_N = (3w\sigma)^{1/3}(4\pi\rho_p^2(1-w)^2)^{-1/3}$. Thus the Coulomb energy in equilibrium varies as the two thirds power of the surface tension.

Although k_{TF} may be large compared with k, the effects of screening may be neglected also in calculating the change in the Coulomb energy due to distortion, $\langle\Delta E_C\rangle = \langle E_C\rangle - E_{C0}$, because $\rho_{0n} = 0$ for any nonzero value of n. Thus we find

$$\langle\Delta E_C\rangle = (\rho_p\, r_N)^2 \sum_{j\geq 1}(-1)^j a_j(\xi)\,(kr_c/\pi)^{2j}, \tag{8}$$

where

$$a_j(\xi) = \frac{8}{\pi^3 w^2}\sum_{m=1}^{\infty}\frac{\sin^2(m\pi w)}{m^{4+2j}}\sum_{n=1}^{\infty} n^{2j}\, J_n^2(m\xi), \tag{9}$$

with $\xi = \pi u_0/r_c$. In particular, we obtain $a_1(\xi) = \pi\xi^2\,(1-w)^2/3$ and $a_2(\xi) = \pi\xi^2\,(1-w)^2\,[(1+2w-2w^2)\,\pi^2/45 + \xi^2/4]$. The leading Coulomb contribution to the energy due to distortion is $-E_{C0}(ku_0)^2/2$. This exactly cancels the corresponding surface term in Eq. (5), as it must on general grounds. By comparing the sum of the Coulomb (Eq. (8)) and surface (Eq. (5)) contributions to the energy due to distortion with the general expression (4) we find

$$K_1 = 2\,E_{C0}\,(1+2w-2w^2)\,r_c^2/15. \tag{10}$$

The curvature correction to this expression is $4bE_{C0}$, which may be neglected since r_c is typically ~ 10 fm.

In the calculations above we assumed that the thickness of a slab *measured along* Oz remains unperturbed. In reality, the slab will be thinner near the extrema of $u(x)$ and thicker near the extrema of $\partial u/\partial x$. Allowance for this effect, however, yields a contribution to $\langle E_d\rangle$ containing an additional small factor $\sim (k^2 + k_{TF}^2)\, r_c^2$ relative to the terms already considered.

Finally, we remark that, even though at finite temperatures strict long range positional order of layered phases will be destroyed according to the standard Landau-Peierls argument, our theoretical estimates of elastic constants should be a good approximation at low temperatures.

Now we turn to the columnar phase. We assume the rods to have a circular cross section of radius r_N and define r_c by the condition that the density of rods per unit area is $1/\pi r_c^2$. Then the scaling laws that lead to Eq. (2) hold also for this case.

The lowest energy configuration is one with rods on a two-dimensional triangular lattice. We choose Oz to be the axis of the $\mathrm{D_h}$ symmetry. Displacements are then described by a two-dimensional vector $\mathbf{u} = (u_x, u_y)$, and the energy of deformation is

$$
\begin{aligned}
E_d = & \frac{B}{2}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)^2 + \frac{C}{2}\left[\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right)^2 \right. \\
& \left. + \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)^2\right] + \frac{K_3}{2}\left(\frac{\partial^2 \mathbf{u}}{\partial z^2}\right)^2 \\
& + B'\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)\left(\frac{\partial \mathbf{u}}{\partial z}\right)^2 + \frac{B''}{2}\left(\frac{\partial \mathbf{u}}{\partial z}\right)^4 .
\end{aligned} \tag{11}
$$

The first two lines of this formula reproduce Eq. (7.28) of Ref. [3]. The constant B is associated with uniform transverse compression or dilation, C with transverse shear (Fig. 1c), and K_3 with bending.

The third line contains higher-order terms. The last one may be important when amplitudes of longitudinal deformations exceed $(K_3/B'')^{1/2}$. The term with B' provides a non-linear coupling of transverse and longitudinal deformations, which causes, among other effects, a breakdown of linear elasticity and hydrodynamics at large scales (see §§8.3–8.4 of Ref. [3]). In the case of smectics, an analogous coupling is provided by the cross term coming from the expression in square brackets in Eq. (1).

The terms B, B', and B'' stem from the energy increase due to the change of the cross section of the unit cell. A longitudinal shear $\partial u_x/\partial z = \Delta$ accompanied by uniform transverse compression $\partial u_x/\partial x = \partial u_y/\partial y = \Delta'/2$ maintains the triangular lattice but leads to a change of the lattice spacing proportional to $(1+\Delta')^{1/2}/(1+\Delta^2)^{1/4}$. Then Eqs. (2) and (11) yield $B = \frac{3}{2}E_{C0}$, $B' = -\frac{3}{4}E_{C0}$, and $B'' = \frac{3}{8}E_{C0}$.

The two other constants are associated with deformations that are not a simple scaling of the triangular lattice. We estimate them in a way analogous to that for smectics. Imposing a perturbation of the form $\mathbf{u}(z) = \mathbf{u}_0 \cos kz$ and Fourier transforming, we arrive at the expression

$$
\langle E_C \rangle = \sum_{lmn}{}' \frac{2\pi w \rho_p^2 J_n^2(\mathbf{k}_{lm}\cdot\mathbf{u}_0)}{k_{TF}^2 + k_{lm}^2 + (nk)^2}\left(\frac{2J_1(k_{lm}r_N)}{k_{lm}r_N}\right)^2 , \tag{12}
$$

where $\mathbf{k}_{lm} = l\mathbf{k}_1 + m\mathbf{k}_2$ is a reciprocal lattice vector. In the absence of transverse deformation, $\mathbf{k}_1$ and $\mathbf{k}_2$ are vectors of length $(8\pi/\sqrt{3})^{1/2} r_c^{-1}$, with an angle $\pi/3$ between them (Fig. 1c; we shall assume that $\mathbf{k}_2$ is directed along Oy). With $u_0 = 0$, only the term $n = 0$ survives, and neglecting k_{TF}, we recover Eq. (10) of Ref. [6] for E_{C0}. The analytic formula obtained by replacing the hexagonal unit cell by an equivalent cylinder [4],

$$
E_{C0} = (\pi/2)\,(\rho_p r_N)^2 w\,(\ln(1/w) - 1 + w), \tag{13}
$$

turns out to be very accurate at the values of w at which the "spaghetti" phase is expected[1]: it underestimates E_{C0} for the hexagonal cell by less than 0.6% at $w < 0.35$, which is several times smaller than the difference due to inclusion of a realistic screening wavenumber $k_{TF} \approx 0.4\, r_c^{-1}$. Since for the columnar case the surface energy is $E_\sigma = 2w\sigma/r_N$, one finds $r_N = (2\sigma)^{1/3}(\pi\rho_p^2[\ln(1/w) - 1 + w])^{-1/3}$ in equilibrium.

A transverse shear $\partial u_x/\partial y = \Delta$ changes k_{1y} by $(2\pi\sqrt{3})^{1/2}\Delta/r_c$ (Fig. 1c). Numerical summation of the series (12) and identification of the term quadratic in Δ yields the elastic constant C, shown in Fig. 2a. In the relevant range of w, it can be approximated as $\log_{10}(C/E_{C0}) \approx 2.1(w - 0.3)$ (dashed line). (This estimate of C is only a first approximation, since we have kept the cross sections of rods circular, whereas in reality they can adjust their form to the shear).

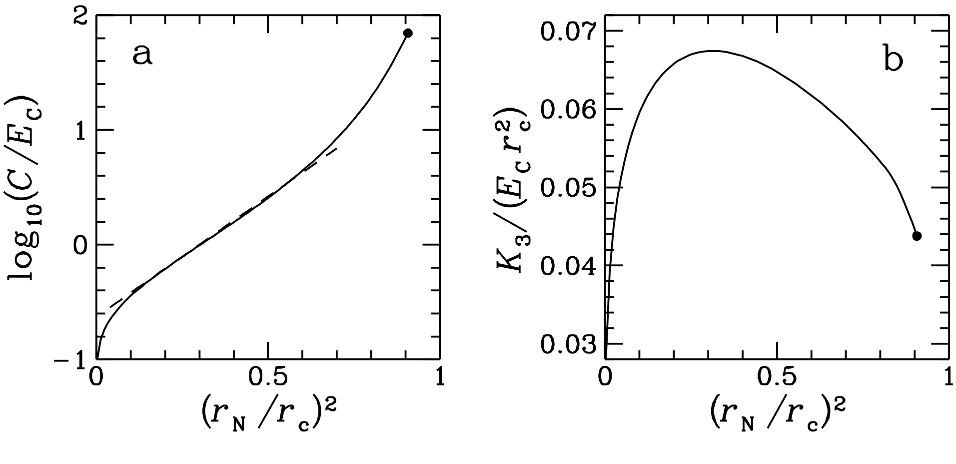

Fig. 2. Shear constant C and bend constant K_3 of the columnar phase.

Finally, using a finite $\mathbf{u}_0$ in Eq. (12), performing the summation and identifying the term proportional to $k^4u_0^2$, we determine the bend constant K_3, which is plotted in Fig. 2b. In the physically relevant range of w, $K_3 = (0.064 - 0.067)\, E_{C0} r_c^2$. The term proportional to $k^4u_0^4$ yields for B'' the value obtained earlier.

The fact that matter in neutron stars near the boundary of the core has the elastic properties of liquid crystals rather than a crystalline solid will have important consequences for a number of aspects of neutron star behavior. First, the maximum elastic energy that can be stored in the crust will be reduced, and this should be taken into account in models of glitch phenomena and starquakes that depend on the deviations of the figure of the star from that

[1] The columnar phase is favored for w between a lower limit of 0.15–0.20 and an upper limit of 0.30–0.35 (and an upper limit of w for the layered phase is about 0.60–0.65) [4–7].

for a fluid (see, for example, Ref. [12]). It is not possible to make a quantitative estimate of the reduction of the elastic rigidity of the outer parts of the star without a detailed model for the orientation of the liquid-crystal phases, but since these phases are estimated to make up roughly half of the matter by mass outside the core, one might expect the effective elastic rigidity to be reduced by roughly a factor of 2. Since the"plate tectonics" of liquid crystals is likely to be very different from that of crystalline solids, a second problem where the elastic properties will be crucial is in models of the evolution of neutron star magnetic fields that invoke such processes [13]. Other applications are to the propagation of elastic distortions, and to the energy of defects in the liquid-crystal structure.

In our discussion above we assumed that in the absence of deformation the spacings of rods and slabs had their equilibrium values. However, in reality, this may not be the case, since to alter the spacing requires a major rearrangement of the proton distribution. Just how easy it is to do this depends on how slabs and rods are connected to each other, and the related question of what defects are present. Should the spacing be smaller than its equilibrium value, there will be a positive contribution to the elastic energy of the form $(E_\sigma/2 - E_C)(\nabla_\perp u)^2$, which gives a restoring force linear in the deformation. Should the spacing be larger, this contribution will be negative, which gives a tendency to rotation and leads to spontaneous deformation analogous to the Helfrich effect in conventional liquid crystals [3]. Eventually the spacing will approach equilibrium, either by spontaneous reconnection of rods and slabs or by motion of defects.

To summarize, the above considerations indicate that matter near the boundary with the core of a neutron star has the elastic properties of a liquid crystal, rather than a conventional solid. To distinguish this region from that further out in the star, we consider it appropriate to refer to it as the "mantle" rather than as part of the "crust". It is clear that the properties of neutron stars need to be reconsidered in light of this new understanding.

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