Diffusion in superfluid Fermi mixtures: General formalism

O. A. Goglichidze [*] and M. E. Gusakov
Ioffe Institute, Politekhnicheskaya 26, 194021 St. Petersburg, Russia

(Received 17 May 2023; accepted 15 August 2023; published 30 August 2023)

With neutron star applications in mind, we developed a theory of diffusion in mixtures of superfluid, strongly interacting Fermi liquids. By employing the Landau theory of Fermi liquids, we determined matrices that relate the currents of different particle species, their momentum densities, and the partial entropy currents to each other. Using these results, and applying the quasiclassical kinetic equation for the Bogoliubov excitations, we derived general expressions for the diffusion coefficients, which properly incorporate all the Fermi-liquid effects and depend on the momentum transfer rates between different particle species. The developed framework can be used as a starting point for systematic calculations of the diffusion coefficients (as well as other kinetic coefficients) in superfluid Fermi mixtures, particularly, in superfluid neutron stars.

DOI: 10.1103/PhysRevC.108.025814

I. INTRODUCTION

The present work aims at developing a framework for studying the transport properties (in particular, diffusion) in superfluid strongly interacting Fermi mixtures and at elucidating the role of the Fermi-liquid effects in shaping these properties. Although this problem is quite general, its solution is of particular relevance to the physics of neutron stars (NSs), and we will always keep NSs in mind when discussing it below.

Neutron stars are compact objects with a mass of about $M \approx 1.4 M_\odot$ and a radius $R \approx 12$ km ($M_\odot$ is the solar mass). The density in their inner layers exceeds the nuclear saturation density, 2.8×10^{14} g cm^{-3}, making these objects unique astrophysical laboratories for studying superdense matter and testing such fundamental physical theories as the theory of strong interactions, general relativity theory, and many-body quantum theory [1]. Despite the fact that NSs were discovered more than 50 years ago, the equation of state and even the composition of matter in their deepest layers are still not well understood. Various theoretical models predict a composition ranging from a purely nucleon one (neutrons, protons, and electrons with admixture of muons) to nucleon-hyperon and quark matter. Laboratory studies of such dense and strongly degenerate matter are not feasible and the only way of testing the theories is to compare NS observations with predictions from theoretical models.

An important and often crucial feature of not too hot NSs is the presence of baryon superfluidity/superconductivity in their interiors (see, e.g., Ref. [2,3]). Superfluidity of baryons (e.g., neutrons and protons in the case of the simplest NS matter composition) has a dramatic impact on the NS dynamics by substantially modifying the (magneto)hydrodynamic equations [4–10] and strongly affecting the dissipative properties of NS matter (see, e.g., the review [11]).

One of the potentially interesting dissipative mechanisms in NSs is associated with particle diffusion. Indeed, since the NS matter consists of a number of different strongly interacting particle species, the departure from the diffusion equilibrium can lead to effective dissipation of the mechanical (and magnetic) energy through the diffusion currents (see, e.g., [12]). The diffusion coefficients in a normal (nonsuperfluid) mixture of Fermi liquids were calculated by Anderson *et al.* [13]. Later, Yakovlev and Shalybkov [14,15], focusing mainly on the related mechanism of the electric conductivity, outlined the derivation of the expression for the diffusion tensor in the presence of the magnetic field. When calculating the momentum transfer rates, these authors adopted the free-particle model [16]. Subsequently, their calculations were improved by taking into account nuclear in-medium effects [17–19]. The diffusion and electrical conductivity were also analyzed for quark matter in Refs. [20,21]. An importance of the diffusion effects was revealed for the evolution of the magnetic field in NSs in Refs. [22–33]. Moreover, recently it was argued that, under certain circumstances, the particle diffusion may become a leading dissipative agent for damping of neutron star oscillations [34].

Most of the works mentioned above were devoted to studying nonsuperfluid Fermi mixtures. Since NS matter is generally superfluid and superconducting, the results obtained in these works are of limited scope. Cooper pairing of particles in a mixture leads to several important effects that can potentially affect diffusion. First of all, superfluidity leads to the appearance of the energy gap in the dispersion relation for the elementary Bogoliubov thermal excitations. An impact of this effect on the diffusion coefficients was analyzed in Ref. [35]. Second, the number of Bogoliubov excitations is not conserved in the collisions (see, e.g., [36]), which significantly complicates all calculations related to particle scatterings. Third, the superconductivity (i.e., the superfluidity of charged particles) noticeably modifies the screening properties of a mixture and affects the electromagnetic interactions between different charged particle species [20,37–39]. All these effects

[*]goglichidze@gmail.com

 ©2023 American Physical Society

substantially complicate the collision integral. Note, however, that in a strongly interacting Fermi mixture, besides the dissipative interaction described by the collision integral, one should also account for the nondissipative interaction between particles, the so called Fermi-liquid effects. To our best knowledge, the influence of these effects on the particle diffusion in superfluid mixtures has not been studied in the literature. Still, the theory of transport processes in such systems cannot be developed in a consistent way without taking Fermi-liquid effects into account. The aim of the present paper is to fill this gap and to introduce a formalism allowing one to calculate the main transport coefficients, in particular, diffusion in strongly interacting superfluid Fermi mixtures within the framework of the Landau Fermi-liquid theory.

The paper is organized as follows. Section II provides basic definitions and notations. It also briefly discusses, following Ref. [40], the method of deriving the superfluid entrainment matrix, as this method shares many common features with the approach adopted in the present work. In Sec. III, the expressions for the normal currents are obtained for a simplified problem ignoring the dissipative interaction between different particle species. In Sec. IV, the same problem is considered in the framework of kinetic theory. The relation between the normal currents and the chemical potential gradients is found, the expressions for the diffusion coefficients are derived, and the entropy generation equation is presented. Section V generalizes the results obtained above to charged mixtures. Section VI presents summary of our results. The paper also contains a number of Appendices. In Appendix A, the various "entrainment matrices" introduced in the paper are given in different limiting cases. Appendices B and C describe, respectively, the general equations of the relativistic superfluid hydrodynamics and the same equations in the limit of small fluid velocities. In Appendix D we discuss the effective interaction Hamiltonian for Bogoliubov excitations. Finally, Appendix E presents the collision integrals for Bogoliubov excitations, as well as the formal derivation of the expressions for the momentum transfer rates.

Throughout this paper we will use the system of units in which the Planck constant $\hbar$, the Boltzmann constant k_B, and the normalization volume V equal unity ($\hbar = k_B = V = 1$). However, we will not set the speed of light c equal to 1, since we will be dealing mostly with the nonrelativistic hydrodynamic velocities.

II. SUPERFLUID CURRENTS IN A FERMI-LIQUID MIXTURE

Let us consider a mixture of two interacting superfluid Fermi liquids which we label by the indices "n" and "p". In spite of the obvious association with neutrons and protons, up to Sec. V, we will assume that both constituents are uncharged fluids. In what follows, the indices α, α', and α'' run over particle species. The index β labels the particle species different from the species α ($\beta \neq \alpha$). We deal only with spin-unpolarized matter. This allows us to disregard the spin dependence of various quantities and treat them as spin-averaged functions whenever possible.

One of the key features of the hydrodynamics and kinetics of superfluid mixtures is the so-called entrainment effect, which manifests itself in the fact that superfluid currents are, generally, not parallel to superfluid velocities [41]. In particular, in the case of a two-component nonrelativistic mixture, the currents can be represented as

$$\mathbf{J}_n = (\rho_n - \rho_{nn} - \rho_{np})\,\mathbf{V}_q + \rho_{nn}\,\mathbf{V}_{sn} + \rho_{np}\,\mathbf{V}_{sp}, \qquad (1)$$

$$\mathbf{J}_p = (\rho_p - \rho_{pp} - \rho_{pn})\,\mathbf{V}_q + \rho_{pp}\,\mathbf{V}_{sp} + \rho_{pn}\,\mathbf{V}_{sn}, \qquad (2)$$

where $\mathbf{J}_\alpha$ is the mass-current density ($\alpha = n, p$), $\mathbf{V}_{s\alpha}$ is the superfluid velocity, $\mathbf{V}_q$ is the velocity of thermal excitations (normal liquid component), and $\rho_{\alpha\alpha'}$ is the so-called Andreev-Bashkin matrix (also known as entrainment or mass-density matrix). The elements of this matrix were calculated for both nonrelativistic and relativistic mixtures in a series of papers (see, e.g., Refs. [40,42–49]). In this section we briefly outline the calculation of the entrainment matrix based on the relativistic Landau Fermi-liquid theory [50], closely following the work of Gusakov *et al.* [40].

In the case of relativistic fluids, it is more convenient to work with the particle current densities $\mathbf{j}_\alpha$ instead of the mass current densities $\mathbf{J}_\alpha$. The former can be represented as [45]

$$\mathbf{j}_\alpha = \left(n_\alpha - \sum_{\alpha'} \mu_{\alpha'} Y_{\alpha\alpha'}\right)\mathbf{u} + c^2 \sum_{\alpha'} Y_{\alpha\alpha'} \mathbf{Q}_{\alpha'}, \qquad (3)$$

where $\mathbf{u}$ is the spatial component of the four-velocity u^μ, normalized by condition $u^\mu u_\mu = -c^2$, and describing the motion of normal liquid component; n_α and μ_α are, respectively, the number density and relativistic chemical potential of particle species α measured in the frame, in which $u^\mu = (c, 0, 0, 0)$. Finally, $\mathbf{Q}_\alpha$ is the half Cooper-pair momentum. In the nonrelativistic limit,

$$\mathbf{u} = \mathbf{V}_q, \qquad \rho_{\alpha\alpha'} = m_\alpha m_{\alpha'} c^2 Y_{\alpha\alpha'}, \qquad (4)$$

where m_α is the bare mass of particle species α.

A. Basic definitions

Throughout the paper we assume that $Q_\alpha/p_{F\alpha} \ll 1$, as well as that $Q_\alpha/m_\alpha \ll c$, where $p_{F\alpha}$ is the Fermi momentum for particle species α. In this case, the energy density E of the system can be represented as [40,51]

$$E - \sum_\alpha \breve{\mu}_\alpha n_\alpha = \sum_{\mathbf{p}\sigma\alpha} \left(\varepsilon_0^{(\alpha)}(\mathbf{p} + \mathbf{Q}_\alpha) - \breve{\mu}_\alpha\right)\left(\mathcal{N}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} - \theta^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}\right) + \frac{1}{2}\sum_{\mathbf{p}\mathbf{p}'\sigma\sigma'\alpha\alpha'} f^{\alpha\alpha'}(\mathbf{p} + \mathbf{Q}_\alpha, \mathbf{p}' + \mathbf{Q}_{\alpha'})\left(\mathcal{N}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} - \theta^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}\right)$$
$$\times \left(\mathcal{N}^{(\alpha')}_{\mathbf{p}'+\mathbf{Q}_{\alpha'}} - \theta^{(\alpha')}_{\mathbf{p}'+\mathbf{Q}_{\alpha'}}\right) - \sum_{\mathbf{p}\alpha} \Delta^{(\alpha)}_{\mathbf{p}} u^{(\alpha)}_{\mathbf{p}} v^{(\alpha)}_{\mathbf{p}}\left(1 - \mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} - \mathcal{F}^{(\alpha)}_{-\mathbf{p}+\mathbf{Q}_\alpha}\right). \qquad (5)$$

Here, $\mathcal{N}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ and $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ are, respectively, the distribution functions for Landau quasiparticles and Bogoliubov excitations of particle species α, $\theta^{(\alpha)}_{\mathbf{p}} = \theta(p_{F_\alpha} - |\mathbf{p}|)$, $\theta(x)$ is the step function, $\varepsilon^{(\alpha)}_0(\mathbf{p})$ is the first variation of the system energy density E with respect to the distribution function $\mathcal{N}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ calculated for the normal Fermi mixture, $f^{\alpha\alpha'}(\mathbf{p}, \mathbf{p}')$ is the second variation, also called the (spin-averaged) Landau quasiparticle interaction function [52], $\breve{\mu}_\alpha$ is the nonequilibrium analog of the chemical potential μ_α (to be specified below), $\Delta^{(\alpha)}_{\mathbf{p}}$ is the Fourier component of the superfluid order parameter, and $u^{(\alpha)}_{\mathbf{p}}$ and $v^{(\alpha)}_{\mathbf{p}}$ are the Bogoliubov coherence factors related by the normalization condition,

$$u^{(\alpha)2}_{\mathbf{p}} + v^{(\alpha)2}_{\mathbf{p}} = 1. \tag{6}$$

The expression (5) formally contains summations over the spin indices σ and σ'. However, since the matter is assumed to be spin unpolarized, we omit these indices in the distribution functions. The quasiparticle distribution function $\mathcal{N}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ is related to the corresponding distribution function for Bogoliubov excitations $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ by the equation

$$\mathcal{N}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = v^{(\alpha)2}_{\mathbf{p}} + u^{(\alpha)2}_{\mathbf{p}}\, \mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} - v^{(\alpha)2}_{\mathbf{p}}\, \mathcal{F}^{(\alpha)}_{-\mathbf{p}+\mathbf{Q}_\alpha}. \tag{7}$$

The Bogoliubov coherence factors can be found by minimizing the quantity $E - \sum_\alpha \breve{\mu}_\alpha n_\alpha$ with respect to $u^{(\alpha)}_{\mathbf{p}}$, taking into account the condition (6) and treating the distribution functions $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ as fixed parameters [51].[1] The result is

$$u^{(\alpha)2}_{\mathbf{p}} = \frac{1}{2}\left(1 + \frac{H^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} + H^{(\alpha)}_{-\mathbf{p}+\mathbf{Q}_\alpha}}{2\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} + H^{(\alpha)}_{-\mathbf{p}+\mathbf{Q}_\alpha} - H^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}}\right), \tag{8}$$

where

$$\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = \frac{\delta\big(E - \sum_\alpha \breve{\mu}_\alpha n_\alpha\big)}{\delta \mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}} = \frac{1}{2}\left(H^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} - H^{(\alpha)}_{-\mathbf{p}+\mathbf{Q}_\alpha}\right) + \sqrt{\frac{1}{4}\big(H^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} + H^{(\alpha)}_{-\mathbf{p}+\mathbf{Q}_\alpha}\big)^2 + \Delta^{(\alpha)2}_{\mathbf{p}}} \tag{9}$$

is the energy of a Bogoliubov excitation, while

$$H^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = \varepsilon^{(\alpha)}_0(\mathbf{p}+\mathbf{Q}_\alpha) - \breve{\mu}_\alpha + \sum_{\mathbf{p}'\sigma'\alpha'} f^{\alpha\alpha'}(\mathbf{p}+\mathbf{Q}_\alpha, \mathbf{p}'+\mathbf{Q}_{\alpha'}) \times \left(\mathcal{N}^{(\alpha')}_{\mathbf{p}'+\mathbf{Q}_{\alpha'}} - \theta^{(\alpha')}_{\mathbf{p}'+\mathbf{Q}_{\alpha'}}\right) \tag{10}$$

is the quantity that formally coincides with the energy of a Landau quasiparticle in a nonsuperfluid matter [43,52]. It is easy to see that the Bogoliubov coherence factors are even in $\mathbf{p}$: $u^{(\alpha)}_{\mathbf{p}} = u^{(\alpha)}_{-\mathbf{p}}$, $v^{(\alpha)}_{\mathbf{p}} = v^{(\alpha)}_{-\mathbf{p}}$. The nonequilibrium chemical potential $\breve{\mu}_\alpha$ is determined from the requirement that, for a given (generally, nonequilibrium) distribution function of Bogoliubov excitations $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$, the summation of the quasiparticle distribution function $\mathcal{N}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ over the quantum states gives the particle number density,

$$n_\alpha = \sum_{\mathbf{p}\sigma} \mathcal{N}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}. \tag{11}$$

In the vicinity of the Fermi surface the absolute values of the arguments of the function $f^{\alpha\alpha'}(\mathbf{p}, \mathbf{p}')$ can be approximately put equal to $p \approx p_{F_\alpha}$ while the function itself can be expanded into Legendre polynomials $P_l(\cos\Theta)$:

$$f^{\alpha\alpha'}(\mathbf{p}, \mathbf{p}') = \sum_l f^{\alpha\alpha'}_l P_l(\cos\Theta), \tag{12}$$

where Θ is the angle between the vectors $\mathbf{p}$ and $\mathbf{p}'$, $f^{\alpha\alpha'}_l$ are the symmetric Landau parameters ($f^{\alpha\alpha'}_l = f^{\alpha'\alpha}_l$). The Landau parameters and the particle effective mass, defined as

$$m^*_\alpha = \frac{p_{F\alpha}}{v_{F\alpha}}, \tag{13}$$

are related by the equation [45]

$$\frac{\mu_\alpha}{m^*_\alpha c^2} = 1 - \sum_{\alpha'} \frac{G_{\alpha\alpha'}\mu_{\alpha'}}{n_\alpha c^2}. \tag{14}$$

In these formulas $v_{F\alpha}$ is the Fermi velocity and the symmetric matrix $G_{\alpha\alpha'}$ is given by

$$G_{\alpha\alpha'} = \frac{1}{9\pi^4} p^2_{F\alpha} p^2_{F\alpha'} f^{\alpha\alpha'}_1. \tag{15}$$

The Fermi momentum $p_{F\alpha}$ is expressed through the particle number density by the standard formula: $p_{F\alpha} = (3\pi^2 n_\alpha)^{1/3}$. The equilibrium distribution function for Bogoliubov excitations can be found from minimization of the thermodynamic potential

$$F = E - \sum_\alpha \mu_\alpha n_\alpha - TS \tag{16}$$

with respect to $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$. Here T is the temperature, S is the system entropy density, and $E - \sum_\alpha \mu_\alpha n_\alpha$ is given by the expression (5), where the chemical potential $\breve{\mu}_\alpha$, taken in equilibrium, is denoted as μ_α. The entropy density S is given by the standard combinatorial expression

$$S = \sum_{\mathbf{p}\sigma\alpha} \sigma^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}, \tag{17}$$

where

$$\sigma^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = -\big(1 - \mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}\big) \ln\big(1 - \mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}\big) - \mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} \ln \mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}. \tag{18}$$

Taking the variation of F and equating the result to zero, one obtains the standard Fermi-Dirac distribution function [45]

$$\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = \frac{1}{1 + e^{\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}/T}}. \tag{19}$$

This is the equilibrium distribution function for Bogoliubov thermal excitations in the reference frame, in which the normal velocity $\mathbf{u}$ vanishes, $\mathbf{u} = 0$. The equilibrium distribution function in an arbitrary frame can be calculated by minimizing the thermodynamic potential (16) with an additional

[1] We emphasize that the vectors $\mathbf{Q}_\alpha$ should also be treated fixed when varying all the thermodynamic potentials considered in the present paper.

constraint that fixes the momentum density associated with Bogoliubov excitations; see Sec. III for details.

Throughout this paper, we make the assumption that all the pairing gaps $\Delta_{\mathbf{p}}^{(\alpha)}$ are isotropic, meaning they do not depend on the direction of the vector $\mathbf{p}$. This assumption requires further clarification. As mentioned in the Introduction, neutron stars serve as one of the primary applications for the results obtained in this study. It is widely accepted that neutrons in the cores of neutron stars form Cooper pairs in the triplet $^3P_2$ state (see, however, Ref. [53] for an alternative viewpoint). In such a state, the energy gap is anisotropic (see, e.g., Ref. [54]), and it depends on the direction of the quantization axis. The presence of this additional preferred direction significantly complicates the analysis, leading to the diffusion and other coefficients obtained in the paper becoming tensors rather than scalars. To avoid this difficulty, it is usually assumed (e.g., Refs. [40,43,55–57]) that the matter of the neutron star cores consists of a collection of microscopic domains with randomly oriented quantization axes (see, however, Refs. [47,58]). Then, after averaging over the volume containing a large number of domains, the preferred direction will disappear. In this case, it is reasonable to expect that accounting for the microscopic anisotropy of the energy gap does not result in conceptually new effects and can be treated within the formalism developed for the isotropic gaps. The only thing that needs to be done is to establish a relation between a given anisotropic gap and the effective isotropic gap (see Sec. II B below). This is the strategy we choose to adopt in the present work, which aligns with the method used in calculating the entrainment matrix components in Refs. [40,43]. The authors of the recent work [49] also restrict themselves to considering the isotropic neutron gaps. As pointed out by Leinson [58], who studied the anisotropic gaps, such an approach can be helpful in the case of small hydrodynamic velocities, which is relevant to our study. A similar approach is commonly employed in calculating transport coefficients [55–57], while the method of averaging the angle-dependent neutron gap (not related to domain averaging) is often used to simplify first-principle calculations of the gap itself (see, e.g., Refs. [2,59,60]).

B. Calculation of the relativistic entrainment matrix

Throughout this paper we work under the assumption that it is always possible to choose a reference frame in which all the hydrodynamic velocities in the system are small, in particular, $Q_\alpha/p_{F\alpha} \ll 1$.[2] Physically, this means that all the relative velocities are assumed to be small. Restricting ourselves to the linear approximation in $Q_\alpha/p_{F\alpha}$, we can rewrite the “energy” (10) as

$$H^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = \varepsilon^{(\alpha)}_{\mathbf{p}} + \Delta H^{(\alpha)}_{\mathbf{p}}, \tag{20}$$

where $\varepsilon^{(\alpha)}_{\mathbf{p}}$ is, by definition, the energy $H^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ in the system *without superfluid currents* ($\mathbf{Q}_{\alpha'} = 0$) and the linear dependence on $\mathbf{Q}_{\alpha'}$ is encoded in the second term, $\Delta H^{(\alpha)}_{\mathbf{p}}$. In the absence of superfluid currents, the first term in the expression (10) is of the order $\sim T + \Delta^{(\alpha)}_{\mathbf{p}}$, while the second term $\sim f^{\alpha\alpha'}(\mathbf{p}_{F\alpha}, \mathbf{p}'_{F\alpha'}) n_{\alpha'} (T^2 + \Delta^{(\alpha)2}_{\mathbf{p}})/\mu^2_\alpha$ [43]. Therefore, the contribution of the second term in the expression (10) to the energy $\varepsilon^{(\alpha)}_{\mathbf{p}}$ is negligibly small and one can write $\varepsilon^{(\alpha)}_{\mathbf{p}} = \varepsilon^{(\alpha)}_0(\mathbf{p}) - \breve{\mu}_\alpha$. The second term in the expansion (20), generally, can be presented as[3]

$$\Delta H^{(\alpha)}_{\mathbf{p}} = \sum_{\alpha'} \gamma_{\alpha\alpha'} \mathbf{p}\mathbf{Q}_{\alpha'}, \tag{21}$$

where $\gamma_{\alpha\alpha'}$ is a matrix to be determined below.

The energy of Bogoliubov excitations and the distribution functions can be expanded as well:

$$\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = E^{(\alpha)}_{\mathbf{p}} + \Delta H^{(\alpha)}_{\mathbf{p}}, \tag{22}$$

$$\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = \mathfrak{f}^{(\alpha)}_{\mathbf{p}} + \frac{\partial \mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}}\, \Delta H^{(\alpha)}_{\mathbf{p}}, \tag{23}$$

$$\mathcal{N}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = n^{(\alpha)}_{\mathbf{p}} + \frac{\partial \mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}} \Delta H^{(\alpha)}_{\mathbf{p}}, \tag{24}$$

where $E^{(\alpha)}_{\mathbf{p}}$ is the Bogoliubov excitation energy, while $\mathfrak{f}^{(\alpha)}_{\mathbf{p}}$ and $n^{(\alpha)}_{\mathbf{p}}$ are the distribution functions for Bogoliubov excitations and the Landau quasiparticles in the absence of superfluid currents. These quantities are given by the following well-known expressions (see, e.g., Ref. [43]):

$$E^{(\alpha)}_{\mathbf{p}} = \sqrt{\varepsilon^{(\alpha)2}_{\mathbf{p}} + \Delta^{(\alpha)2}_{\mathbf{p}}}, \tag{25}$$

$$\mathfrak{f}^{(\alpha)}_{\mathbf{p}} = \frac{1}{1 + e^{E^{(\alpha)}_{\mathbf{p}}/T}}, \tag{26}$$

$$n^{(\alpha)}_{\mathbf{p}} = v^{(\alpha)2}_{\mathbf{p}} + \left(u^{(\alpha)2}_{\mathbf{p}} - v^{(\alpha)2}_{\mathbf{p}}\right)\mathfrak{f}^{(\alpha)}_{\mathbf{p}}. \tag{27}$$

Substituting (20) and (24) into (10), one obtains, with the accuracy to the terms linear in $\mathbf{Q}_\alpha$,

$$\Delta H^{(\alpha)}_{\mathbf{p}} = \frac{\mathbf{p}\mathbf{Q}_\alpha}{m^*_\alpha} + \sum_{\mathbf{p}'\sigma'\alpha'} f^{\alpha\alpha'}(\mathbf{p},\mathbf{p}') \times \left(\frac{\partial \mathfrak{f}^{(\alpha')}_{\mathbf{p}'}}{\partial E^{(\alpha')}_{\mathbf{p}'}} \Delta H^{(\alpha')}_{\mathbf{p}'} - \frac{\partial \theta^{(\alpha')}_{\mathbf{p}'}}{\partial \mathbf{p}'} \mathbf{Q}_{\alpha'} \right). \tag{28}$$

We have already neglected the terms $\sim (T^2 + \Delta^{(\alpha)2}_{\mathbf{p}})/\mu^2_\alpha$ in this expression. The functions in the parentheses have a sharp maximum near the Fermi surface of particle species α' (at $p \sim p_{F\alpha'}$), so that the sums in Eq. (28) can be approximately

[2]For the sake of brevity, the velocity $\mathbf{u}$ as well as the momenta $\mathbf{Q}_\alpha$ are further referred to as the hydrodynamic velocities.

[3]Note that this representation is general as long as the system does not contain any vectors other than $\mathbf{Q}_\alpha$. In particular, Eq. (21) is not correct if at least one of the paring gaps is treated as anisotropic (see Ref. [43] for more details). Replacing the anisotropic gap with an effective isotropic one allows us to avoid this complication.

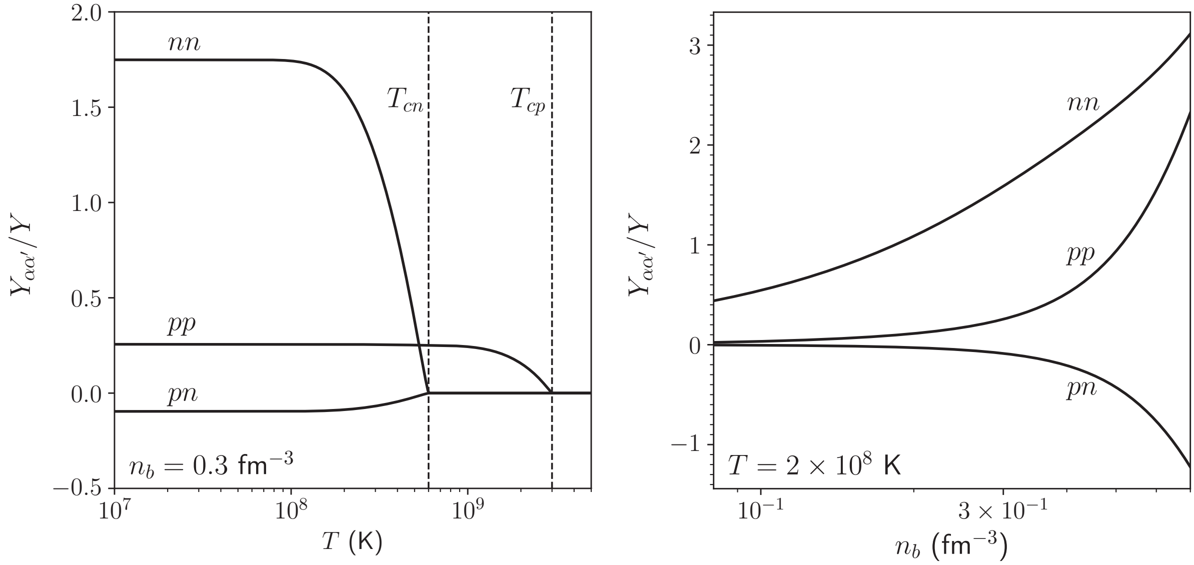

FIG. 1. The coefficients $Y_{\alpha\alpha'}$ normalized by $Y = 10^{41}$ cm^{-3} erg^{-1} as functions of the temperature T (left panel) and the baryon number density n_b (right panel). The critical temperatures are chosen to be $T_{cn} = 6\times10^8$ K and $T_{cp} = 3\times10^9$ K (shown by the vertical dashed lines in the left panel). The left panel is plotted for $n_b = 0.3$ fm^{-3}, the right panel is plotted for $T = 2\times10^8$ K.

calculated as

$$\sum_{\mathbf{p}'\sigma'} f^{\alpha\alpha'}(\mathbf{p},\mathbf{p}')\frac{\partial \mathfrak{f}^{(\alpha')}_{\mathbf{p}'}}{\partial E^{(\alpha')}_{\mathbf{p}'}}\Delta H^{(\alpha')}_{\mathbf{p}'} = -\frac{G_{\alpha\alpha'}}{n_\alpha} m^*_{\alpha'}\Phi_{\alpha'}\Delta H^{(\alpha')}_{\mathbf{p}}, \quad (29)$$

$$\sum_{\mathbf{p}'\sigma'} f^{\alpha\alpha'}(\mathbf{p},\mathbf{p}')\frac{\partial \theta^{(\alpha')}_{\mathbf{p}'}}{\partial \mathbf{p}'}\mathbf{Q}_{\alpha'} = -\frac{G_{\alpha\alpha'}}{n_\alpha}\mathbf{p}\,\mathbf{Q}_{\alpha'}. \quad (30)$$

For the same reason, we have

$$\sum_{\mathbf{p}\sigma} p^i p^k \frac{\partial \mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}} = -m^*_\alpha n_\alpha \Phi_\alpha \delta^{ij}. \quad (31)$$

This equality is extensively used in the rest of the paper. The function Φ_α in Eqs. (29) and (31) is defined as

$$\Phi_\alpha = -\frac{\pi^2}{m^*_\alpha p_{F\alpha}}\sum_{\mathbf{p}\sigma}\frac{\partial \mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}}. \quad (32)$$

It changes from $\Phi_\alpha = 0$ at $T = 0$ to $\Phi_\alpha = 1$ at $T \geqslant T_{c\alpha}$, where $T_{c\alpha}$ is the critical temperature for transition of particle species $\alpha = n, p$ into the superfluid state.

Plugging Eqs. (21), (29), and (30) into the expression (28), one can obtain a system of equations on $\gamma_{\alpha\alpha'}$:

$$\gamma_{\alpha\alpha'} = \frac{\delta_{\alpha\alpha'}}{m^*_\alpha} + \frac{G_{\alpha\alpha'}}{n_\alpha} - \sum_{\alpha''}\frac{G_{\alpha\alpha''}}{n_\alpha} m^*_{\alpha''}\Phi_{\alpha''}\gamma_{\alpha''\alpha'}. \quad (33)$$

The solution to these equations is

$$\gamma_{\alpha\alpha} = \frac{(n_\alpha + G_{\alpha\alpha}m^*_\alpha)(n_\beta + G_{\beta\beta}m^*_\beta\Phi_\beta) - G^2_{\alpha\beta}m^*_\alpha m^*_\beta\Phi_\beta}{m^*_\alpha \mathcal{S}}, \quad (34)$$

$$\gamma_{\alpha\beta} = \frac{G_{\alpha\beta}n_\beta(1-\Phi_\beta)}{\mathcal{S}}, \quad (35)$$

where $\beta \neq \alpha$ and

$$\mathcal{S} = (n_n + G_{nn}m^*_n\Phi_n)(n_p + G_{pp}m^*_p\Phi_p) - G^2_{np}m^*_n m^*_p \Phi_n\Phi_p. \quad (36)$$

To calculate the particle current density, one can use the standard “nonsuperfluid” formula [49,51,61]

$$\mathbf{j}_\alpha = \sum_{\mathbf{p}\sigma}\frac{\partial H^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}}{\partial \mathbf{p}}\,\mathcal{N}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}. \quad (37)$$

Plugging Eqs. (20), (21), and (24) into this formula and using the expression (31), one obtains Eq. (3) with

$$Y_{\alpha\alpha'} = \frac{\gamma_{\alpha\alpha'}n_\alpha}{c^2}(1-\Phi_\alpha). \quad (38)$$

It is easy to see that the matrix $Y_{\alpha\alpha'}$ is symmetric. Figure 1 shows the behavior of the coefficients $Y_{\alpha\alpha'}$ as functions of temperature T and baryon number density, $n_b = n_n + n_p$. It is plotted for neutron star matter composed of protons, neutrons, and electrons assuming the BSk24 equation of state [62]. The Landau parameters and the functions Φ_α [see Eq. (32)] were calculated as described in Ref. [63]; see this reference for more details. Following Refs. [43,55], the effective neutron gap was taken to be equal to the minimum value of the angle-dependent $^3P_2$ gap with the projection of the total angular momentum of a pair $m_J = 0$ (see Appendix A in Ref. [43] for details).

The particle current density (37) can alternatively be represented as [51]

$$\mathbf{j}_\alpha = c^2\sum_{\alpha'}Y^0_{\alpha\alpha'}\left[\mathbf{Q}_{\alpha'} + \frac{1}{n_{\alpha'}}\sum_{\mathbf{p}\sigma}\mathbf{p}\mathcal{F}^{(\alpha')}_{\mathbf{p}+\mathbf{Q}_{\alpha'}}\right], \quad (39)$$

where the coefficients

$$Y^0_{\alpha\alpha'} = \delta_{\alpha\alpha'}\frac{n_\alpha}{m^*_\alpha c^2} + \frac{G_{\alpha\alpha'}}{c^2} \quad (40)$$

are equal to the matrix elements $Y_{\alpha\alpha'}$ taken at $T = 0$. On the other hand, the momentum density of particle species α equals [51]

$$\mathbf{P}_\alpha = \sum_{\mathbf{p}\sigma}(\mathbf{p} + \mathbf{Q}_\alpha)\mathcal{N}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = n_\alpha \mathbf{Q}_\alpha + \sum_{\mathbf{p}\sigma}\mathbf{p}\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}. \quad (41)$$

Using Eqs. (39) and (41) together with the relation (14), one can easily verify that

$$\sum_\alpha \mathbf{P}_\alpha = \sum_\alpha \frac{\mu_\alpha}{c^2}\mathbf{j}_\alpha, \quad (42)$$

that is, the total momentum density coincides with the total mass-current density in the system, the expected result.

Note that the expressions (37) and (39) for particle current densities, as well as the expression (41) for the momentum densities, are quite general and can be used even if the system is *not* in complete thermodynamic equilibrium [51] (i.e., $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ in these equations is not necessarily the Fermi-Dirac distribution function). We will make use of this fact in the subsequent presentation.

III. NORMAL CURRENTS IN A SUPERFLUID MIXTURE OF FERMI LIQUIDS

All the calculations in the previous section were done in the frame of reference comoving with the normal liquid component. In other words, it was implicitly assumed that thermal excitations of both species, "n" and "p", move with identical *normal* (nonsuperfluid) velocity. How should the results of the previous section be modified if we assume that these velocities differ for the "n" and "p" species? In the present section we will try to address this question for an idealized system, in which normal velocities of two species differ, but the dissipative interaction (due to collisions) between them is absent. At the same time, we will allow both species to interact with a heat bath so that the system can be described with a single temperature. A similar but more general problem (allowing for dissipative interaction between the two species) will be considered in Sec. IV within the kinetic theory. It will be demonstrated that this problem, formulated in the language of kinetic theory, is directly related to the diffusion in superfluid mixtures.

As it is discussed in the previous section, we assume that there always exists a reference frame in which the velocities of all the particle species are small. In the system with two different normal currents it is convenient to work in this frame and minimize the thermodynamic potential

$$\tilde{F} = E - \sum_\alpha \mu_{\alpha 1} n_\alpha - TS - \sum_\alpha \mathbf{P}_\alpha \mathbf{V}_{q\alpha}, \quad (43)$$

where $\mathbf{P}_\alpha$ is the momentum density of particle species α given by Eq. (41); the vectors $\mathbf{V}_{q\alpha}$ are abstract Lagrange multipliers whose physical meaning will be clarified below and $\mu_{\alpha 1}$ are some chemical potentials in the chosen reference frame (also Lagrange multipliers, required to keep the total number of particles α fixed). The last term in the right-hand side of the expression (43) indicates that our thermodynamic potential is minimized at fixed momentum densities of the "n" and "p" particle species. Using the definition (41), the thermodynamic potential (43) can be rewritten as

$$E - \sum_\alpha \mu_\alpha n_\alpha - TS - \sum_\alpha \mathcal{P}_\alpha \mathbf{V}_{q\alpha}, \quad (44)$$

where $\mathcal{P}_\alpha \equiv \sum_{\mathbf{p}\sigma}\mathbf{p}\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ and we introduced the new chemical potential $\mu_\alpha \equiv \mu_{\alpha 1} + \mathbf{Q}_\alpha \mathbf{V}_{q\alpha}$.

There is one point to be made here before moving on. All the thermodynamic variables appearing in the hydrodynamic equations are usually defined (measured) in the reference frame comoving with the fluid. When a mixture of few superfluids is considered, one usually chooses the frame comoving with the normal (nonsuperfluid) liquid component (see, e.g., Ref. [41]). In particular, the chemical potentials entering the formula (3) are assumed to be the same as those arising in the second law of thermodynamics (B7) written down in the comoving reference frame. However, allowing for the two independent normal velocity fields, the standard definition of the comoving frame loses its meaning and we are forced to work in a more general reference frame. This means, in particular, that the chemical potentials μ_α entering the expression (44) are not the same potentials as those appearing in the formulas (3) and (B7). However, since these potentials are scalars with respect to spatial transformations, the difference between chemical potentials measured in different reference frames can depend only on bilinear combinations of the hydrodynamic velocities. Thus, this difference is quadratically small by assumption (see the beginning of Sec. II B) and can be ignored in the subsequent analysis.

Minimizing the thermodynamic potential (44) by calculating the variation of the energy density E, the number densities n_α, and the quantities $\mathcal{P}_\alpha$ with respect to variation of the distribution function $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$, one obtains

$$\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = \frac{1}{1 + e^{(\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} - \mathbf{p}\mathbf{V}_{q\alpha})/T}}. \quad (45)$$

As we just emphasized, we work in a reference frame where all the hydrodynamic velocities are small. Therefore, assuming that $Q_\alpha/p_{F\alpha}$ and $V_{q\alpha}/v_{F\alpha} \ll 1$,[4] all the quantities in this expression can be expanded in powers of $\mathbf{Q}_\alpha$ and $\mathbf{V}_{q\alpha}$. In this way, the expressions (23) and (24) should be replaced with

$$\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = \mathfrak{f}^{(\alpha)}_{\mathbf{p}} + \frac{\partial \mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}}\big(\Delta H^{(\alpha)}_{\mathbf{p}} - \mathbf{p}\mathbf{V}_{q\alpha}\big), \quad (46)$$

$$\mathcal{N}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = n^{(\alpha)}_{\mathbf{p}} + \frac{\partial \mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}}\big(\Delta H^{(\alpha)}_{\mathbf{p}} - \mathbf{p}\mathbf{V}_{q\alpha}\big), \quad (47)$$

where we made use of the expansion (22) for the energy $\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$. The quantity $\Delta H^{(\alpha)}_{\mathbf{p}}$ in Eq. (22) now, generally,

[4]Further it will be argued that $\mathbf{V}_{q\alpha} \approx \mathbf{u}$.

depends on both the vectors $\mathbf{Q}_\alpha$ and $\mathbf{V}_{q\alpha}$, and can be written as

$$\Delta H_{\mathbf{p}}^{(\alpha)} = \sum_{\alpha'} (\gamma_{\alpha\alpha'} \mathbf{p}\mathbf{Q}_{\alpha'} + K_{\alpha\alpha'} \mathbf{p}\mathbf{V}_{q\alpha'}). \tag{48}$$

Plugging Eqs. (20), (47), and (48) into (10), and using (again, as in Sec. II B) Eqs. (29) and (30), one obtains, besides Eqs. (33), the equations for the coefficients $K_{\alpha\alpha'}$:

$$K_{\alpha\alpha'} = \frac{G_{\alpha\alpha'} m^*_{\alpha'}}{n_\alpha} - \sum_{\alpha''} \frac{G_{\alpha\alpha''}}{n_\alpha} m^*_{\alpha''} \Phi_{\alpha''} K_{\alpha''\alpha'}. \tag{49}$$

The solution to these equations is

$$K_{\alpha\alpha} = \frac{G_{\alpha\alpha} m^*_\alpha \Phi_\alpha (n_\beta + G_{\beta\beta} m^*_\beta \Phi_\beta) - G^2_{\alpha\beta} m^*_\alpha m^*_\beta \Phi_\alpha \Phi_\beta}{\mathcal{S}}, \tag{50}$$

$$K_{\alpha\beta} = \frac{G_{\alpha\beta} m^*_\beta n_\beta \Phi_\beta}{\mathcal{S}}, \tag{51}$$

where we recall that $\beta \neq \alpha$ and $\mathcal{S}$ is given by the expression (36).

Now one can calculate the particle current density $\mathbf{j}_\alpha$. Substituting Eqs. (20), (47), and (48) into (37), one obtains

$$\mathbf{j}_\alpha = R_{\alpha\alpha} \mathbf{V}_{q\alpha} + R_{\alpha\beta} \mathbf{V}_{q\beta} + c^2 Y_{\alpha\alpha} \mathbf{Q}_\alpha + c^2 Y_{\alpha\beta} \mathbf{Q}_\beta, \tag{52}$$

where the elements of the superfluid entrainment matrix $Y_{\alpha\alpha'}$ are given by Eqs. (38) and the coefficients $R_{\alpha\alpha'}$ are defined as

$$R_{\alpha\alpha} = n_\alpha K_{\alpha\alpha}(1 - \Phi_\alpha) + n_\alpha \Phi_\alpha, \tag{53}$$

$$R_{\alpha\beta} = n_\alpha K_{\alpha\beta}(1 - \Phi_\alpha). \tag{54}$$

This set of coefficients is, to our best knowledge, introduced in the literature for the first time. In analogy to the matrix $Y_{\alpha\alpha'}$, we call the coefficients $R_{\alpha\alpha'}$ the *normal entrainment matrix*. Alternatively, the matrix $R_{\alpha\alpha'}$ can be expressed through $\gamma_{\alpha\alpha'}$ as

$$R_{\alpha\alpha'} = n_{\alpha'} m^*_{\alpha'} \Phi_{\alpha'} \gamma_{\alpha'\alpha}. \tag{55}$$

It can be verified that the obtained coefficients satisfy the following *sum rule*:

$$\mu_\alpha Y_{\alpha\alpha} + \mu_\beta Y_{\alpha\beta} + R_{\alpha\alpha} + R_{\alpha\beta} = n_\alpha, \tag{56}$$

which is the generalization to finite temperatures of the sum rule derived in Ref. [45]. This sum rule is related to particle number conservation. Indeed, if one introduces the superfluid velocities according to the definitions, $\mathbf{V}_{s\alpha} \equiv \mathbf{Q}_\alpha c^2/\mu_\alpha$,[5] and assumes that all the fluid components move with one and the same velocity, i.e., $\mathbf{V}_{qn} = \mathbf{V}_{qp} = \mathbf{V}_{sn} = \mathbf{V}_{sp} = \mathbf{u}$, then [in view of Eq. (56)] the expression (52) will reduce to $\mathbf{j}_\alpha = n_\alpha \mathbf{u}$, as expected. One can obtain another useful relation from Eq. (56) by using the definitions (38), (53), and (54):

$$\frac{\mu_\alpha}{c^2} \gamma_{\alpha\alpha} + \frac{\mu_\beta}{c^2} \gamma_{\alpha\beta} + K_{\alpha\alpha} + K_{\alpha\beta} = 1. \tag{57}$$

[5]Note that, for a relativistic equation of state, the superfluid velocity introduced this way does not obey the potentiality condition, curl $\mathbf{V}_{s\alpha} \neq 0$. This is in contrast to the vectors $\mathbf{Q}_\alpha$, which are proportional (with constant coefficient) to the gradient of the superfluid order parameter phase [see expression (B11)], and thus satisfy the potentiality condition.

Setting $\mathbf{V}_{qn} = \mathbf{V}_{qp} = \mathbf{u}$ in Eq. (52) and taking into account Eq. (56), one derives [cf. Eq. (3)]

$$\mathbf{j}_{\alpha,0} = n_{q\alpha} \mathbf{u} + c^2 \sum_{\alpha'} Y_{\alpha\alpha'} \mathbf{Q}_{\alpha'}, \tag{58}$$

where

$$n_{q\alpha} = n_\alpha - \sum_{\alpha'} \mu_{\alpha'} Y_{\alpha\alpha'} = \sum_{\alpha'} R_{\alpha\alpha'}. \tag{59}$$

Therefore, when the vectors $\mathbf{V}_{qn}$ and $\mathbf{V}_{qp}$ coincide, they have the meaning of the normal velocity of thermal excitations. In turn, the quantity $n_{q\alpha}$ can be interpreted as the normal density of particle species α. Note, however, that in a strongly interacting mixture $n_{q\alpha}$ can become negative; see the discussion at the end of the present section.

If one of the constituents, say "n", is nonsuperfluid ($\Phi_n = 1$), then the particle current densities reduce to

$$\mathbf{j}_n = n_n \mathbf{V}_{qn}, \quad \mathbf{j}_p = R_{pp} \mathbf{V}_{qp} + R_{pn} \mathbf{V}_{qn} + c^2 Y_{pp} \mathbf{Q}_p; \tag{60}$$

the values of the matrix elements in this limit are given in Appendix A. In the case of a completely nonsuperfluid mixture, they further simplify as

$$\mathbf{j}_n = n_n \mathbf{V}_{qn}, \quad \mathbf{j}_p = n_p \mathbf{V}_{qp}. \tag{61}$$

Thus, in the nonsuperfluid mixture, the vectors $\mathbf{V}_{q\alpha}$ have the meaning of the hydrodynamic velocities of the corresponding particle species. In the superfluid mixture the interpretation of these vectors is more complicated; see the end of Sec. IV for a discussion.

On the other hand, if one of the species, say "p", is in the regime of strong superfluidity ($\Phi_p \to 0$, $T \ll T_{cp}$), then one has for the currents

$$\mathbf{j}_n = R_{nn} \mathbf{V}_{qn} + c^2 Y_{nn} \mathbf{Q}_n + c^2 Y_{np} \mathbf{Q}_p, \tag{62}$$

$$\mathbf{j}_p = R_{pn} \mathbf{V}_{qn} + c^2 Y_{pn} \mathbf{Q}_n + c^2 Y_{pp} \mathbf{Q}_p. \tag{63}$$

Again, the relevant matrix elements are given in Appendix A. As one can see, the Fermi-liquid effects lead to the finite normal current density of particle species "p" even at $T \ll T_{cp}$.

One can notice that the matrix $R_{\alpha\alpha'}$, in contrast to the matrix $Y_{\alpha\alpha'}$, is not symmetric. This fact, however, does not point to some defect of the theory since the vectors $\mathbf{V}_{q\alpha}$, in contrast to the vectors $\mathbf{Q}_\alpha$, are not momenta. To symmetrize the expression (52), we introduce the momentum densities $\mathbf{P}_{q\alpha}$, associated with the normal fluid motions instead of the vectors $\mathbf{V}_{q\alpha}$. Using Eq. (41), they are defined as

$$\mathbf{P}_{q\alpha} = \sum_{\mathbf{p}\sigma} \mathbf{p} \mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}\Big|_{\mathbf{Q}_n=\mathbf{Q}_p=0}. \tag{64}$$

Now, with the help of Eq. (39), one can express the particle current densities through the momentum variables $\mathbf{P}_{q\alpha}/n_\alpha$:

$$\mathbf{j}_\alpha = c^2 Y^0_{\alpha\alpha} \frac{\mathbf{P}_{q\alpha}}{n_\alpha} + c^2 Y^0_{\alpha\beta} \frac{\mathbf{P}_{q\beta}}{n_\beta} + c^2 Y_{\alpha\alpha} \mathbf{Q}_\alpha + c^2 Y_{\alpha\beta} \mathbf{Q}_\beta. \tag{65}$$

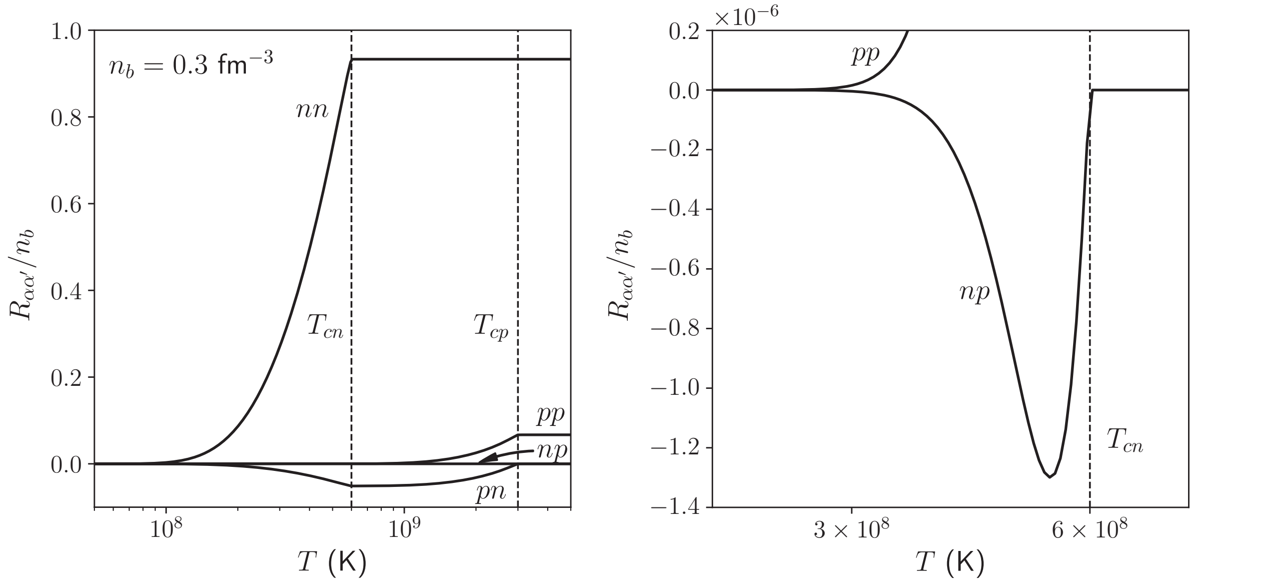

FIG. 2. Left panel: The coefficients $R_{\alpha\alpha'}$ normalized to the baryon density n_b as functions of temperature are shown for $n_b = 0.3$ fm^{-3}, $T_{cp} = 2 \times 10^9$ K, $T_{cn} = 6 \times 10^8$ K, and for the BSk24 equation of state. Right panel: Zoomed-in version of the plot in the left panel near $T \approx T_{cn}$.

In this expression, the matrix $Y^0_{\alpha\alpha'}$, which is a special case of the symmetric matrix $Y_{\alpha\alpha'}$ [see Eq. (40)], is obviously symmetric.

The coefficients $R_{\alpha\alpha'}$, calculated for the same model of neutron-star matter as the matrix $Y_{\alpha\alpha'}$ in Fig. 1, are shown as functions of temperature T in Fig. 2 for $n_b = 0.3$ fm^{-3} and in Fig. 3 for $n_b = 0.5$ fm^{-3}. One can see that all the coefficients vanish when the temperature tends to zero. This is a reasonable result, since there are no temperature excitations in the zero-temperature limit. Considering the vicinity of the critical temperature T_{cn}, one can notice that the behavior of the matrix elements $R_{\alpha\alpha'}$ and $Y_{\alpha\alpha'}$ is different. If $T \geqslant T_{cn}$, both the nondiagonal coefficients $Y_{pn} = Y_{np}$ vanish together with the coefficient Y_{nn}. This is expected, since then the particle species "n" is nonsuperfluid (normal) so that the current density $\mathbf{j}_p$ cannot depend on $\mathbf{Q}_n$. As for the matrix $R_{\alpha\alpha'}$, only the element R_{np} vanishes at $T > T_{cn}$.

The normal number densities $n_{q\alpha}$ are shown in Fig. 4 as functions of temperature for the same equation of state BSk24. The function n_{qn} behaves exactly as one would expect. It equals n_n for $T \geqslant T_{cn}$ and exponentially decreases to zero at $T \ll T_{cn}$. In contrast to n_{qn}, the behavior of the normal density n_{qp} is more counterintuitive. It starts from n_p at $T = T_{cp}$ and then rapidly drops to negative values with decreasing

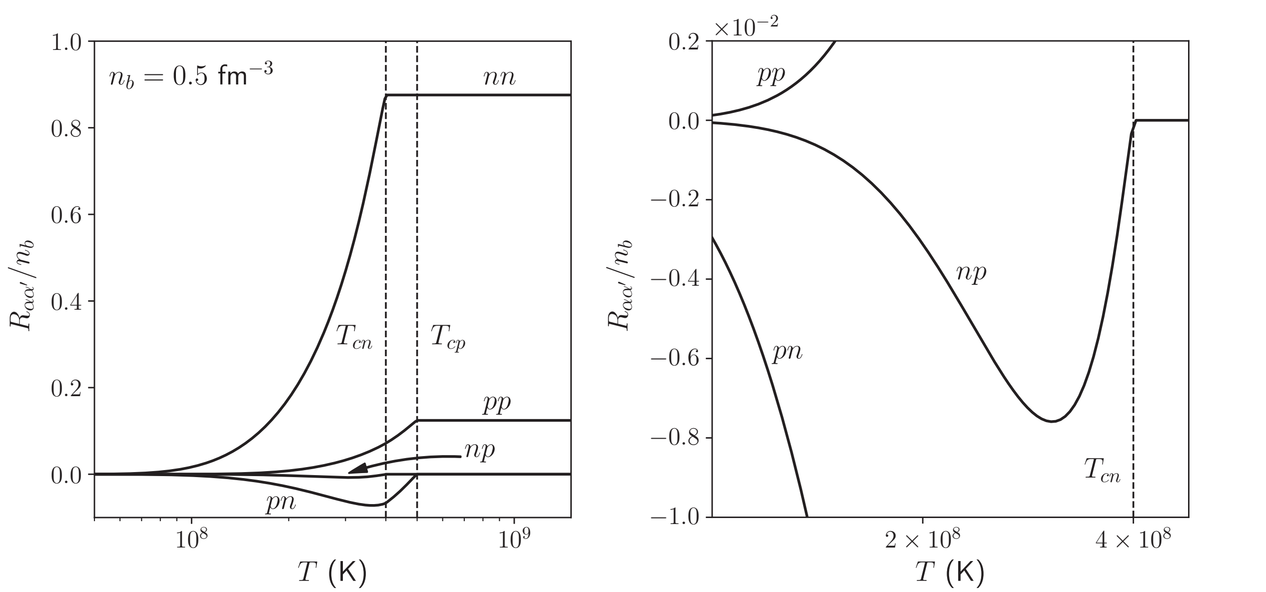

FIG. 3. The same as in Fig. 2 but for $n_b = 0.5$ fm^{-3}, $T_{cp} = 4 \times 10^8$ K, and $T_{cn} = 5 \times 10^8$ K.

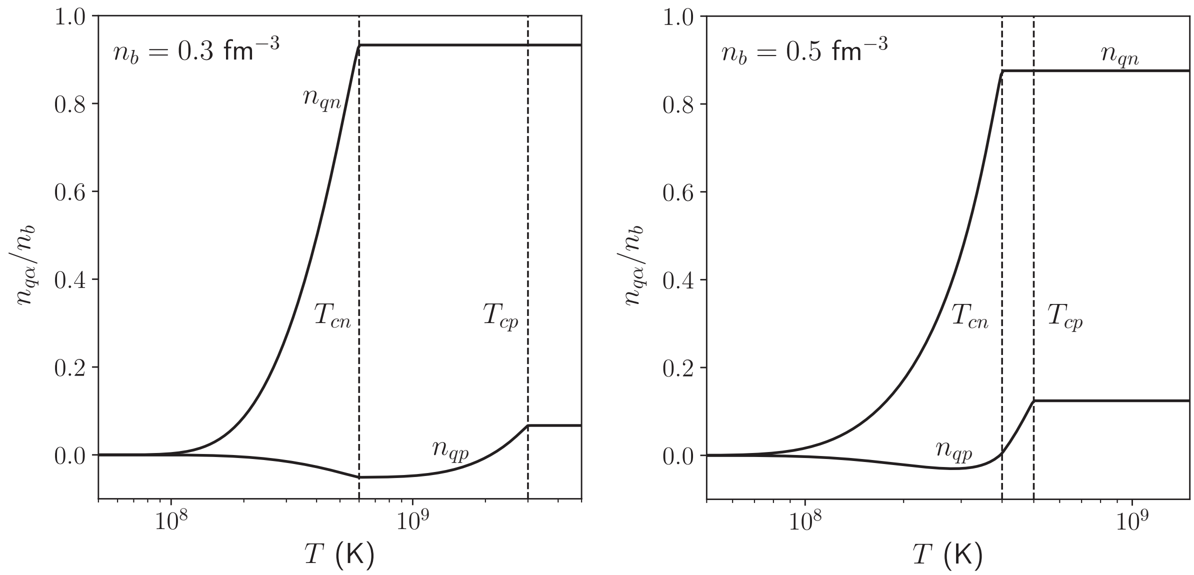

FIG. 4. The normal number densities $n_{q\alpha}$ normalized to the baryon density n_b as functions of temperature plotted for $n_b = 0.3$ fm^{-3}, $T_{cp} = 2 \times 10^9$ K, $T_{cn} = 6 \times 10^8$ K (left panel) and $n_b = 0.5$ fm^{-3}, $T_{cp} = 4 \times 10^8$ K, $T_{cn} = 5 \times 10^8$ K (right panel).

temperature. After reaching a minimum, it begins to increase, approaching zero at $T \to 0$. The negativity of n_{qp} is related to negativity of the coefficient G_{pn} for the chosen equation of state [see Eq. (15) and Eq. (70) below]. This feature, however, does not lead to any unphysical consequences. In particular, the system energy in the presence of particle currents always increases, as shown in what follows.

Assume, for simplicity, that the only currents generated in the system are the normal ones, i.e., $\mathbf{Q}_n = \mathbf{Q}_p = 0$. Then, in the linear approximation, one can express $\mathbf{V}_{q\alpha}$ through $\mathbf{P}_{q\alpha}$ as

$$\mathbf{V}_{q\alpha} = \sum_{\alpha'} M_{\alpha\alpha'} \mathbf{P}_{q\alpha'}, \tag{66}$$

where the elements of the matrix $M_{\alpha\alpha'}$ can be calculated by equating Eqs. (52) and (65), and using the expression (109) for the determinant of the matrix $R_{\alpha\alpha'}$:

$$M_{\alpha\alpha'} = \frac{n_\alpha \delta_{\alpha\alpha'} + G_{\alpha\alpha'} m_\alpha^* \Phi_\alpha}{n_\alpha n_{\alpha'} m_\alpha^* \Phi_\alpha}. \tag{67}$$

Now, using the fact that the thermodynamic potential (44) is stationary with respect to variations of n_α, S, and $\mathcal{P}_\alpha = \mathbf{P}_{q\alpha}$, one has $dE = \sum_\alpha \mu_\alpha dn_\alpha + T dS + \sum_\alpha \mathbf{V}_{q\alpha} d\mathbf{P}_{q\alpha}$, and hence [see Eq. (66)]

$$E \approx E_0(n_\alpha, S) + \frac{1}{2} \sum_{\alpha\alpha'} M_{\alpha\alpha'} \mathbf{P}_{q\alpha} \mathbf{P}_{q\alpha'}, \tag{68}$$

where $E_0(n_\alpha, S)$ is the energy density of the system in the absence of currents. To make the system energy with currents larger than that without currents, the matrix $M_{\alpha\alpha'}$ should be positive-definite, which implies

$$\frac{n_\alpha}{m_\alpha^*} + G_{\alpha\alpha} \Phi_\alpha > 0, \quad \mathcal{S} > 0. \tag{69}$$

An equivalent set of constraints has been obtained, in particular, in Ref. [44] from the requirement of stability of superfluid Fermi mixture at $T = 0$ with respect to spontaneous generation of superfluid currents in the system. The conditions (69) do not constrain the sign of the coefficient G_{np}. On the other hand, in the limit $T \ll T_{cp}$, one obtains [cf. Eq. (A31)]

$$n_{qp} \approx R_{pn} \approx \frac{n_n m_n^* G_{np} \Phi_n}{n_n + G_{nn} m_n^* \Phi_n}, \tag{70}$$

according to which the quantity n_{qp} has the same sign as the coefficient G_{np}.

IV. DIFFUSION

In the previous section, we obtained the expression for particle current densities under the assumption that the dissipative interaction between different particle species can be neglected. If one wants to allow for such interaction, one should work within the framework of transport theory. In this section, we demonstrate the close relation of the approach developed in Sec. III to the diffusion theory of particles in a superfluid Fermi mixture.

A. Basic equations

Let us suppose that the characteristic wavenumber of our problem is $\mathfrak{q}$, while the characteristic frequency is ω. In the long-wavelength/small-frequency ("quasiclassical") limit ($\mathfrak{q} v_{F\alpha},\ \omega \ll \Delta_\alpha$), one can formulate the Boltzmann-like kinetic equation for the Bogoliubov excitations [51,64]:

$$\frac{\partial \mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}}{\partial t} + \frac{\partial \mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}}{\partial \mathbf{p}} \frac{\partial \mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}}{\partial \mathbf{r}} - \frac{\partial \mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}}{\partial \mathbf{r}} \frac{\partial \mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}}{\partial \mathbf{p}} = I_\alpha[\mathcal{F}], \tag{71}$$

where $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ is the distribution function of Bogoliubov excitations, $\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ is the energy of a Bogoliubov excitation given by Eq. (9), $I^{(\alpha)}[\mathcal{F}]$ is the collision integral (see Appendix E), and $\mathcal{F}$ denotes the set of the distribution functions $\mathcal{F}^{(n)}_{\mathbf{p}+\mathbf{Q}_n}$ and $\mathcal{F}^{(p)}_{\mathbf{p}+\mathbf{Q}_p}$. This equation should be supplemented with the continuity equations for particle species $\alpha = n, p$. In what follows, we do not account for the chemical reactions between the species "n" and "p", assuming that the number of particles of each species is conserved separately. Thus, the continuity equations can be represented as

$$\frac{\partial n_\alpha}{\partial t} + \mathrm{div}\mathbf{j}_\alpha = 0, \tag{72}$$

where the particle current density $\mathbf{j}_\alpha$ can still be calculated with the expressions (37) or (39). It is worth noting that Eq. (72) cannot be obtained from the kinetic equation (71). To derive it from the microphysical theory, one needs to consider the full system of kinetic equations for Landau quasiparticles [51,64,65]. In addition to Eqs. (71) and (72), one also needs an equation describing the evolution of the superfluid component. For an uncharged mixture this "superfluid" equation takes the form [51,64]

$$\frac{\partial \mathbf{Q}_\alpha}{\partial t} = -\nabla \breve{\mu}_\alpha, \tag{73}$$

where $\breve{\mu}_\alpha$ is the nonequilibrium analog of the corresponding chemical potential of particle species α, which has already been introduced in the expression (5).

B. Thermodynamic equilibrium

First of all, we need to determine the equilibrium state to which dissipative corrections will be sought. Besides the usual thermodynamic variables (e.g., temperature and chemical potentials), the equilibrium state of a superfluid Fermi mixture is generally characterized by the velocity $\mathbf{u}$ of normal part of the mixture (Bogoliubov thermal excitations) and by the superfluid currents for each particle species [5]. These supercurrents can exist in the system without any dissipation until they reach some critical values (see, e.g., Refs. [66–68]). The equilibrium distribution function corresponding to this situation has already been found in Sec. III. Indeed, as discussed in Sec. III, when the vectors $\mathbf{V}_{qn}$ and $\mathbf{V}_{qp}$ are equal to each other, they have the meaning of the normal velocity $\mathbf{u}$. In this case, corresponding to complete thermodynamic equilibrium, the equilibrium distribution function of Bogoliubov excitations takes the form [see Eq. (45) with $\mathbf{V}_{q\alpha} = \mathbf{u}$]

$$\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0} = \frac{1}{1 + e^{(\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0} - \mathbf{p}\,\mathbf{u})/T}}, \tag{74}$$

where

$$\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0} = E^{(\alpha)}_{\mathbf{p}} + \mathbf{p}\sum_{\alpha'}\gamma_{\alpha\alpha'}\boldsymbol{w}_{\alpha'} + \mathbf{p}\mathbf{u} \tag{75}$$

is the equilibrium energy of Bogoliubov excitations and $\boldsymbol{w}_\alpha = [\mathbf{Q}_\alpha - (\mu_\alpha/c^2)\mathbf{u}]$ is the vector proportional to the difference between the superfluid and normal velocities.

To obtain Eq. (75) one needs to substitute Eq. (48) into (22), set $\mathbf{V}_{qn} = \mathbf{V}_{qp} = \mathbf{u}$, and apply the relation (57). Plugging (75) into the distribution function (74), one finally gets

$$\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0} = \frac{1}{1 + e^{\{E^{(\alpha)}_{\mathbf{p}} + \mathbf{p}\sum_{\alpha'}\gamma_{\alpha\alpha'}\boldsymbol{w}_{\alpha'}\}/T}}. \tag{76}$$

The equilibrium particle current density can be calculated by setting $\mathbf{V}_{qn} = \mathbf{V}_{qp} = \mathbf{u}$ in Eq. (52), and is given by the expression (58).

C. Perturbations of the thermodynamic equilibrium

Let us now assume a small departure from the thermodynamic equilibrium. We restrict ourselves to considering the corrections caused by the small gradients of macroscopic variables. In this case, the formal small parameter is the Knudsen number $\mathcal{K} = \ell \mathrm{q}$, where ℓ is the mean free path of Bogoliubov excitations.

To find an approximate solution to Eqs. (71)–(73), we, following Chapman-Enskog method (see, e.g., Ref. [69]), expand the distribution function for Bogoliubov excitations in the powers of the small parameter $\mathcal{K}$:

$$\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = \overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0} + \bar{\mathfrak{f}}^{(\alpha)}_1. \tag{77}$$

Here the function $\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}$ satisfies the equation

$$I^{(\alpha)}\big[\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}, \overline{\mathcal{F}}^{(\beta)}_{\mathbf{p}+\mathbf{Q}_\beta,0}\big] = 0 \tag{78}$$

(see Appendix E for details) and $\bar{\mathfrak{f}}^{(\alpha)}_1$ is the small correction to be found below. The function $\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}$ is the Fermi-Dirac distribution function,

$$\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0} = \frac{1}{1 + \mathrm{e}^{(\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} - \mathbf{p}\,\mathbf{u})/T}}, \tag{79}$$

where all the thermodynamic variables, as well as the hydrodynamic velocities, can generally depend on time and space coordinates, and

$$\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} = \mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0} + \Delta\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} \tag{80}$$

is the *local* Bogoliubov excitation energy, which depends on the nonequilibrium distribution functions $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ of all particle species. In the Landau theory of Fermi liquids the energy $\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ generally differs from the equilibrium energy $\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}$, given by Eq. (75) [52]. The difference $\Delta\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ between these energies should be found simultaneously (and self-consistently) with the function $\bar{\mathfrak{f}}^{(\alpha)}_1$; see Sec. IV D for details.

Let us transform the left-hand side of Eq. (71) to a form more suitable for the subsequent analysis. Plugging the expansion (77) into (71), we get

$$\frac{\partial \overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}}{\partial \mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}}\Bigg[\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial t}+\mathbf{p}\sum_{\alpha'}\gamma_{\alpha\alpha'}\frac{\partial \boldsymbol{w}_{\alpha'}}{\partial t}+\mathbf{p}\sum_{\alpha'}\boldsymbol{w}_{\alpha'}\frac{\partial \gamma_{\alpha\alpha'}}{\partial t}-\frac{E^{(\alpha)}_{\mathbf{p}}+\mathbf{p}\sum_{\alpha'}\gamma_{\alpha\alpha'}\boldsymbol{w}_{\alpha'}}{T}\frac{\partial T}{\partial t}-\left(\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial \mathbf{p}}+\sum_{\alpha'}\gamma_{\alpha\alpha'}\boldsymbol{w}_{\alpha'}\right)$$
$$\times\frac{E^{(\alpha)}_{\mathbf{p}}+\mathbf{p}\sum_{\alpha'}\gamma_{\alpha\alpha'}\boldsymbol{w}_{\alpha'}}{T}\nabla T-\left(\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial \mathbf{p}}+\sum_{\alpha'}\gamma_{\alpha\alpha'}\boldsymbol{w}_{\alpha'}\right)\nabla(\mathbf{p}\mathbf{u})+(\mathbf{u}\nabla)\left(E^{(\alpha)}_{\mathbf{p}}+\mathbf{p}\sum_{\alpha'}\gamma_{\alpha\alpha'}\boldsymbol{w}_{\alpha'}\right)\Bigg]+\text{derivatives of }\Delta\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$$
$$+\text{derivatives of }\bar{\mathfrak{f}}^{(\alpha)}_1=I^{(\alpha)}[\mathcal{F}]. \tag{81}$$

First of all, we note that the expression (81) contains a number of terms with various derivatives of $\bar{\mathfrak{f}}^{(\alpha)}_1$ and $\Delta\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$, which we do not write down explicitly. There terms are quadratically small in the parameter $\mathcal{K}$ and, therefore, they should be omitted. Further, choosing n_α and S as the independent thermodynamic variables,[6] we substitute the relations

$$\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial t}=\sum_{\alpha'}\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial n_{\alpha'}}\frac{\partial n_{\alpha'}}{\partial t}+\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial S}\frac{\partial S}{\partial t}, \tag{82}$$

$$\frac{\partial T}{\partial t}=\sum_{\alpha'}\frac{\partial T}{\partial n_{\alpha'}}\frac{\partial n_{\alpha'}}{\partial t}+\frac{\partial T}{\partial S}\frac{\partial S}{\partial t}, \tag{83}$$

$$\frac{\partial \gamma_{\alpha\alpha'}}{\partial t}=\sum_{\alpha'}\frac{\partial \gamma_{\alpha\alpha'}}{\partial n_{\alpha'}}\frac{\partial n_{\alpha'}}{\partial t}+\frac{\partial \gamma_{\alpha\alpha'}}{\partial S}\frac{\partial S}{\partial t}, \tag{84}$$

$$\frac{\partial \mu_\alpha}{\partial t}=\sum_{\alpha'}\frac{\partial \mu_\alpha}{\partial n_{\alpha'}}\frac{\partial n_{\alpha'}}{\partial t}+\frac{\partial \mu_\alpha}{\partial S}\frac{\partial S}{\partial t} \tag{85}$$

into Eq. (81). To calculate the time derivatives of n_α, we make use of the continuity equations (72), where the particle current density is given by Eq. (58). In order to calculate the time derivative of the entropy we apply the entropy equation (130), which has a natural form and will be derived from the kinetic equation in Sec. IV E.

Besides the expansion in Knudsen number we should work in the linear approximation in hydrodynamic velocities. Moreover, following the standard approach (see, e.g., Refs. [4,12,71]), we will omit all the terms in Eq. (81) that explicitly depend on the velocity difference $\boldsymbol{w}_\alpha$ (but not on its derivatives).[7] To simplify the calculations, we also consider a certain point in space where $\mathbf{u}=0$ at a given moment of time. Clearly, this assumption does not lead to any loss of generality, since it can always be fulfilled by choosing an appropriate inertial reference frame.

To calculate the time derivative of the momentum $\mathbf{Q}_\alpha$, we should use the superfluid equation (73). Before using it, it is necessary to relate the chemical potentials $\breve{\mu}_\alpha$ and μ_α. Generally, they differ for two reasons. First, as discussed above, the chemical potential itself can be defined in various ways (e.g., in different reference frames). This difference is of the second order smallness in hydrodynamic velocities and can be neglected (see Sec. III). Second, the quantity $\breve{\mu}_\alpha$ may contain dissipative corrections caused by the departure of the distribution functions $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ from those defined in local thermodynamic equilibrium. However, these differences are small, $\sim\mathcal{O}(\mathcal{K})$, and should be neglected in the Chapman-Enskog method after substituting Eq. (73) into the left-hand side of the kinetic equation. Thus, for our purposes, one could use Eq. (73) with $\breve{\mu}_\alpha$ replaced by μ_α.

With these comments in mind, Eq. (81) can be rewritten as

$$-\frac{\partial \mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}}\Bigg[\sum_{\alpha'}\gamma_{\alpha\alpha'}\frac{\mu_{\alpha'}}{c^2}\mathbf{p}\frac{\partial \mathbf{u}}{\partial t}+\sum_{\alpha'}\gamma_{\alpha\alpha'}\mathbf{p}\nabla\mu_{\alpha'}$$
$$-\sum_{\alpha'}\left(\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial n_{\alpha'}}+\frac{E^{(\alpha)}_{\mathbf{p}}}{T}\frac{\partial T}{\partial n_{\alpha'}}\right)\mathrm{div}\mathbf{j}_{\alpha'}$$
$$+S\left(\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial S}-\frac{E^{(\alpha)}_{\mathbf{p}}}{T}\frac{\partial T}{\partial S}\right)\mathrm{div}\mathbf{u}+\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial \mathbf{p}}\nabla(\mathbf{p}\mathbf{u})$$
$$+\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial \mathbf{p}}\frac{E^{(\alpha)}_{\mathbf{p}}}{T}\nabla T\Bigg]=I_\alpha[\mathcal{F}]. \tag{86}$$

It remains just to exclude the time derivative of the velocity $\mathbf{u}$. To this aim, let us multiply Eq. (86) by $\mathbf{p}$ and sum the result over the quantum states. Using Eqs. (31) and (55), we find

$$\sum_{\alpha'}R_{\alpha'\alpha}\frac{\mu_{\alpha'}}{c^2}\frac{\partial \mathbf{u}}{\partial t}+\sum_{\alpha'}R_{\alpha'\alpha}\nabla\mu_{\alpha'}+S_\alpha\nabla T=\sum_{\mathbf{p}\sigma}\mathbf{p}I_\alpha[\mathcal{F}]. \tag{87}$$

Note that some terms in Eq. (86) disappeared after the summation because they are antisymmetric in $\mathbf{p}$. In Eq. (87), S_α is the "partial entropy density" for particle species α,

$$S_\alpha=\sum_{\mathbf{p}\sigma}\sigma^{(\alpha)}_{\mathbf{p},0}, \tag{88}$$

where $\sigma^{(\alpha)}_{\mathbf{p},0}$ is given by the expression (18) with the current-free equilibrium distribution function (26) instead of $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$.

[6] Strictly speaking, besides n_α and S any thermodynamic quantity in a superfluid matter will also generally depend on the velocity difference squared, $\boldsymbol{w}_\alpha^2=[\mathbf{Q}_\alpha-(\mu_\alpha/c^2)\mathbf{u}]^2$ (see, e.g., Ref. [4] for a detailed discussion and Ref. [70] for a relativistic generalization). However, since we work in the linear approximation in hydrodynamic velocities, this dependence can be ignored.

[7] Accounting for these terms would lead to a substantial increase in the number of transport coefficients, even for a one-constituent superfluid liquid (see, e.g., Ref. [72]). However, as argued in the literature [12], these terms are small in comparison to the retained ones in the majority of applications. Since obtaining the most general form of the (linearized) transport equation is not among the goals of the present paper, we restrict ourselves to neglecting these terms in what follows.

It can be shown that

$$\sum_{\mathbf{p}\sigma}\sigma^{(\alpha)}_{\mathbf{p},0}=-\frac{1}{3}\sum_{\mathbf{p}\sigma}p\frac{E^{(\alpha)}_{\mathbf{p}}}{T}\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial p}\frac{\partial \mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}}. \tag{89}$$

From this equation it becomes apparent that the last term in the left-hand side of Eq. (87) came from the last term in the left-hand side of Eq. (86). The sum of the right-hand sides of Eq. (87) for all particle species must vanish due to momentum conservation (see Appendix E). As for the left-hand sides, summing them up and using the definition (59) we get[8]

$$\rho_q\frac{\partial \mathbf{u}}{\partial t}=-n_{qn}\nabla\mu_n-n_{qp}\nabla\mu_p-S\nabla T, \tag{90}$$

where

$$\rho_q=\sum_{\alpha}\frac{\mu_\alpha}{c^2}n_{q\alpha}. \tag{91}$$

Equation (90) has the form of the linearized Euler equation, in which the quantity ρ_q plays the role of the density of normal (nonsuperfluid) component of the liquid. However, it was argued in Sec. III that at least one of the normal number densities $n_{q\alpha}$ can be negative. In spite of this, the quantity ρ_q appears to be always non-negative. Indeed, substituting the definitions (59), (55), (34), and (35) into (91), and making use of the relation (14), one can represent ρ_q as

$$\rho_q=\frac{n_n n_p}{\mathcal{S}}[n_n m^*_n\Phi_n(1-\Phi_p)+n_p m^*_p\Phi_p(1-\Phi_n)]+\frac{\mathcal{S}_{\rm nsf}}{\mathcal{S}}\frac{n_n\mu_n+n_p\mu_p}{c^2}\Phi_n\Phi_p, \tag{92}$$

where $\mathcal{S}$ is given by Eq. (36) and $\mathcal{S}_{\rm nsf}$ is the value of $\mathcal{S}$ in a nonsuperfluid mixture [see Eq. (A16)]. According to the stability constraint (69), $\mathcal{S}>0$ and, consequently, $\mathcal{S}_{\rm nsf}>0$. Thus, the normal density $\rho_q\geqslant 0$ and vanishes only at $T=0$, as expected.

Plugging the time derivative from Eq. (90) into (86), one arrives at

$$\begin{aligned}-\frac{\partial \mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}}\Bigg\{&\sum_{\alpha'}\left[\gamma_{\alpha\alpha'}-\left(\gamma_{\alpha\alpha}\frac{\mu_\alpha}{c^2}+\gamma_{\alpha\beta}\frac{\mu_\beta}{c^2}\right)\frac{n_{q\alpha'}}{\rho_q}\right]\mathbf{p}\,\nabla\mu_{\alpha'}\\&-\left[\left(\gamma_{\alpha\alpha}\frac{\mu_\alpha}{c^2}+\gamma_{\alpha\beta}\frac{\mu_\beta}{c^2}\right)\frac{S}{\rho_q}\mathbf{p}-\frac{E^{(\alpha)}_{\mathbf{p}}}{T}\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial \mathbf{p}}\right]\nabla T\\&+\sum_{\alpha'}\left(\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial n_{\alpha'}}-\frac{E^{(\alpha)}_{\mathbf{p}}}{T}\frac{\partial T}{\partial n_{\alpha'}}\right)\mathrm{div}\mathbf{j}_{\alpha'}\\&+S\left(\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial S}-\frac{E^{(\alpha)}_{\mathbf{p}}}{T}\frac{\partial T}{\partial S}\right)\mathrm{div}\mathbf{u}+\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial \mathbf{p}}\nabla(\mathbf{p}\mathbf{u})\Bigg\}=I_\alpha[\mathcal{F}].\end{aligned} \tag{93}$$

This equation can be rewritten in a more canonical form as

$$\begin{aligned}-\frac{\partial \mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}}\Bigg\{&\sum_{\alpha'}\left[\gamma_{\alpha\alpha'}-\left(\gamma_{\alpha\alpha}\frac{\mu_\alpha}{c^2}+\gamma_{\alpha\beta}\frac{\mu_\beta}{c^2}\right)\frac{n_{q\alpha'}}{\rho_q}\right]\mathbf{p}\,\nabla\mu_{\alpha'}-\left[\left(\gamma_{\alpha\alpha}\frac{\mu_\alpha}{c^2}+\gamma_{\alpha\beta}\frac{\mu_\beta}{c^2}\right)\frac{S}{\rho_q}\mathbf{p}-\frac{E^{(\alpha)}_{\mathbf{p}}}{T}\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial \mathbf{p}}\right]\nabla T\\&+\left[\sum_{\alpha'}n_{\alpha'}\left(\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial n_{\alpha'}}-\frac{E^{(\alpha)}_{\mathbf{p}}}{T}\frac{\partial T}{\partial n_{\alpha'}}\right)+S\left(\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial S}-\frac{E^{(\alpha)}_{\mathbf{p}}}{T}\frac{\partial T}{\partial S}\right)+\frac{1}{3}\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial \mathbf{p}}\mathbf{p}\right]\mathrm{div}\mathbf{u}\\&+\sum_{\alpha'}\left(\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial n_{\alpha'}}-\frac{E^{(\alpha)}_{\mathbf{p}}}{T}\frac{\partial T}{\partial n_{\alpha'}}\right)\mathrm{div}(\mathbf{j}_{\alpha'}-n_{\alpha'}\mathbf{u})\\&+\frac{1}{2}\left(\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial p^i}p^j-\frac{1}{3}\delta^{ij}\frac{\partial E^{(\alpha)}_{\mathbf{p}}}{\partial \mathbf{p}}\mathbf{p}\right)\left(\frac{\partial u^i}{\partial r^j}+\frac{\partial u^j}{\partial r^i}-\frac{2}{3}\delta^{ij}\mathrm{div}\mathbf{u}\right)\Bigg\}=I_\alpha[\mathcal{F}].\end{aligned} \tag{94}$$

This is the general equation allowing one to study different transport processes in strongly interacting superfluid Fermi mixtures, such as the thermal conductivity, bulk and shear viscosities, and particle diffusion. In nonsuperfluid matter one has $\gamma_{\alpha\alpha}\to 1/m^*_\alpha$ and $\gamma_{\alpha\beta}\to 0$, and thus the in-medium effects in the left-hand side of Eq. (94) manifest themselves only through the renormalization of the particle masses, when m_α is replaced with m^*_α. In contrast, in a superfluid mixture, Eq. (94) explicitly depends on *all* the Landau parameters $f_1^{\alpha\alpha'}$ through the elements of the matrix $\gamma_{\alpha\alpha'}$ see Eqs. (34) and (35).

D. Diffusion currents

In the present paper we focus on the diffusion of particles produced by the gradients of chemical potentials. Assuming all other gradients in Eq. (94) vanish, we find

$$-\frac{\partial \mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}}\sum_{\alpha'}\left[\gamma_{\alpha\alpha'}-\left(\gamma_{\alpha\alpha}\frac{\mu_\alpha}{c^2}+\gamma_{\alpha\beta}\frac{\mu_\beta}{c^2}\right)\frac{n_{q\alpha'}}{\rho_q}\right]\mathbf{p}\,\nabla\mu_{\alpha'}=I_\alpha[\mathcal{F}]. \tag{95}$$

Let us multiply this equation by $\mathbf{p}$ and sum the result over quantum states. Using the expressions (31) and (55) together

[8]One can arrive at the same equation from the phenomenological hydrodynamics. To do this, one needs to substitute Eq. (C19) into (C10), accounting for the relation (56) and setting the electrical charges of all particle species to zero.

with the definitions (59) and (91), we obtain[9]

$$\frac{\mathbb{R}}{\rho_q}\frac{\mu_n\mu_p}{c^2}\left(\frac{\nabla\mu_n}{\mu_n}-\frac{\nabla\mu_p}{\mu_p}\right)=\sum_{\mathbf{p}\sigma}\mathbf{p}\,I_n[\mathcal{F}], \tag{96}$$

$$-\frac{\mathbb{R}}{\rho_q}\frac{\mu_n\mu_p}{c^2}\left(\frac{\nabla\mu_n}{\mu_n}-\frac{\nabla\mu_p}{\mu_p}\right)=\sum_{\mathbf{p}\sigma}\mathbf{p}\,I_p[\mathcal{F}], \tag{97}$$

where $\mathbb{R}=R_{nn}R_{pp}-R_{np}R_{pn}$.

Our immediate goal will be to find the corrections $\bar{\mathfrak{f}}_1^{(\alpha)}$ to the equilibrium distribution functions (77), and to calculate the particle current densities using the expression (37). Instead of the corrections $\bar{\mathfrak{f}}_1^{(\alpha)}$ it is convenient to introduce the functions ϕ_α through the relation [52]

$$\bar{\mathfrak{f}}_1^{(\alpha)}=-\frac{\partial\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}}{\partial\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}}\phi_\alpha=\frac{\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}\big(1-\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}\big)}{T}\phi_\alpha. \tag{98}$$

Following Refs. [15,73], we look for ϕ_α in the form[10]

$$\phi_\alpha=\mathbf{p}\mathbf{V}_{i\alpha}(\mathbf{p}), \tag{99}$$

where the vector $\mathbf{V}_{i\alpha}(\mathbf{p})$ is assumed to be a smooth function of $\mathbf{p}$. In a highly degenerate matter, collisions occur mostly between particles in the vicinity of the corresponding Fermi surfaces, where the derivative $-\partial\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}/\partial\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}$ has a sharp maximum, while the function $\mathbf{V}_{i\alpha}(\mathbf{p})$ is expected to vary only slightly. Hence, to a first approximation one may treat $\mathbf{V}_{i\alpha}$ as a constant vector. In the present paper, following Refs. [15,73], we will work within this approximation.

As was already mentioned above, besides the correction $\bar{\mathfrak{f}}_1^{(\alpha)}$, one also needs to determine the correction $\Delta\mathfrak{E}_{\mathbf{p}+\mathbf{Q}_\alpha}$ to the equilibrium Bogoliubov excitation energy [see the expression (80)]. To do this, we make use of the self-consistency relation (10), exactly as was done in Sec. III. Working in the linear approximation in vectors $\mathbf{V}_{i\alpha}$, the energy correction can be presented as

$$\Delta\mathfrak{E}^{(\alpha)}_{\mathbf{p}}=\mathbf{p}\sum_{\alpha'}\tilde{K}_{\alpha\alpha'}\mathbf{V}_{i\alpha'}, \tag{100}$$

where the coefficients $\tilde{K}_{\alpha\alpha'}$ are yet to be determined. Substituting this expression together with expressions (98) and (99) into the distribution function (77) and linearizing the result with respect to $\mathbf{V}_{i\alpha}$, one gets

$$\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}\approx\mathfrak{f}^{(\alpha)}_{\mathbf{p}}+\frac{\partial\mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}}\mathbf{p}\sum_{\alpha'}\gamma_{\alpha\alpha'}\boldsymbol{w}_{\alpha'}+\frac{\partial\mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}}\mathbf{p}\sum_{\alpha'}\tilde{K}_{\alpha\alpha'}\mathbf{V}_{i\alpha'}$$
$$-\frac{\partial\mathfrak{f}^{(\alpha)}_{\mathbf{p}}}{\partial E^{(\alpha)}_{\mathbf{p}}}\mathbf{p}\mathbf{V}_{i\alpha}. \tag{101}$$

Out of the thermodynamic equilibrium the Bogoliubov coherence factors are still given by Eqs. (6) and (8) [51,64]. Hence, the distribution functions $\mathcal{N}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ and $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ are still related by the expression (7). Then, linearizing Eq. (8) and using the fact that Bogoliubov coherence factors are even, while the representation (100) is odd with respect to transformation $\mathbf{p}\to-\mathbf{p}$, one can show that $\Delta H^{(\alpha)}_{\mathbf{p}}=\Delta\mathfrak{E}^{(\alpha)}_{\mathbf{p}}$, where $\Delta H^{(\alpha)}_{\mathbf{p}}$ is the nonequilibrium correction to the quantity $H^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$, given by Eq. (10). Plugging all the expansions into the expression (10), one can finally obtain the equations for the coefficients $\tilde{K}_{\alpha\alpha'}$; the detailed calculation is similar to that presented in Sec. III. As a consequence, the resulting equations coincides with Eq. (49), hence the matrix elements $\tilde{K}_{\alpha\alpha'}$ coincide with $K_{\alpha\alpha'}$, and are given by the expressions (50) and (51). The fact that $\tilde{K}_{\alpha\alpha'}=K_{\alpha\alpha'}$ allows us to use the expansion (22) with $\Delta H^{(\alpha)}_{\mathbf{p}}$ given by Eq. (48), where by $\mathbf{V}_{q\alpha}$ one should understand the sum $\mathbf{u}+\mathbf{V}_{i\alpha}$, i.e., $\mathbf{V}_{q\alpha}=\mathbf{u}+\mathbf{V}_{i\alpha}$.

Now, substituting the expansion (101) into Eq. (37) and using the relation (57) together with the definitions (38), (53), (54), and (59), one obtains

$$\mathbf{j}_\alpha=R_{\alpha\alpha}\mathbf{V}_{q\alpha}+R_{\alpha\beta}\mathbf{V}_{q\beta}+c^2Y_{\alpha\alpha}\mathbf{Q}_\alpha+c^2Y_{\alpha\beta}\mathbf{Q}_\beta \tag{102}$$

$$=n_{q\alpha}\mathbf{u}+c^2Y_{\alpha\alpha}\mathbf{Q}_\alpha+c^2Y_{\alpha\beta}\mathbf{Q}_\beta+\Delta\mathbf{j}_\alpha. \tag{103}$$

Here, the expression in the first line is completely identical to that in Eq. (52). In the second line the same result is presented in the form more suitable for establishing the connection with the phenomenological hydrodynamics. Equation (103) introduces the diffusion currents according to definition

$$\Delta\mathbf{j}_\alpha=R_{\alpha\alpha}\mathbf{V}_{i\alpha}+R_{\alpha\beta}\mathbf{V}_{i\beta}. \tag{104}$$

Inverting the relations (104), one obtains

$$\mathbf{V}_{in}=\frac{R_{pp}\Delta\mathbf{j}_n-R_{np}\Delta\mathbf{j}_p}{\mathbb{R}},\quad \mathbf{V}_{ip}=\frac{R_{nn}\Delta\mathbf{j}_p-R_{pn}\Delta\mathbf{j}_n}{\mathbb{R}}. \tag{105}$$

To complete the derivation one also needs to define the comoving reference frame in which $\mathbf{u}=0$. This can be done in a number of ways. We prefer to choose the so-called Landau-Lifshitz frame, in which the additional momentum density caused by the dissipative currents (diffusion currents in our case) equals zero [12]. Using Eq. (42), this requirement translates into

$$\sum_\alpha\frac{\mu_\alpha}{c^2}\Delta\mathbf{j}_\alpha=0. \tag{106}$$

A similar condition is used in the study of nonsuperfluid mixtures in Refs. [15,74].

Let us now consider the right-hand side of Eqs. (96) and (97). After substitution of the distribution functions (77) together with expressions (98) and (99) into the collision integral, it can be represented in the following form (see

[9]The same equations can be obtained if one substitute the time derivative from Eq. (90) into Eqs. (87), where ∇T is set to zero.

[10]It should be stressed that if a set of functions $\mathbf{V}_{i\alpha}(\mathbf{p})$ satisfies the transport equation, then another set $\mathbf{V}_{i\alpha}(\mathbf{p})+\mathbf{a}$, where $\mathbf{a}$ is an arbitrary vector, would satisfy the same equation (see also Appendix E). From the physical point of view, this ambiguity is related to the fact that the definition of the velocity $\mathbf{u}$ in the dissipative hydrodynamics is not unique (see, e.g., Ref. [12]). Hence, an additional constraint should be applied to the solution. The choice of the constraint is discussed after the expression (104).

Appendix E for details):

$$\sum_{\mathbf{p}\sigma} \mathbf{p}\, I_\alpha[\mathcal{F}] = -J_{\alpha\beta}(\mathbf{V}_{i\alpha} - \mathbf{V}_{i\beta}), \tag{107}$$

where $J_{\alpha\beta}$ are the momentum transfer rates ($J_{\alpha\beta} = J_{\beta\alpha}$).[11] Use of Eqs. (96), (97), (105), and (107) allows us to find

$$\Delta\mathbf{j}_p = -\frac{\mu_p\mu_n^2}{c^4\rho_q^2}\frac{\mathbb{R}^2}{J_{np}}\left(\frac{\nabla\mu_p}{\mu_p} - \frac{\nabla\mu_n}{\mu_n}\right),$$
$$\Delta\mathbf{j}_n = \frac{\mu_n\mu_p^2}{c^4\rho_q^2}\frac{\mathbb{R}^2}{J_{np}}\left(\frac{\nabla\mu_p}{\mu_p} - \frac{\nabla\mu_n}{\mu_n}\right). \tag{108}$$

The determinant of the matrix $R_{\alpha\alpha'}$ can be represented, with the help of Eqs. (34)–(36) and (55), as

$$\mathbb{R} = n_n n_p \Phi_n \Phi_p \frac{\mathcal{S}_{\text{nsf}}}{\mathcal{S}}, \tag{109}$$

where we recall that $\mathcal{S}_{\text{nsf}}$ is the value of the quantity $\mathcal{S}$ taken for a nonsuperfluid mixture [see Eq. (A16)].

The relation (108) allows us to express the diffusion coefficients arising in relativistic superfluid hydrodynamics [10,74] (see Appendix B) in a practical case of low-temperature degenerate isothermal matter ($\nabla T = 0$). For uncharged binary mixture, the expression (C14), written down for both particle species "n" and "p", reads

$$\Delta\mathbf{j}_n = -\mathcal{D}_{nn}\frac{\nabla\mu_n}{T} - \mathcal{D}_{np}\frac{\nabla\mu_p}{T}, \tag{110}$$

$$\Delta\mathbf{j}_p = -\mathcal{D}_{pn}\frac{\nabla\mu_n}{T} - \mathcal{D}_{pp}\frac{\nabla\mu_p}{T}. \tag{111}$$

Plugging Eqs. (106) and (B18) into the expression (111) and comparing the result with Eq. (110), we find

$$\mathcal{D}_{nn} = -\frac{\mu_p}{\mu_n}\mathcal{D}_{np}, \quad \mathcal{D}_{pp} = -\frac{\mu_n}{\mu_p}\mathcal{D}_{np}, \tag{112}$$

and

$$\Delta\mathbf{j}_p = \frac{\mathcal{D}_{np}\mu_n}{T}\left(\frac{\nabla\mu_p}{\mu_p} - \frac{\nabla\mu_n}{\mu_n}\right). \tag{113}$$

Finally, comparing Eq. (113) with (108), we arrive at

$$\mathcal{D}_{np} = -\frac{\mu_p\mu_n T}{c^4\rho_q^2}\frac{\mathbb{R}^2}{J_{np}}. \tag{114}$$

This is one of the central results of our work showing how the Fermi-liquid effects and superfluidity manifest themselves in the diffusion coefficients of the mixture. To expose the Fermi-liquid effects in this formula, let us consider a few special cases. If one deals with a mixture of two independent Fermi liquids ($G_{\alpha\beta} = 0$), the diffusion coefficient takes the form

$$\mathcal{D}_{np} = -\frac{\mu_p\mu_n n_{qn}^2 n_{qp}^2}{(\mu_n n_{qn} + \mu_p n_{qp})^2}\frac{T}{J_{np}}, \tag{115}$$

where (see Appendix A)

$$n_{q\alpha} = R_{\alpha\alpha} = \frac{n_\alpha m_\alpha^* c^2 \Phi_\alpha}{m_\alpha^* c^2 \Phi_\alpha + \mu_\alpha(1 - \Phi_\alpha)}. \tag{116}$$

If all the Fermi-liquid effects can be neglected, Eq. (115) remains valid, while the normal density reduces to

$$n_{q\alpha} = n_\alpha \Phi_\alpha. \tag{117}$$

Hence, the Fermi-liquid effects in the diffusion coefficient $\mathcal{D}_{np}$ arise by replacing $n_{qn}^2 n_{qp}^2$ with $\mathbb{R}^2$ in the numerator, while employing the general expression (59) for the normal densities in the denominator. In the case of a nonsuperfluid mixture, the relation (114) turns into (cf. Eq. (113) in Ref. [74])[12]

$$\mathcal{D}_{np} = -\frac{\mu_p\mu_n n_n^2 n_p^2 T}{(\mu_n n_n + \mu_p n_p)^2 J_{np}}. \tag{118}$$

E. Entropy

Let us now derive the equation for the entropy density. Multiplying Eq. (71) by $\ln[(1 - \mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha})/\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}]$ and summing the result over the quantum states and particle species indices, one obtains

$$\frac{\partial S}{\partial t} + \text{div}\left(\sum_{\mathbf{p}\sigma\alpha} \frac{\partial \mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}}{\partial \mathbf{p}} \sigma^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}\right) = \Gamma_s, \tag{119}$$

where

$$\Gamma_s = -\sum_{\mathbf{p}\sigma\alpha} \ln\left(\frac{\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}}{1 - \mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}}\right) I_\alpha[\mathcal{F}] \tag{120}$$

is the entropy production rate.

Let us first consider the expression (120). Plugging the expansion (77) together with Eq. (98) into it and linearizing the result with respect to ϕ_α, one gets

$$\Gamma_s = -\sum_{\mathbf{p}\sigma\alpha} \ln\left(\frac{\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}}{1 - \overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}}\right) I_\alpha[\mathcal{F}] - \frac{1}{T}\sum_{\mathbf{p}\sigma\alpha} \phi_\alpha I_\alpha[\mathcal{F}]$$
$$= \frac{1}{T}\sum_{\mathbf{p}\sigma\alpha} \left(\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} - \mathbf{p}\mathbf{u}\right) I_\alpha[\mathcal{F}] - \frac{1}{T}\sum_{\mathbf{p}\sigma\alpha} \phi_\alpha I_\alpha[\mathcal{F}]. \tag{121}$$

The first term in this expression equals zero due to the energy and momentum conservation in collision events. Substitution

[11]Taking this property into account, one can easily see that Eqs. (96) and (97), in fact, coincide.

[12]The expression in Ref. [74] contains an additional constant c in the denominator because the diffusion currents in that paper have the dimension of number density.

of Eq. (95) into the second term gives

$$\Gamma_s = \frac{1}{T}\sum_{\alpha\alpha'}\left[\gamma_{\alpha\alpha'} - \left(\gamma_{\alpha\alpha}\frac{\mu_\alpha}{c^2} + \gamma_{\alpha\beta}\frac{\mu_\beta}{c^2}\right)\frac{n_{q\alpha'}}{\rho_q}\right]\nabla\mu_{\alpha'}\sum_{\mathbf{p}\sigma}\mathbf{p}\phi_\alpha$$
$$\times\frac{\partial \mathfrak{f}_{\mathbf{p}}^{(\alpha)}}{\partial E_{\mathbf{p}}^{(\alpha)}} = \frac{1}{T}\frac{\mathbb{R}}{\rho_q}\frac{\mu_n\mu_p}{c^2}\left(\frac{1}{m_n^* n_n\Phi_n}\sum_{\mathbf{p}\sigma}\mathbf{p}\phi_n\frac{\partial \mathfrak{f}_{\mathbf{p}}^{(n)}}{\partial E_{\mathbf{p}}^{(n)}}\right.$$
$$\left. - \frac{1}{m_p^* n_p\Phi_p}\sum_{\mathbf{p}\sigma}\mathbf{p}\phi_p\frac{\partial \mathfrak{f}_{\mathbf{p}}^{(p)}}{\partial E_{\mathbf{p}}^{(p)}}\right)\left(\frac{\nabla\mu_n}{\mu_n} - \frac{\nabla\mu_p}{\mu_p}\right), \tag{122}$$

where to obtain the second equality we used Eq. (55). Now, plugging expression (99) together with Eq. (105) into (122), and using Eqs. (31) and (59), one finds

$$\Gamma_s = \frac{1}{T}\frac{\mu_n\mu_p}{c^2}\frac{n_{qn}\Delta\mathbf{j}_p - n_{qp}\Delta\mathbf{j}_n}{\rho_q}\left(\frac{\nabla\mu_n}{\mu_n} - \frac{\nabla\mu_p}{\mu_p}\right). \tag{123}$$

In view of the definition (91) and the relation (106), one can transform Eq. (123) into

$$\Gamma_s = -\frac{\mu_p}{T}\left(\frac{\nabla\mu_p}{\mu_p} - \frac{\nabla\mu_n}{\mu_n}\right)\Delta\mathbf{j}_p. \tag{124}$$

Expressing $\Delta\mathbf{j}_p$ from Eq. (108) one verifies that $\Gamma_s \geqslant 0$, as it should be. Applying again the relation (106) to the expression (124), one gets

$$\Gamma_s = -\frac{1}{T}\sum_\alpha \Delta\mathbf{j}_\alpha\nabla\mu_\alpha. \tag{125}$$

In this form, the entropy production rate coincides with the phenomenological relation (C20) for uncharged mixture at constant T. In the limit of nonrelativistic equation of state, the entropy production rate reduces to the standard expression [12]:

$$\Gamma_s = -\frac{1}{T}\left(\frac{\nabla\mu_n}{m_n} - \frac{\nabla\mu_p}{m_p}\right)m_p\Delta\mathbf{j}_p. \tag{126}$$

Let us now turn to the calculation of the entropy current density [see the term under divergence in Eq. (119)]. Linearizing it using Eqs. (18), (22), (46), and (48), one obtains[13]

$$\sum_{\mathbf{p}\sigma\alpha}\frac{\partial\mathfrak{E}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}}{\partial\mathbf{p}}\sigma_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} \approx \sum_{\mathbf{p}\sigma\alpha}\left[\frac{\partial\Delta H_{\mathbf{p}}^{(\alpha)}}{\partial\mathbf{p}}\sigma_{\mathbf{p},0}^{(\alpha)} + \frac{E_{\mathbf{p}}^{(\alpha)}}{T}\frac{\partial E_{\mathbf{p}}^{(\alpha)}}{\partial\mathbf{p}}\frac{\partial\mathfrak{f}_{\mathbf{p}}^{(\alpha)}}{\partial E_{\mathbf{p}}^{(\alpha)}}\right.$$
$$\left.\times\left(\Delta H_{\mathbf{p}}^{(\alpha)} - \mathbf{p}\mathbf{V}_{q\alpha}\right)\right], \tag{127}$$

where $\sigma_{\mathbf{p},0}^{(\alpha)}$ is the entropy density calculated for the distribution function (26). Substituting Eq. (48) into (127) and accounting for Eqs. (89), one sees that the first two terms in the expression (127) cancel out. Thus, the entropy current density becomes

$$\sum_{\mathbf{p}\sigma\alpha}\frac{\partial\mathfrak{E}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}}{\partial\mathbf{p}}\sigma_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} = \sum_\alpha S_\alpha\mathbf{V}_{q\alpha}, \tag{128}$$

where S_α is the partial entropy density given by the expression (88). Consequently, the entropy generation equation (119) takes the form

$$\frac{\partial S}{\partial t} + \mathrm{div}(S_n\mathbf{V}_{qn} + S_p\mathbf{V}_{qp}) = \Gamma_s. \tag{129}$$

One sees that the vectors $\mathbf{V}_{q\alpha} = \mathbf{u} + \mathbf{V}_{i\alpha}$ have the meaning of the corresponding *partial entropy velocities*. If the dissipative terms can be ignored, this equation reduces to the standard nondissipative form [12]

$$\frac{\partial S}{\partial t} + \mathrm{div}(S\mathbf{u}) = 0. \tag{130}$$

For a deeper understanding of the nature of the velocities $\mathbf{V}_{q\alpha}$ let us sum Eq. (71) over the quantum states. As a result, we obtain the "continuity equation" for the Bogoliubov excitations that has the form

$$\frac{\partial}{\partial t}\sum_{\mathbf{p}\sigma}\mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} + \mathrm{div}\left(\sum_{\mathbf{p}\sigma}\frac{\partial\mathfrak{E}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}}{\partial\mathbf{p}}\mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}\right) = \sum_{\mathbf{p}\sigma}I_\alpha[\mathcal{F}]. \tag{131}$$

Note that the number of Bogoliubov excitations is not necessarily conserved (see Appendix E for details). Hence, the right-hand side of Eq. (131) is generally nonzero. Linearizing the current density of Bogoliubov excitations, one gets

$$\sum_{\mathbf{p}\sigma}\frac{\partial\mathfrak{E}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}}{\partial\mathbf{p}}\mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} \approx \sum_{\mathbf{p}\sigma}\left[\frac{\partial\Delta H_{\mathbf{p}}^{(\alpha)}}{\partial\mathbf{p}}\mathfrak{f}_{\mathbf{p}}^{(\alpha)} + \frac{\partial E_{\mathbf{p}}^{(\alpha)}}{\partial\mathbf{p}}\frac{\partial\mathfrak{f}_{\mathbf{p}}^{(\alpha)}}{\partial E_{\mathbf{p}}^{(\alpha)}}\right.$$
$$\left.\times\left(\Delta H_{\mathbf{p}}^{(\alpha)} - \mathbf{p}\mathbf{V}_{q\alpha}\right)\right]. \tag{132}$$

It is easy to see that the first two terms under the sum here can be combined into the total derivative over $\mathbf{p}$. Hence, these terms disappear after the summation and one finally obtains

$$\sum_{\mathbf{p}\sigma}\frac{\partial\mathfrak{E}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}}{\partial\mathbf{p}}\mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} \approx \sum_{\mathbf{p}\sigma}\mathfrak{f}_{\mathbf{p}}^{(\alpha)}\,\mathbf{V}_{q\alpha}. \tag{133}$$

Thus, one can say that the partial entropy moves together with the Bogoliubov excitations. This result might be expected, since the distribution function $\mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}$ of these excitations determines the partial entropy in the system [see Eqs. (17) and (88)]. However, the nontrivial result is that in a strongly interacting mixture the partial entropy current is generally *not collinear* with the normal current of the corresponding particle species [see the first two terms in Eq. (52)]. This property is in sharp contrast with the case of a nonsuperfluid mixture, where these currents are always collinear, so that

$$\mathbf{j}_\alpha = n_\alpha\mathbf{V}_{q\alpha}. \tag{134}$$

To understand why the normal current densities in a superfluid mixture generally depend on both the velocities $\mathbf{V}_{qn}$ and $\mathbf{V}_{qp}$, consider, for simplicity, a point in space where $\mathbf{Q}_n = \mathbf{Q}_p = 0$.

[13]We remind the reader that, as argued in Sec. IV, these equations can be used if one identifies $\mathbf{V}_{q\alpha}$ with $\mathbf{u} + \mathbf{V}_{i\alpha}$.

Then, substituting Eqs. (20) and (47) into (37) and linearizing the obtained expression with respect to $\mathbf{V}_{q\alpha}$, one gets

$$\begin{aligned}\mathbf{j}_\alpha &= \sum_{\mathbf{p}\sigma}\left(\frac{\partial\varepsilon_{\mathbf{p}}^{(\alpha)}}{\partial\mathbf{p}}\frac{\partial\mathfrak{f}_{\mathbf{p}}^{(\alpha)}}{\partial E_{\mathbf{p}}}\Delta H_{\mathbf{p}}^{(\alpha)} + \frac{\partial\Delta H_{\mathbf{p}}^{(\alpha)}}{\partial\mathbf{p}}n_{\mathbf{p}}^{(\alpha)}\right)\\ &\quad - \sum_{\mathbf{p}\sigma}\frac{\partial\varepsilon_{\mathbf{p}}^{(\alpha)}}{\partial\mathbf{p}}\frac{\partial\mathfrak{f}_{\mathbf{p}}^{(\alpha)}}{\partial E_{\mathbf{p}}}\mathbf{p}\mathbf{V}_{q\alpha}\\ &= \sum_{\mathbf{p}\sigma}\left(\frac{\partial\varepsilon_{\mathbf{p}}^{(\alpha)}}{\partial\mathbf{p}}\frac{\partial\mathfrak{f}_{\mathbf{p}}^{(\alpha)}}{\partial E_{\mathbf{p}}} - \frac{\partial n_{\mathbf{p}}^{(\alpha)}}{\partial\mathbf{p}}\right)\Delta H_{\mathbf{p}}^{(\alpha)}\\ &\quad - \sum_{\mathbf{p}\sigma}\frac{\partial\varepsilon_{\mathbf{p}}^{(\alpha)}}{\partial\mathbf{p}}\frac{\partial\mathfrak{f}_{\mathbf{p}}^{(\alpha)}}{\partial E_{\mathbf{p}}}\mathbf{p}\mathbf{V}_{q\alpha},\end{aligned} \tag{135}$$

where the second equality is obtained after integration by parts. In the nonsuperfluid mixture, the first sum in this equation vanishes and one arrives at the usual expression for the particle current density (see, e.g., Ref. [52]). In fact, it is not necessary to know the energy correction $\Delta H_{\mathbf{p}}^{(\alpha)}$ to calculate $\mathbf{j}_\alpha$ in this case. In contrast, in the superfluid mixture, this sum becomes comparable to the second one and hence additional terms from $\Delta H_{\mathbf{p}}^{(\alpha)}$, containing both the vectors $\mathbf{V}_{qn}$ and $\mathbf{V}_{qp}$, come into play [see Eq. (48)].

V. CHARGED MIXTURES

In this section we generalize the results of the previous section to charged mixtures (e.g., neutrons, protons, and electrons). For simplicity, we assume that the magnetic field in the system is absent. If the particle species α possesses an electrical charge e_α, the "superfluid" equation (73) should be replaced with [51]

$$\frac{\partial\mathbf{Q}_\alpha}{\partial t} = -\nabla\breve{\mu}_\alpha + e_\alpha\mathbf{E}, \tag{136}$$

where $\mathbf{E}$ is the electric field. As before, the difference between $\breve{\mu}_\alpha$ and μ_α will be ignored in the subsequent analysis. For the sake of convenience, let us introduce the vectors $\mathbf{b}_\alpha = \nabla\mu_\alpha - e_\alpha\mathbf{E}$, so that Eq. (136) becomes

$$\frac{\partial\mathbf{Q}_\alpha}{\partial t} = -\mathbf{b}_\alpha. \tag{137}$$

For our particular system $e_n = 0$, $e_p = q$, where q is the elementary charge. To make the electrical quasineutrality possible, we should add a third constituent labeled "e" (e.g., electrons) with $e_e = -q$. This constituent is assumed to be composed of noninteracting fermions (an ideal Fermi gas). The kinetic equation for the "e" constituent takes the form [75]

$$\frac{\partial\mathcal{N}_{\mathbf{p}}^{(e)}}{\partial t} + \frac{\partial\varepsilon_{\mathbf{p}}^{(e)}}{\partial\mathbf{p}}\frac{\partial\mathcal{N}_{\mathbf{p}}^{(e)}}{\partial\mathbf{r}} - e_e\mathbf{E}\frac{\partial\mathcal{N}_{\mathbf{p}}^{(e)}}{\partial\mathbf{p}} = I^{(e)}[\mathcal{F},\mathcal{N}], \tag{138}$$

where by $\mathcal{F}$ and $\mathcal{N}$ we denote, respectively, the set of distribution functions $\mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}$ ($\alpha = n,\ p$) and $\mathcal{N}_{\mathbf{p}}^{(e)}$. The partial entropy associated with the particle species "e" is given by

$$S_e = \sum_{\mathbf{p}\sigma}\sigma_{\mathbf{p}}^{(e)}, \tag{139}$$

where

$$\sigma_{\mathbf{p}}^{(e)} = -\left(1-\mathcal{N}_{\mathbf{p}}^{(e)}\right)\ln\left(1-\mathcal{N}_{\mathbf{p}}^{(e)}\right) - \mathcal{N}_{\mathbf{p}}^{(e)}\ln\mathcal{N}_{\mathbf{p}}^{(e)}. \tag{140}$$

The approximate solution of Eq. (138) is sought in the form

$$\mathcal{N}_{\mathbf{p}}^{(e)} = \mathcal{N}_{\mathbf{p},0}^{(e)} + \mathfrak{n}_1^{(e)} = \mathcal{N}_{\mathbf{p},0}^{(e)} - \frac{\partial\mathcal{N}_{\mathbf{p},0}^{(e)}}{\partial\varepsilon_{\mathbf{p}}^{(e)}}\phi_e, \tag{141}$$

where the equilibrium distribution function is given by the usual Fermi-Dirac distribution,

$$\mathcal{N}_{\mathbf{p},0}^{(e)} = \frac{1}{1+e^{(\varepsilon_{\mathbf{p}}^{(e)}-\mu_e-\mathbf{p}\,\mathbf{u})/T}}, \tag{142}$$

Note that, in contrast to superfluids "n" and "p", the departure from the equilibrium does not modify the energy $\varepsilon_{\mathbf{p}}^{(e)}$, since it is independent of the distribution functions.

Repeating the derivation of Eq. (87) from Sec. IV, one obtains for $\alpha = n, p$

$$\sum_{\alpha'=n,p}R_{\alpha'\alpha}\frac{\mu_{\alpha'}}{c^2}\frac{\partial\mathbf{u}}{\partial t} + \sum_{\alpha'=n,p}R_{\alpha'\alpha}\mathbf{b}_{\alpha'} + S_\alpha\nabla T = \sum_{\mathbf{p}\sigma}\mathbf{p}I_\alpha[\mathcal{F},\mathcal{N}]. \tag{143}$$

In turn, similar derivation for particles "e" yields

$$n_e\frac{\mu_e}{c^2}\frac{\partial\mathbf{u}}{\partial t} + n_e\mathbf{b}_e + S_e\nabla T = \sum_{\mathbf{p}\sigma}\mathbf{p}I_e[\mathcal{F},\mathcal{N}]. \tag{144}$$

Our purpose is to find the connection between the dissipative currents and the vectors $\mathbf{b}_\alpha$. To this aim, one can again assume the ansatz (99) for the superfluid species "n" and "p", and the similar ansatz

$$\phi_e = \mathbf{p}\,\mathbf{V}_{ie} \tag{145}$$

for the species "e". Here, as before, the dependence of the vectors $\mathbf{V}_{i\alpha}$ on the momentum variable $\mathbf{p}$ is neglected. Hence, following the derivation of Sec. IV D, one arrives at the following expressions for the diffusion current densities [cf. Eq. (104)]:

$$\Delta\mathbf{j}_n = R_{nn}\mathbf{V}_{in} + R_{np}\mathbf{V}_{ip}, \tag{146}$$

$$\Delta\mathbf{j}_p = R_{pn}\mathbf{V}_{in} + R_{pp}\mathbf{V}_{ip}, \tag{147}$$

$$\Delta\mathbf{j}_e = n_e\mathbf{V}_{ie}. \tag{148}$$

Plugging now expansions (77) and (141) into the right-hand sides of Eqs. (143) and (144), in which the temperature gradient is set to zero, one finds

$$\begin{aligned}&\left(R_{nn}\frac{\mu_n}{c^2} + R_{pn}\frac{\mu_p}{c^2}\right)\frac{\partial\mathbf{u}}{\partial t} + R_{nn}\mathbf{b}_n + R_{pn}\mathbf{b}_p\\ &\quad = J_{pn}\mathbf{w}_{pn} + J_{en}\mathbf{w}_{en},\end{aligned} \tag{149}$$

$$\begin{aligned}&\left(R_{pp}\frac{\mu_p}{c^2} + R_{np}\frac{\mu_n}{c^2}\right)\frac{\partial\mathbf{u}}{\partial t} + R_{pp}\mathbf{b}_p + R_{np}\mathbf{b}_n\\ &\quad = -J_{pn}\mathbf{w}_{pn} - J_{pe}\mathbf{w}_{pe},\end{aligned} \tag{150}$$

$$n_e\frac{\mu_e}{c^2}\frac{\partial\mathbf{u}}{\partial t} + n_e\mathbf{b}_e = -J_{en}\mathbf{w}_{en} + J_{pe}\mathbf{w}_{pe}, \tag{151}$$

where we introduced a notation

$$\mathbf{w}_{\alpha\beta} = \mathbf{V}_{i\alpha} - \mathbf{V}_{i\beta} \tag{152}$$

and where $J_{\alpha\beta}$ are the momentum transfer rates (see Appendix E). To exclude the time derivative of the velocity $\mathbf{u}$ from these equations, one can sum them up, which results in

$$\rho_q \frac{\partial \mathbf{u}}{\partial t} = -n_{qn}\mathbf{b}_n - n_{qp}\mathbf{b}_p - n_e\mathbf{b}_e, \tag{153}$$

where the normal energy density ρ_q now equals

$$\rho_q = \frac{\mu_n}{c^2}n_{qn} + \frac{\mu_p}{c^2}n_{qp} + \frac{\mu_e}{c^2}n_e. \tag{154}$$

After $\partial\mathbf{u}/\partial t$ is excluded, only two out of three of Eqs. (149)–(151) remain independent. Taking, for instance, Eqs. (150) and (151), using Eq. (153), and excluding $\mathbf{w}_{pe}$ with the identity $\mathbf{w}_{pe} \equiv \mathbf{w}_{pn} - \mathbf{w}_{en}$, one obtains

$$-\frac{\mu_p\mu_n\mathbb{R} + \mu_e\mu_p n_e R_{pp}}{c^2\rho_q}\left(\frac{\mathbf{b}_n}{\mu_n} - \frac{\mathbf{b}_p}{\mu_p}\right) + \frac{\mu_e\mu_n n_e R_{np} + \mu_e\mu_p n_e R_{pp}}{c^2\rho_q}\left(\frac{\mathbf{b}_n}{\mu_n} - \frac{\mathbf{b}_e}{\mu_e}\right) = -(J_{pn} + J_{pe})\mathbf{w}_{pn} + J_{pe}\mathbf{w}_{en}, \tag{155}$$

$$\frac{\mu_e\mu_p n_e n_{qp}}{c^2\rho_q}\left(\frac{\mathbf{b}_n}{\mu_n} - \frac{\mathbf{b}_p}{\mu_p}\right) - \mu_e n_e\frac{\mu_n n_{qn} + \mu_p n_{qp}}{c^2\rho_q}\left(\frac{\mathbf{b}_n}{\mu_n} - \frac{\mathbf{b}_e}{\mu_e}\right) = J_{pe}\mathbf{w}_{pn} - (J_{en} + J_{pe})\mathbf{w}_{en}. \tag{156}$$

Equations (155)–(156) allow us to express the vectors $\mathbf{w}_{\alpha\beta}$ through the vectors $\mathbf{d}_\alpha$. In turn, Eqs. (146)–(148) together with the condition (106) relate the vectors $\Delta\mathbf{j}_\alpha$ and $\mathbf{w}_{\alpha\beta}$:

$$\Delta\mathbf{j}_n = -\frac{(\mu_p\mathbb{R} - \mu_e n_e R_{np})\mathbf{w}_{pn} + \mu_e n_e n_{qn}\mathbf{w}_{en}}{c^2\rho_q}, \tag{157}$$

$$\Delta\mathbf{j}_p = \frac{(\mu_n\mathbb{R} + \mu_e n_e R_{pp})\mathbf{w}_{pn} - \mu_e n_e n_{qp}\mathbf{w}_{en}}{c^2\rho_q}, \tag{158}$$

$$\Delta\mathbf{j}_e = n_e\frac{(c^2\rho_q - \mu_e n_e)\mathbf{w}_{en} - (\mu_p R_{pp} + \mu_n R_{np})\mathbf{w}_{pn}}{c^2\rho_q}. \tag{159}$$

In the case of isothermal matter, the phenomenological expression for the diffusion currents is [cf. Eq. (C14)]

$$\Delta\mathbf{j}_\alpha = -\sum_{\alpha'}\mathcal{D}_{\alpha\alpha'}\frac{\mathbf{b}_{\alpha'}}{T}. \tag{160}$$

The relation (106) allows us to exclude the diagonal diffusion coefficients $\mathcal{D}_{\alpha\alpha}$ and, after accounting for Eq. (B18), one arrives at

$$\Delta\mathbf{j}_\alpha = \sum_{\beta\neq\alpha}\mathcal{D}_{\alpha\beta}\left(\frac{\mathbf{b}_\beta}{\mu_\beta} - \frac{\mathbf{b}_\alpha}{\mu_\alpha}\right). \tag{161}$$

Now, plugging the quantities $\mathbf{w}_{\alpha\alpha'}$ obtained as a solution to Eqs. (155)–(156) into Eqs. (158)–(159) and comparing the result with the phenomenological expression (161), one can find the formulas for the diffusion coefficients $\mathcal{D}_{\alpha\alpha'}$. We prefer not to present these lengthy expressions here, but they can be easily found if necessary. Clearly, the procedure described above can be easily extended to a mixture of arbitrary number of species.

What remains to be done is to find an expression for the entropy production rate. Repeating the derivation from Sec. IV E, one gets

$$\Gamma_s = -\frac{1}{T}\sum_{\mathbf{p}\sigma\alpha}\mathbf{p}\mathbf{V}_{i\alpha}I_\alpha[\mathcal{F},\mathcal{N}] = -\frac{1}{T}\sum_{\mathbf{p}\sigma}(\mathbf{w}_{pn}\mathbf{p}I_p[\mathcal{F},\mathcal{N}] + \mathbf{w}_{en}\mathbf{p}I_e[\mathcal{F},\mathcal{N}]). \tag{162}$$

Here, the first equality coincides with Eq. (121) except that the summation is now performed over the three particle species, $\alpha = n,\ p,\ e$. In the second equality, we used the fact that $\sum_{\mathbf{p}\sigma\alpha}\mathbf{p}I_\alpha[\mathcal{F},\mathcal{N}] = 0$. Plugging in the left-hand side of Eqs. (155) and (156) instead of $\sum_{\mathbf{p}\sigma}\mathbf{p}I_\alpha[\mathcal{F},\mathcal{N}]$ and using expressions (158) and (159), we arrive at

$$\Gamma_s = -\frac{1}{T}\left[\mu_p\Delta\mathbf{j}_p\left(\frac{\mathbf{b}_p}{\mu_p} - \frac{\mathbf{b}_n}{\mu_n}\right) + \mu_e\Delta\mathbf{j}_e\left(\frac{\mathbf{b}_e}{\mu_e} - \frac{\mathbf{b}_n}{\mu_n}\right)\right]. \tag{163}$$

To bring this expression into the form of the phenomenological Eq. (C20), one needs just to apply the relation (106). To verify that the entropy production rate (163) is non-negative, one should substitute the right-hand sides of Eqs. (155) and (156) instead of $\sum_{\mathbf{p}\sigma}\mathbf{p}I_\alpha[\mathcal{F},\mathcal{N}]$ into Eq. (162). The resulting quadratic form

$$\Gamma_s = \frac{1}{T}\left(J_{pn}\mathbf{w}_{pn}^2 + J_{en}\mathbf{w}_{en}^2 + J_{pe}\mathbf{w}_{pe}^2\right) \tag{164}$$

is obviously positive-definite.

VI. SUMMARY

In this paper we have developed a general formalism for studying particle diffusion in superfluid mixtures of strongly interacting Fermi liquids. Our results can be summarized as follows:

(1) The diffusion in superfluid mixtures is manifested through the modification of the expressions for the normal currents of all particle species $\alpha = n, p$ in the mixture. These normal currents can be introduced into the theory by minimizing the thermodynamic potential (44). They can be expressed in terms of the velocities $\mathbf{V}_{q\alpha}$, which are conjugate variables to the momenta of species α. In the approximation of small velocities, the normal component of each current density $\mathbf{j}_\alpha$ is a linear combination of the velocities $\mathbf{V}_{qn}$ and $\mathbf{V}_{qp}$. The coefficients in this linear combinations constitute the normal entrainment matrix, $R_{\alpha\alpha'}$, introduced in this work for the first time.

(2) The velocities $\mathbf{V}_{q\alpha}$ can be interpreted as partial entropy velocities for particle species α or, equivalently,

as the velocities of the flow of Bogoliubov excitations α. These velocities become independent and self-contained variables in superfluid mixtures, being generally not collinear to the corresponding normal current densities, as well as to the momentum densities, associated with the motion of thermal excitations of different particle species. Here we find some resemblance to the Carter's formalism (see, e.g., Ref. [76]), in which the entropy current is treated on an equal footing with other conserved currents of a multifluid mixture.

(3) To study diffusion effects in superfluid mixtures, we applied the Boltzmann-like kinetic equation for Bogoliubov thermal excitations, supplementing it with the continuity and superfluid equations for each particle species. Using the Chapman-Enskog method and assuming the standard ansatz (99) for the nonequilibrium correction to the distribution function, we again arrived at the velocities $\mathbf{V}_{q\alpha}$. That is, these velocities are natural variables for describing diffusion processes in superfluid Fermi mixtures. In particular, the friction force between different particle species "n" and "p" appears to be directly proportional to $\mathbf{V}_{qn} - \mathbf{V}_{qp}$ [see, e.g., Eq. (107)]. Somewhat loosely, one may say that the "friction of entropy currents" produces heat (the entropy velocity $\mathbf{V}_{q\alpha}$ is generally not equal to the normal fluid velocity of particle species α, so that this result is nontrivial).

(4) Using transport equations for Bogoliubov excitations, we obtained general expressions for the diffusion coefficients of a mixture of two strongly-interacting Fermi superfluids [see Eqs. (112) and (114)]. The diffusion coefficients depend on the matrix $R_{\alpha\alpha'}$ (which is responsible for the Fermi-liquid/entrainment effects in the mixture) and on the momentum transfer rates $J_{\alpha\beta}$. The general expression for $J_{\alpha\beta}$ is presented in Appendix E and formally has the same structure as $J_{\alpha\beta}$ for a mixture of two weakly interacting superfluid Fermi gases.

(5) There is a quantitative difference between the procedure of calculation of diffusion coefficients for the nonsuperfluid and superfluid Fermi mixtures. In the first case, the inclusion of the Fermi-liquid effects is quite formal. Indeed, after expanding the quasiparticle distribution function in powers of the Knudsen number [cf. Eq. (77)],

$$\mathcal{N}^{(\alpha)}_{\mathbf{p}} = \overline{\mathcal{N}}^{(\alpha)}_{\mathbf{p},0} + \bar{\mathfrak{n}}^{(\alpha)}_1, \tag{165}$$

where [cf. Eq. (79)]

$$\overline{\mathcal{N}}^{(\alpha)}_{\mathbf{p},0} = \frac{1}{1 + e^{(\varepsilon^{(\alpha)}_{\mathbf{p}} + \Delta H^{(\alpha)}_{\mathbf{p}} - \mathbf{p}\,\mathbf{u})/T}}, \tag{166}$$

and substituting this representation into transport equations, the resulting (linearized) equations for the distribution function corrections $\bar{\mathfrak{n}}^{(\alpha)}_1$ will be formally identical to the corresponding equations for a mixture of weakly interacting Fermi gases (see, e.g., Ref. [11]). The expression for the current densities is also identical to its counterpart in weakly interacting mixtures (see the discussion at the end of Sec. IV E). As a consequence, there is no need for calculation of the energy corrections $\Delta H^{(\alpha)}_{\mathbf{p}}$, and the Landau parameters do not appear in the expression (118) for the diffusion coefficients. In contrast, in the superfluid mixture one needs to determine the Bogoliubov excitation energy correction, and the Landau parameters explicitly appear in the diffusion coefficients [see Eq. (114)]. The formal reason for this difference is discussed at the end of Sec. (IV E).

(6) The results discussed above were generalized to the case of charged mixtures in Sec. V. The extension of these results to arbitrary number of particle species in the mixture is straightforward.

Summarizing, the framework for treating particle diffusion developed in the present work opens the way for systematic calculations of diffusion coefficients in superfluid, strongly interacting Fermi mixtures, in particular in superfluid neutron-star matter.

ACKNOWLEDGMENTS

M.E.G. is grateful to the Department of Particle Physics & Astrophysics at the Weizmann Institute of Science for hospitality. This research was supported by the Russian Science Foundation Grant No. 22-12-00048.

APPENDIX A: ELEMENTS OF MATRICES $\gamma_{\alpha\beta}$, $K_{\alpha\beta}$, $Y_{\alpha\beta}$, AND $R_{\alpha\beta}$ IN VARIOUS LIMITING CASES

1. The case of one superfluid and one normal Fermi liquid ($\Phi_n = 1$)

Assume that the species "p" is superfluid, while the species "n" is normal, i.e., the function $\Phi_n = 1$ [see Eq. (32)]. In this limit, the elements of the matrix $\gamma_{\alpha\alpha'}$ reduce to

$$\gamma_{nn} = \frac{1}{m^*_n}, \tag{A1}$$

$$\gamma_{np} = \frac{G_{np} n_p (1 - \Phi_p)}{\mathcal{S}}, \tag{A2}$$

$$\gamma_{pp} = \frac{1}{m^*_p}\frac{\mathcal{S}_{\rm nsf}}{\mathcal{S}}, \tag{A3}$$

$$\gamma_{pn} = 0. \tag{A4}$$

In turn, the elements of the matrix $K_{\alpha\alpha'}$ become

$$K_{nn} = \frac{G_{nn} m^*_n (n_p + G_{pp} m^*_p \Phi_p) - G^2_{np} m^*_n m^*_p \Phi_p}{\mathcal{S}}, \tag{A5}$$

$$K_{np} = \frac{G_{np} m^*_p n_p \Phi_p}{\mathcal{S}}, \tag{A6}$$

$$K_{pp} = \frac{G_{pp} m^*_p \Phi_p (n_n + G_{nn} m^*_n) - G^2_{np} m^*_p m^*_n \Phi_p}{\mathcal{S}}, \tag{A7}$$

$$K_{pn} = \frac{G_{pn} m^*_n n_n}{\mathcal{S}}. \tag{A8}$$

The superfluid entrainment matrix is given by

$$Y_{nn} = Y_{np} = Y_{pn} = 0, \tag{A9}$$

$$Y_{pp} = \frac{n_p}{m_p^* c^2} \frac{\mathcal{S}_{\rm nsf}}{\mathcal{S}} (1 - \Phi_p), \tag{A10}$$

while the normal entrainment matrix equals

$$R_{nn} = n_n, \tag{A11}$$

$$R_{np} = 0, \tag{A12}$$

$$R_{pp} = n_p \frac{\mathcal{S}_{\rm nsf}}{\mathcal{S}} \Phi_p, \tag{A13}$$

$$R_{pn} = \frac{n_n n_p m_n^* (1 - \Phi_p) G_{np}}{\mathcal{S}}. \tag{A14}$$

In these expressions the function $\mathcal{S}$ is given by Eq. (36), which can be represented in the considered limiting case as

$$\mathcal{S} = (n_n + G_{nn} m_n^*)(n_p + G_{pp} m_p^* \Phi_p) - G_{np}^2 m_n^* m_p^* \Phi_p. \tag{A15}$$

Similarly, the function $\mathcal{S}_{\rm nsf}$ coincides with $\mathcal{S}$ calculated for a completely nonsuperfluid mixture (i.e., assuming $\Phi_n = \Phi_p = 1$)

$$\begin{aligned}\mathcal{S}_{\rm nsf} &= (n_n + G_{nn} m_n^*)(n_p + G_{pp} m_p^*) - G_{np}^2 m_n^* m_p^* \\ &= \frac{m_n^* m_p^* c^2}{\mu_n \mu_p} \left[n_n n_p - G_{np} \frac{\mu_n n_n + \mu_p n_p}{c^2} \right].\end{aligned} \tag{A16}$$

Here, the second equality is obtained with the help of Eq. (14). As follows from Eqs. (A11)–(A14), in the case when both species are nonsuperfluid ($\Phi_n = \Phi_p = 1$), one has $R_{nn} = n_n$, $R_{pp} = n_p$, and $R_{np} = R_{pn} = 0$.

2. The case of strong superfluidity of one of the constituents ($\Phi_p \to 0$)

Assume now that particle species "p" is strongly superfluid, $\Phi_p \to 0$. Then the matrix $\gamma_{\alpha\alpha'}$ simplifies to

$$\gamma_{nn} = \frac{n_n + G_{nn} m_n^*}{m_n^* (n_n + G_{nn} m_n^* \Phi_n)}, \tag{A17}$$

$$\gamma_{np} = \frac{G_{np}}{n_n + G_{nn} m_n^* \Phi_n}, \tag{A18}$$

$$\gamma_{pp} = \frac{(n_p + G_{pp} m_p^*)(n_n + G_{nn} m_n^* \Phi_n) - G_{np}^2 m_n^* m_p^* \Phi_n}{m_p^* n_p (n_n + G_{nn} m_n^* \Phi_n)}, \tag{A19}$$

$$\gamma_{pn} = \frac{n_n G_{np} (1 - \Phi_n)}{n_p (n_n + G_{nn} m_n^* \Phi_n)}, \tag{A20}$$

while the matrix $K_{\alpha\alpha'}$ reduces to

$$K_{nn} = \frac{G_{nn} m_n^* \Phi_n}{(n_n + G_{nn} m_n^* \Phi_n)}, \tag{A21}$$

$$K_{np} = 0, \tag{A22}$$

$$K_{pp} = 0, \tag{A23}$$

$$K_{pn} = \frac{G_{pn} m_n^* n_n \Phi_n}{n_p (n_n + G_{nn} m_n^* \Phi_n)}. \tag{A24}$$

The superfluid entrainment matrix becomes

$$Y_{nn} = \frac{n_n (n_n + G_{nn} m_n^*)}{c^2 m_n^* (n_n + G_{nn} m_n^* \Phi_n)} (1 - \Phi_n), \tag{A25}$$

$$Y_{pp} = \frac{(n_p + G_{pp} m_p^*)(n_n + G_{nn} m_n^* \Phi_n) - G_{np}^2 m_n^* m_p^* \Phi_n}{c^2 m_p^* (n_n + G_{nn} m_n^* \Phi_n)}, \tag{A26}$$

$$Y_{pn} = Y_{np} = \frac{n_n G_{np} (1 - \Phi_n)}{c^2 (n_n + G_{nn} m_n^* \Phi_n)}, \tag{A27}$$

while the normal entrainment matrix is

$$R_{nn} = \frac{n_n (n_n + G_{nn} m_n^*) \Phi_n}{(n_n + G_{nn} m_n^* \Phi_n)}, \tag{A28}$$

$$R_{np} = 0, \tag{A29}$$

$$R_{pp} = 0, \tag{A30}$$

$$R_{pn} = \frac{n_n m_n^* G_{np} \Phi_n}{n_n + G_{nn} m_n^* \Phi_n}. \tag{A31}$$

In the limit when both particle species are strongly superfluid, $\Phi_n \to 0$ and $\Phi_p \to 0$, one has from these expressions $R_{nn} = R_{pp} = R_{np} = R_{pn} = 0$.

3. The case of two independent Fermi liquids ($G_{np} = G_{pn} = 0$)

Assume now that our mixture is composed of two superfluid Fermi liquids, which do not "feel" each other in a sense that the Landau parameters $f_1^{np} = f_1^{pn} = 0$, i.e., $G_{np} = G_{pn} = 0$; see Eq. (15).

In this case,

$$\gamma_{\alpha\alpha} = \frac{1}{m_\alpha^*} \frac{n_\alpha + G_{\alpha\alpha} m_\alpha^*}{n_\alpha + G_{\alpha\alpha} m_\alpha^* \Phi_\alpha} = \frac{1}{m_\alpha^* \Phi_\alpha + \frac{\mu_\alpha}{c^2} (1 - \Phi_\alpha)}, \tag{A32}$$

$$K_{\alpha\alpha} = \frac{G_{\alpha\alpha} m_\alpha^* \Phi_\alpha}{n_\alpha + G_{\alpha\alpha} m_\alpha^* \Phi_\alpha} = \frac{\left(m_\alpha^* - \frac{\mu_\alpha}{c^2}\right) \Phi_\alpha}{m_\alpha^* \Phi_\alpha + \frac{\mu_\alpha}{c^2} (1 - \Phi_\alpha)}, \tag{A33}$$

$$\gamma_{\alpha\beta} = K_{\alpha\beta} = 0, \tag{A34}$$

where we applied the relation (14). In turn, the matrices $Y_{\alpha\alpha'}$ and $R_{\alpha\alpha'}$ reduce to

$$Y_{\alpha\alpha} = \frac{n_\alpha (1 - \Phi_\alpha)}{m_\alpha^* c^2 \Phi_\alpha + \mu_\alpha (1 - \Phi_\alpha)}, \tag{A35}$$

$$R_{\alpha\alpha} = \frac{n_\alpha m_\alpha^* c^2 \Phi_\alpha}{m_\alpha^* c^2 \Phi_\alpha + \mu_\alpha (1 - \Phi_\alpha)}, \tag{A36}$$

$$Y_{\alpha\beta} = R_{\alpha\beta} = 0. \tag{A37}$$

As it is expected, the nondiagonal elements of all matrices vanish in this case.

APPENDIX B: RELATIVISTIC HYDRODYNAMICS OF SUPERFLUID MIXTURES

In this Appendix, we briefly describe the phenomenological relativistic hydrodynamics of superfluid mixtures, since we refer to some of its equations in the main text of the paper. We use the version of hydrodynamics developed in Refs. [8,10,70,71,74]. Despite this hydrodynamics taking into account an extremely rich set of various dissipative phenomena, here we restrict ourselves to considering its simplified version, which only allows for diffusion as a dissipative mechanism. The effects of diffusion have been incorporated into the relativistic hydrodynamics of normal and superfluid mixtures in Refs. [10,74]. We will follow these works in what follows. For simplicity, we assume that there are no vortices in the system and also ignore the effects related to polarization and magnetization of the medium. In addition, we neglect possible chemical reactions converting different particle species into each other. With these reservations, the equations of superfluid relativistic hydrodynamics consist of

(i) The energy-momentum conservation law

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu}=0, \tag{B1}$$

where

$$T^{\mu\nu}=\frac{E+P}{c^2}u^\mu u^\nu+Pg^{\mu\nu}+\sum_{\alpha\alpha'}Y_{\alpha\alpha'}\big[c^2 w^\mu_{(\alpha)}w^\nu_{(\alpha')}+\mu_\alpha w^\mu_{(\alpha')}u^\nu+\mu_{\alpha'}w^\nu_{(\alpha)}u^\mu\big]+T^{\mu\nu}_{\rm EM}. \tag{B2}$$

(ii) The particle conservation law for each species α

$$\frac{\partial j^\mu_{(\alpha)}}{\partial x^\mu}=0, \tag{B3}$$

where

$$j^\mu_{(\alpha)}=n_\alpha u^\mu+\sum_{\alpha'}Y_{\alpha\alpha'}w^\mu_{(\alpha')}+\Delta j^\mu_{(\alpha)}. \tag{B4}$$

(iii) The constraints on the four-velocities and four-currents:

$$u_\mu w^\mu_{(\alpha)}=0, \tag{B5}$$

$$u_\mu \Delta j^\mu_{(\alpha)}=0. \tag{B6}$$

(iv) The second law of thermodynamics

$$dE=TdS+\sum_\alpha \mu_\alpha dn_\alpha+\sum_{\alpha\alpha'}\frac{Y_{\alpha\alpha'}}{2}d\big(w_{(\alpha)\mu}w^\mu_{(\alpha')}\big). \tag{B7}$$

(v) The Maxwell equations

$$\partial_\nu F^{\mu\nu}=\frac{4\pi}{c}J^\mu,\quad \partial_\mu F^{\nu\xi}+\partial_\xi F^{\mu\nu}+\partial_\nu F^{\xi\mu}=0. \tag{B8}$$

In the formulas above, $T^{\mu\nu}_{\rm EM}$ is the standard electromagnetic energy-momentum tensor, $F^{\mu\nu}$ is the electromagnetic field tensor, $J^\nu=\sum_\alpha e_\alpha j^\mu_{(\alpha)}$ is the charge current density, e_α is the charge of particle species α, $w^\mu_{(\alpha)}$ is the superfluid four-vector given below, $\Delta j^\mu_{(\alpha)}$ is the dissipative correction to the particle current density, E is the energy density in the comoving frame of reference $[u^\mu=(c,0,0,0)]$, P is the pressure given by standard formula

$$P=-E+TS+\sum_\alpha n_\alpha\mu_\alpha, \tag{B9}$$

and $g^{\mu\nu}=\mathrm{diag}(-1,1,1,1)$ is the Minkowski metrics. In the above expressions and in the text below, a summation is assumed over repeated space-time indices μ, ν, and ξ. Note, however, that the sum over the particle species indices α and α' will be written explicitly to facilitate comparison with the main text of the paper. The superfluid four-velocity can be represented as

$$w^\mu_{(\alpha)}=Q^\mu_{(\alpha)}-\frac{\mu_\alpha}{c^2}u^\mu, \tag{B10}$$

where we introduced the half Cooper-pair four-momentum

$$Q^\mu_{(\alpha)}=\frac{1}{2}\partial^\mu\Phi_\alpha-\frac{e_\alpha}{c}A^\mu, \tag{B11}$$

Φ_α is the phase of the Cooper-pair condensate wave function, and A^μ is the four-potential of the electromagnetic field.

Equations (B1)–(B11) allow one to obtain the entropy generation equation,

$$\partial_\mu S^\mu=\Gamma_s, \tag{B12}$$

where

$$S^\mu=Su^\mu-\sum_\alpha\frac{\mu_\alpha}{T}\Delta j^\mu_{(\alpha)} \tag{B13}$$

is the entropy four-current and

$$\Gamma_s=-\sum_\alpha \Delta j_{(\alpha)\mu}d^\mu_{(\alpha)} \tag{B14}$$

is the entropy generation rate. In this expression

$$d^\mu_{(\alpha)}=(\partial^\mu+u^\mu u^\nu\partial_\nu)\Big(\frac{\mu_\alpha}{T}\Big)-\frac{e_\alpha E^\mu}{T}, \tag{B15}$$

where

$$E^\mu=\frac{u_\nu}{c}F^{\mu\nu}. \tag{B16}$$

If the gradients are small and there is no preferred direction, the current corrections $\Delta j^\mu_{(\alpha)}$ can be presented as

$$\Delta j^\mu_{(\alpha)}=-\sum_{\alpha'}\mathcal{D}_{\alpha\alpha'}d^\mu_{(\alpha')}. \tag{B17}$$

It follows from the Onsager principle that the diffusion coefficients $\mathcal{D}_{\alpha\alpha'}$ must be symmetric,

$$\mathcal{D}_{\alpha\alpha'}=\mathcal{D}_{\alpha'\alpha}. \tag{B18}$$

APPENDIX C: THE NONRELATIVISTIC LIMIT OF SUPERFLUID HYDRODYNAMICS

Let us consider a reference frame in which all the hydrodynamic velocities are nonrelativistic, i.e. $|u^i|$, $|c^2Q^i_\alpha/\mu_\alpha|\ll c$, where the index i runs over the spatial coordinates (as in Sec. II, we only consider fluid motions for which such reference frame does exist). Our aim is to find the form of hydrodynamic equations appropriate for this frame. Using the relation (B5), one can rewrite the expression (B10) as

$$w^0_{(\alpha)}=\frac{\mathbf{u}\mathbf{Q}_\alpha}{u^0}-\frac{\mu_\alpha}{c^2}\frac{\mathbf{u}^2}{u^0}, \tag{C1}$$

$$\boldsymbol{w}_\alpha=\mathbf{Q}_\alpha-\frac{\mu_\alpha}{c^2}\mathbf{u}, \tag{C2}$$

where $\mathbf{u}$, $\boldsymbol{w}_\alpha$, and $\mathbf{Q}_\alpha$ are the spatial parts of the four-vectors u^μ, $w^\mu_{(\alpha)}$, and $Q^\mu_{(\alpha)}$. Substituting these expressions together with the expansion

$$u^0 \approx c\left(1+\frac{1}{2}\frac{\mathbf{u}^2}{c^2}\right) \tag{C3}$$

into (B2) and neglecting the terms $\sim\mathbf{u}/c$ in that equation, one gets

$$T^{i0} \approx c\rho_q \mathbf{u} + c\sum_{\alpha\alpha'} Y_{\alpha\alpha'}\mu_\alpha \mathbf{Q}_{\alpha'} + T^{i0}_{\mathrm{EM}}, \tag{C4}$$

$$T^{ij} \approx \rho_q u^i u^j + c^2 \sum_{\alpha\alpha'} Y_{\alpha\alpha'} Q^i_\alpha Q^j_{\alpha'} + P\delta^{ij} + T^{ij}_{\mathrm{EM}}, \tag{C5}$$

where δ^{ij} is the the Kronecker delta and the following notation has been introduced:

$$\rho_q = \frac{TS}{c^2} + \sum_\alpha \frac{\mu_\alpha}{c^2}\left(n_\alpha - \sum_{\alpha'} Y_{\alpha\alpha'}\mu_{\alpha'}\right). \tag{C6}$$

This combination arises in expressions (C4)–(C5) after one accounts for the definition (B9). Equation (B1) can be represented as

$$\frac{1}{c}\frac{\partial T^{i0}}{\partial t} + \frac{\partial T^{ij}}{\partial x^j} = 0, \tag{C7}$$

where one should employ the expressions (C4) and (C5). Using Eqs. (B7) and (B9), one gets the following Gibbs-Duhem relation

$$dP \approx S dT + \sum_\alpha n_\alpha d\mu_\alpha + \sum_{\alpha\alpha'} \frac{Y_{\alpha\alpha'}}{2} d(\boldsymbol{w}_\alpha \boldsymbol{w}_{\alpha'}), \tag{C8}$$

where we have neglected the terms containing $w^0_{(\alpha)} w^0_{(\alpha')} \sim (\mathbf{u}^2/c^2)\boldsymbol{w}_\alpha \boldsymbol{w}_{\alpha'}$ [see Eqs. (C1) and (C2)].

In the present paper, we work in the linear approximation in hydrodynamic velocities. Thus, it is instructive to write down the phenomenological equations in the same approximation. Plugging Eqs. (C4) and (C5) into (C7) and neglecting the terms quadratic in $\mathbf{u}$ and $\mathbf{Q}_\alpha$, we obtain

$$\rho_q \frac{\partial \mathbf{u}}{\partial t} + \sum_{\alpha\alpha'} \mu_\alpha Y_{\alpha\alpha'} \frac{\partial \mathbf{Q}_{\alpha'}}{\partial t} + \nabla P = -\frac{\partial T^{\mu\nu}_{\mathrm{EM}}}{\partial x^\nu}. \tag{C9}$$

Here we used the fact that the time derivatives of the quantities ρ_q, $Y_{\alpha\alpha'}$, and μ_α are of the linear order smallness in hydrodynamic velocities [see Eqs. (82)–(85) and the discussion afterwards]. Expressing the pressure gradient using Eq. (C8), and neglecting the small terms $\propto \boldsymbol{w}_\alpha \boldsymbol{w}_{\alpha'}$, we present Eq. (C9) in the final form

$$\rho_q \frac{\partial \mathbf{u}}{\partial t} + \sum_{\alpha\alpha'} \mu_\alpha Y_{\alpha\alpha'} \frac{\partial \mathbf{Q}_{\alpha'}}{\partial t} + T\nabla S + \sum_\alpha n_\alpha(\nabla\mu_\alpha - e_\alpha \mathbf{E}) = 0. \tag{C10}$$

To derive this equation, we noted that the right-hand side of Eq. (C9) equals the Lorentz force (see, e.g., Ref. [77]). Since the magnetic field in the present paper is neglected, we only keep the electrical part of the Lorentz force in Eq. (C10).

In the nonrelativistic limit the particle current density acquires the form

$$j^0_{(\alpha)} = cn_\alpha, \quad \mathbf{j}_\alpha = Y_{\alpha\alpha}\mathbf{Q}_\alpha + Y_{\alpha\beta}\mathbf{Q}_\beta + n_{q\alpha}\mathbf{u} + \Delta\mathbf{j}_{(\alpha)}. \tag{C11}$$

To derive Eq. (C11), one needs to plug Eq. (B10) into (B4) and take into account Eqs. (B5) and (B6). The continuity equation (B3) transforms into

$$\frac{\partial n_\alpha}{\partial t} + \mathrm{div}\mathbf{j}_\alpha = 0. \tag{C12}$$

The vectors $\Delta\mathbf{j}_{(\alpha)}$ are the linear functions of the spatial components of the four-vectors $d^\mu_{(\alpha)}$ [see Eq. (B17)]. In the nonrelativistic limit, $E^\mu = (0, \mathbf{E})$, where $\mathbf{E}$ is the electric field. Hence, one can write

$$d^0_{(\alpha)} = 0, \quad \mathbf{d}_{(\alpha)} = \nabla\left(\frac{\mu_\alpha}{T}\right) - \frac{e_\alpha \mathbf{E}}{T} \tag{C13}$$

and, in view of the relation (B17), one has

$$\Delta\mathbf{j}_\alpha = -\sum_{\alpha'} \mathcal{D}_{\alpha\alpha'}\left[\nabla\left(\frac{\mu_\alpha}{T}\right) - \frac{e_\alpha \mathbf{E}}{T}\right]. \tag{C14}$$

Since, according to the expressions (B10) and (B11), the superfluid four-velocity contains the gradient of the scalar function $\partial^\mu\Phi_\alpha$, it obviously satisfies the following constraint:

$$\frac{\partial}{\partial x_\mu}\left(w^\nu_{(\alpha)} + \frac{e_\alpha}{c}A^\nu + \frac{\mu_\alpha}{c^2}u^\nu\right) - \frac{\partial}{\partial x_\nu}\left(w^\mu_{(\alpha)} + \frac{e_\alpha}{c}A^\mu + \frac{\mu_\alpha}{c^2}u^\mu\right) = 0. \tag{C15}$$

Assuming that the index $\mu = 0$ and the index ν runs over the spatial coordinates, one gets from this equation, after using Eq. (C2),

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{Q}_\alpha + \frac{e_\alpha}{c}\mathbf{A}\right) + \nabla\left(w^0_{(\alpha)} + \frac{e_\alpha}{c}A^0 + \frac{\mu_\alpha}{c^2}u^0\right) = 0. \tag{C16}$$

Plugging Eqs. (C1) and (C3) into (C16) one finally arrives at

$$\frac{\partial \mathbf{Q}_\alpha}{\partial t} + \nabla\left[\mu_\alpha\left(1 - \frac{1}{2}\frac{\mathbf{u}^2}{c^2}\right) + \mathbf{u}\mathbf{Q}_\alpha\right] - e_\alpha\mathbf{E} = 0, \tag{C17}$$

where the electric field is expressed through the components of the four-potential:

$$\mathbf{E} = -\nabla A^0 - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{C18}$$

In the linear approximation in hydrodynamic velocities this equation reduces to[14]

$$\frac{\partial \mathbf{Q}_\alpha}{\partial t} + \nabla\mu_\alpha - e_\alpha \mathbf{E} = 0. \tag{C19}$$

Substituting the vector (C13) into Eq. (B14), one obtains the nonrelativistic expression for the entropy production rate

$$\Gamma_s = -\sum_\alpha \Delta\mathbf{j}_\alpha \left[\nabla\left(\frac{\mu_\alpha}{T}\right) - \frac{e_\alpha \mathbf{E}}{T}\right]. \tag{C20}$$

APPENDIX D: THE EFFECTIVE INTERACTION HAMILTONIAN

Let us consider a mixture of two superfluid Fermi liquids. It is assumed that the scattering of the Landau-liquid quasiparticles can be described by an effective Hamiltonian of the following form:

$$\hat{H}_v = \frac{1}{2}\sum_{1,2,3,4} (1,2|V|3,4)\hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_3 \hat{a}_4. \tag{D1}$$

Here $\hat{a}_k$ and $\hat{a}_k^\dagger$ are, respectively, the destruction and creation operators for a quasiparticle in a quantum state k. Each quantum state k is characterized by the set of quantum numbers $(\mathbf{Q}_{\mathfrak{a}_k} + \mathbf{k}_k, s_k, \mathfrak{a}_k)$, where $\mathbf{Q}_{\mathfrak{a}_k} + \mathbf{k}_k$ is the quasiparticle momentum, $s_k = \pm 1$ is the spin index, and $\mathfrak{a}_k = \pm 1$ is the isospin index ($\mathfrak{a}_k = 1$ can be associated with the particle species "p", while $\mathfrak{a}_k = -1$ with the species "n"). In what follows we also make use of the notation $-k = (\mathbf{Q}_{\mathfrak{a}_k} - \mathbf{k}_k, -s_k, \mathfrak{a}_k)$. We do not specify the dependence of the matrix elements $(1,2|V|3,4)$ on the quantum state variables, but assume that they contain the following Kronecker deltas:

$$(1,2|V|3,4) \to (1,2|V|3,4)\delta_{\mathfrak{a}_1+\mathfrak{a}_2,\mathfrak{a}_3+\mathfrak{a}_4}\delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4}, \tag{D2}$$

ensuring conservation of the total isospin and momentum in particle collisions. Note that, strictly speaking, in our notation the momentum of a quasiparticle equals $\mathbf{Q}_{\mathfrak{a}_k} + \mathbf{k}_k$ and, consequently, the momentum conservation law reads $\mathbf{Q}_{\mathfrak{a}_1} + \mathbf{k}_1 + \mathbf{Q}_{\mathfrak{a}_2} + \mathbf{k}_2 = \mathbf{Q}_{\mathfrak{a}_3} + \mathbf{k}_3 + \mathbf{Q}_{\mathfrak{a}_4} + \mathbf{k}_4$. However, it is easy to see that, in view of the isospin conservation, the vectors $\mathbf{Q}_{\mathfrak{a}_i}$ cancel out.

[14]The superfluid equation (C19) should be compared with the more general, but similar, Eq. (136). Note that Eq. (136) was obtained without linearization with respect to hydrodynamic velocities. The discrepancy between Eqs. (136) and (C19) arises for two reasons. First, the chemical potentials in two equations are defined in different ways (measured in different reference frames). However, as already discussed, this difference is of the second-order smallness in the hydrodynamic velocities and can be ignored in the linear approximation. Second, the nonequilibrium chemical potential $\breve{\mu}_\alpha$ also contains a dissipative correction. For small deviations from the thermodynamic equilibrium this correction is exclusively due to the bulk viscosity. However, all the viscous terms were omitted in the phenomenological hydrodynamics of Appendix B, on which the derivation of Eq. (C19) is based. The form of the superfluid equation with the dissipative correction is given in Ref. [71].

Taking into account the commutation rules for operators $\hat{a}_k$ and $\hat{a}_k^\dagger$, one can antisymmetrize the expression (D1) and present it as

$$\hat{H}_v = \frac{1}{4}\sum_{1,2,3,4} \langle 1,2|V|3,4\rangle_a \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_3 \hat{a}_4, \tag{D3}$$

where the matrix elements $\langle 1,2|V|3,4\rangle_a$ are given by[15]

$$\begin{aligned}\langle 1,2|V|3,4\rangle_a = \tfrac{1}{2}[&(1,2|V|3,4) - (1,2|V|4,3)\\ &- (2,1|V|3,4) + (2,1|V|4,3)]\end{aligned} \tag{D4}$$

and have the following property:

$$\begin{aligned}\langle 1,2|V|3,4\rangle_a &= -\langle 1,2|V|4,3\rangle_a = -\langle 2,1|V|3,4\rangle_a\\ &= \langle 2,1|V|4,3\rangle_a,\end{aligned} \tag{D5}$$

Note that these elements contain the same Kronecker deltas as $(1,2|V|3,4)$, i.e.,

$$\langle 1,2|V|3,4\rangle_a \to \langle 1,2|V|3,4\rangle_a \delta_{\mathfrak{a}_1+\mathfrak{a}_2,\mathfrak{a}_3+\mathfrak{a}_4}\delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4}. \tag{D6}$$

We added a subscript "a" to the matrix element $\langle 1,2|V|3,4\rangle_a$ in order to distinguish it from the elements (D9)–(D13) introduced below.

Let us now make the Bogoliubov transformation for the Landau quasiparticle operators,

$$\hat{a}_k = u_k \hat{b}_k + s_k v_{-k}\hat{b}_{-k}^\dagger, \tag{D7}$$

where the coherence factors are given by Eqs. (6) and (8), and the Bogolubov thermal excitation operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ obey the canonical Fermi commutation relations. Substituting (D7) into (D3), one can represent the Hamiltonian in the form

$$\begin{aligned}\hat{H}_v = \sum_{1,2,3,4}\Big\{&\frac{1}{4}\langle 1,2|V|3,4\rangle_b \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_3 \hat{b}_4\\ &+ \frac{1}{6}\langle 1,2,3|V|4\rangle_b \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_3^\dagger \hat{b}_4 + \frac{1}{6}\langle 1|V|2,3,4\rangle_b \hat{b}_1^\dagger \hat{b}_2 \hat{b}_3 \hat{b}_4\\ &+ \frac{1}{24}\langle 1,2,3,4|V|0\rangle_b \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_3^\dagger \hat{b}_4^\dagger\\ &+ \frac{1}{24}\langle 0|V|1,2,3,4\rangle_b \hat{b}_1 \hat{b}_2 \hat{b}_3 \hat{b}_4,\Big\},\end{aligned} \tag{D8}$$

[15]Note that in Eq. (D4) the matrix elements $(2,1|V|4,3)$ and $(2,1|V|3,4)$ are equal, respectively, to $(2,1|V|4,3) = (1,2|V|3,4)$ and $(2,1|V|3,4) = (1,2|V|4,3)$, so that (D4) can actually be represented as $\langle 1,2|V|3,4\rangle_a = (1,2|V|3,4) - (1,2|V|4,3)$ [78].

where the following coefficients have been introduced:

$$\begin{aligned}\langle 1,2|V|3,4\rangle_b = {}& u_1u_2u_3u_4\langle 1,2|V|3,4\rangle_a + s_1s_2s_3s_4v_1v_2v_3v_4\langle -4,-3|V|-2,-1\rangle_a\\ &- s_2s_4u_1v_2u_3v_4\langle 1,-4|V|3,-2\rangle_a - s_2s_3u_1v_2v_3u_4\langle 1,-3|V|-2,4\rangle_a\\ &- s_1s_4v_1u_2u_3v_4\langle -4,2|V|3,-1\rangle_a - s_1s_3v_1u_2v_3u_4\langle -3,2|V|-1,4\rangle_a,\end{aligned} \tag{D9}$$

$$\begin{aligned}\langle 1,2,3|V|4\rangle_b = {}& s_2u_1v_2u_3u_4\langle 1,3|V|-2,4\rangle_a + s_1s_3s_4v_1u_2v_3v_4\langle -4,2|V|-3,-1\rangle_a\\ &+ s_1v_1u_2u_3u_4\langle 3,2|V|-1,4\rangle_a + s_2s_3s_4u_1v_2v_3v_4\langle -4,1|V|-2,-3\rangle_a\\ &- s_3u_1u_2v_3u_4\langle 1,2|V|-3,4\rangle_a - s_1s_2s_4v_1v_2u_3v_4\langle -4,3|V|-2,-1\rangle_a,\end{aligned} \tag{D10}$$

$$\begin{aligned}\langle 1|V|2,3,4\rangle_b = {}& s_4u_1u_2u_3v_4\langle 1,-4|V|3,2\rangle_a + s_1s_2s_3v_1v_2v_3u_4\langle -2,-3|V|4,-1\rangle_a\\ &+ s_3u_1u_2v_3u_4\langle 1,-3|V|2,4\rangle_a + s_1s_2s_4v_1v_2u_3v_4\langle -4,-2|V|3,-1\rangle_a\\ &- s_2u_1v_2u_3u_4\langle 1,-2|V|3,4\rangle_a - s_1s_3s_4v_1u_2v_3v_4\langle -4,-3|V|2,-1\rangle_a,\end{aligned} \tag{D11}$$

$$\langle 1,2,3,4|V|0\rangle_b = 6s_3s_4u_1u_2v_3v_4\langle 1,2|V|-3,-4\rangle_a, \tag{D12}$$

$$\langle 0|V|1,2,3,4\rangle_b = 6s_1s_2v_1v_2u_3u_4\langle -1,-2|V|3,4\rangle_a. \tag{D13}$$

To obtain these expressions, one should use the relation (D5) together with the commutation relations for the operators $\hat{b}_i$ and $\hat{b}_i^\dagger$. Collecting, for example, the terms with three destruction and one creation operators, one gets

$$\begin{aligned}&\frac{1}{4}\sum_{1,2,3,4}\{s_2u_1v_{-2}u_3u_4\hat{b}_1^\dagger\hat{b}_{-2}\hat{b}_3\hat{b}_4 + s_1v_{-1}u_2u_3u_4\hat{b}_{-1}\hat{b}_2^\dagger\hat{b}_3\hat{b}_4 + s_1s_2s_3v_{-1}v_{-2}v_{-3}u_4\hat{b}_{-1}\hat{b}_{-2}\hat{b}_{-3}^\dagger\hat{b}_4\\ &\quad + s_1s_2s_4v_{-1}v_{-2}u_3u_{-4}\hat{b}_{-1}\hat{b}_{-2}\hat{b}_3\hat{b}_{-4}^\dagger\}\langle 1,2|V|3,4\rangle_a\\ &= \frac{1}{2}\sum_{1,2,3,4}\{-s_2u_1v_2u_3u_4\langle 1,-2|V|3,4\rangle_a + s_1s_2s_3v_1v_2v_3u_4\langle -3,-2|V|-1,4\rangle_a\}\hat{b}_1^\dagger\hat{b}_2\hat{b}_3\hat{b}_4\\ &= \frac{1}{6}\sum_{1,2,3,4}\{s_4u_1u_2u_3v_4\langle 1,-4|V|3,2\rangle_a + s_1s_2s_3v_1v_2v_3u_4\langle -3,-2|V|-1,4\rangle_a\\ &\quad + s_3u_1u_2v_3u_4\langle 1,-3|V|2,4\rangle_a + s_1s_2s_4v_1v_2u_3v_4\langle -4,-2|V|3,-1\rangle_a\\ &\quad - s_2u_1v_2u_3u_4\langle 1,-2|V|3,4\rangle_a - s_1s_3s_4v_1u_2v_3v_4\langle -4,-3|V|2,-1\rangle_a\}\hat{b}_1^\dagger\hat{b}_2\hat{b}_3\hat{b}_4\\ &\equiv \frac{1}{6}\sum_{1,2,3,4}\langle 1|V|2,3,4\rangle_b\,\hat{b}_1^\dagger\hat{b}_2\hat{b}_3\hat{b}_4.\end{aligned} \tag{D14}$$

In the second equality, accounting for the fact that there is a summation over all quantum numbers, we antisymmetrize the sum in order to make the resulting matrix element $\langle 1|V|2,3,4\rangle_b$ antisymmetric with respect to permutations of second, third, and fourth quantum states. The same procedure allows us to obtain the (antisymmetric) matrix element $\langle 1,2,3|V|4\rangle_b$. The coefficients in expression (D8) can be considered as matrix elements for different processes involving Bogoliubov excitations: the element $\langle 1,2|V|3,4\rangle_b$ describes scattering $3,4\to 1,2$, the elements $\langle 1,2,3|V|4\rangle_b$ and $\langle 1|V|2,3,4\rangle_b$ describe decay $4\to 1,2,3$ and coalescence $2,3,4,\to 1$, while the matrix elements $\langle 1,2,3,4|V|0\rangle_b$ and $\langle 0|V|1,2,3,4\rangle_b$ describe creation and destruction of four Bogoliubov excitations. However, since these excitations have positive energy, the last two processes are forbidden by the energy conservation. Hence, these terms can be ignored, in particular, in calculations of Appendix E.

Note once again that the matrix elements (D9)–(D11) are constructed in such a way to make them antisymmetric with respect to permutations of the excitations in the initial state, as well as in the final state. It can also be verified that the element $\langle 1,2,3|V|4\rangle_b$ can be obtained from $\langle 1|V|2,3,4\rangle_b$ by complex conjugation and interchanging the states $1\leftrightarrow 4$ and $2\leftrightarrow 3$.

Substituting Eq. (D6) into (D9)–(D11), one can verify that these matrix elements contain the following momentum Kronecker deltas:

$$\langle 1,2|V|3,4\rangle_b \to \langle 1,2|V|3,4\rangle_b\delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4}, \tag{D15}$$

$$\langle 1|V|2,3,4\rangle_b \to \langle 1|V|2,3,4\rangle_b\delta_{\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4}, \tag{D16}$$

$$\langle 1,2,3|V|4\rangle_b \to \langle 1,2,3|V|4\rangle_b\delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_4-\mathbf{k}_3}. \tag{D17}$$

At the same time, one cannot factor out the isospin Kronecker deltas from these matrix elements because the Bogoliubov transformation involves two operators with different values of momentum, but with the same isospin index $\mathfrak{a}_k$. The important isospin-related property of the matrix elements (D15)–(D17), which holds true in superfluid mixtures, is that they remain nonzero only for transitions with even number of particles of each species. This obvious property is used in Appendix E to obtain the collision integral (E2).

APPENDIX E: MOMENTUM TRANSFER RATES

The total collision integral for particle species α can be represented as a sum

$$I_\alpha = \sum_{\alpha'} I_{\alpha\alpha'}, \tag{E1}$$

where $I_{\alpha\alpha'}$ is the part of the integral describing collisions with particle species α'. In contrast to Landau quasiparticles, the number of Bogoliubov excitations is not necessarily conserved in the collisions. That is why, in addition to scattering, the collision integral contains also the terms describing decay and coalescence events. In terms of the Bogoliubov thermal excitations, the collision integral can be written as

$$\begin{aligned}
I_{\alpha\alpha'} = \sum_{\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4} \Bigg\{ & \frac{1}{1+\delta_{\alpha,\alpha'}} W_{\text{scat.1}}(4_\alpha 3_{\alpha'}|2_{\alpha'}1_\alpha)\delta\big(\mathfrak{E}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha} + \mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}} - \mathfrak{E}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'}} - \mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\big) \\
& \times \big[\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha}\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}}\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'}}\big)\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\big) - \mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'}}\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}}\big)\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha}\big)\big] \\
& + \frac{1-\delta_{\alpha,\alpha'}}{2} W_{\text{scat.2}}(4_{\alpha'}3_{\alpha'}|2_\alpha 1_\alpha)\delta\big(\mathfrak{E}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'}} + \mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}} - \mathfrak{E}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha} - \mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\big) \\
& \times \big[\mathcal{F}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'}}\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}}\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha}\big)\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\big) - \mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\mathcal{F}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha}\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}}\big)\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'}}\big)\big] \\
& + \frac{1-\delta_{\alpha\alpha'}}{2} W_{\text{dec.1}}(4_\alpha|3_{\alpha'}2_{\alpha'}1_\alpha)\delta\big(\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha} + \mathfrak{E}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'}} + \mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}} - \mathfrak{E}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha}\big) \\
& \times \big[\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha}\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\big)\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'}}\big)\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}}\big) - \mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'}}\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}}\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha}\big)\big] \\
& + \frac{1}{1+\delta_{\alpha\alpha'}} W_{\text{dec.2}}(4_{\alpha'}|3_{\alpha'}2_\alpha 1_\alpha)\delta\big(\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha} + \mathfrak{E}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha} + \mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}} - \mathfrak{E}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'}}\big) \\
& \times \big[\mathcal{F}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'}}\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\big)\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha}\big)\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}}\big) - \mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\mathcal{F}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha}\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}}\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'}}\big)\big] \\
& + \frac{1}{2+4\delta_{\alpha,\alpha'}} W_{\text{coal.}}(4_\alpha 3_{\alpha'}2_{\alpha'}|1_\alpha)\delta\big(\mathfrak{E}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha} + \mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}} + \mathfrak{E}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'}} - \mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\big) \\
& \times \big[\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha}\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}}\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'}}\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\big) - \mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'}}\big)\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}}\big)\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha}\big)\big]\Bigg\},
\end{aligned} \tag{E2}$$

Here $W_q(i|f)$ is the differential transition probability for the collision event $i \to f$. In Eq. (E2) we use a slightly different notation for the Bogoliubov excitation quantum state in comparison to that used in Appendix D. Namely, by writing i_α, we explicitly indicate that a given excitation is in the isospin state α, while i stands for the momentum state $\mathbf{Q}_\alpha + \mathbf{p}_i$ [the spin quantum number is not included in i_α since we only consider spin-averaged quantities in Eq. (E2)]. The differential transition probabilities are already summed over the spin states of the Bogoliubov excitations 2_α, 3_α, 4_α and averaged over the spin states of the excitation 1_α. The factors $(1+\delta_{\alpha\alpha'})^{-1}$, $(1-\delta_{\alpha\alpha'})/2$, and $(2+4\delta_{\alpha\alpha'})^{-1}$ are added to prevent double or sixfold counting of the same collision event. In the case of $\alpha' \neq \alpha$, the sets of the scattering and decay events split into two subsets, corresponding to the transition probabilities $W_{\text{scat.1}}$, $W_{\text{scat.2}}$ and $W_{\text{dec.1}}$, $W_{\text{dec.2}}$, respectively. If $\alpha' = \alpha$, both $W_{\text{scat.1}}$ and $W_{\text{scat.2}}$ as well as both $W_{\text{dec.1}}$ and $W_{\text{dec.2}}$ describe the same sets of scattering and decay events. In view of that, the multiplier $(1-\delta_{\alpha\alpha'})/2$ is added to $W_{\text{scat.2}}$ and $W_{\text{dec.1}}$. The delta functions in the expression (E2) ensure the conservation of energy. The differential transition probabilities can be represented in the following form:

$$W_{\text{scat.1}}(4_\alpha 3_{\alpha'}|2_{\alpha'}1_\alpha) = \frac{1}{2}\sum_{\sigma_4,\sigma_3,\sigma_2,\sigma_1} 2\pi\,|\langle \mathbf{Q}_\alpha+\mathbf{p}_1,\sigma_1,\alpha;\ \mathbf{Q}_{\alpha'}+\mathbf{p}_2,\sigma_2,\alpha'|V|\mathbf{Q}_{\alpha'}+\mathbf{p}_3,\sigma_3,\alpha';\ \mathbf{Q}_\alpha+\mathbf{p}_4,\sigma_4,\alpha\rangle_b|^2\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}, \tag{E3}$$

$$W_{\text{scat}.2}(4_{\alpha'}3_{\alpha'}|2_\alpha 1_\alpha) = \frac{1}{2}\sum_{\sigma_4,\sigma_3,\sigma_2,\sigma_1} 2\pi\,|\langle \mathbf{Q}_\alpha + \mathbf{p}_1,\sigma_1,\alpha;\ \mathbf{Q}_\alpha + \mathbf{p}_2,\sigma_2,\alpha|V|\mathbf{Q}_{\alpha'} + \mathbf{p}_3,\sigma_3,\alpha';\ \mathbf{Q}_{\alpha'} + \mathbf{p}_4,\sigma_4,\alpha'\rangle_b|^2\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}, \tag{E4}$$

$$W_{\text{dec}.1}(4_\alpha|3_{\alpha'}2_{\alpha'}1_\alpha) = \frac{1}{2}\sum_{\sigma_4,\sigma_3,\sigma_2,\sigma_1} 2\pi\,|\langle \mathbf{Q}_\alpha + \mathbf{p}_1,\sigma_1,\alpha;\ \mathbf{Q}_{\alpha'} + \mathbf{p}_2,\sigma_2,\alpha';\ \mathbf{Q}_{\alpha'} + \mathbf{p}_3,\sigma_3,\alpha'|V|\mathbf{Q}_\alpha + \mathbf{p}_4,\sigma_4,\alpha\rangle_b|^2\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_4-\mathbf{p}_3}, \tag{E5}$$

$$W_{\text{dec}.2}(4_{\alpha'}|3_{\alpha'}2_\alpha 1_\alpha) = \frac{1}{2}\sum_{\sigma_4,\sigma_3,\sigma_2,\sigma_1} 2\pi\,|\langle \mathbf{Q}_\alpha + \mathbf{p}_1,\sigma_1,\alpha;\ \mathbf{Q}_\alpha + \mathbf{p}_2,\sigma_2,\alpha;\ \mathbf{Q}_{\alpha'} + \mathbf{p}_3,\sigma_3,\alpha'|V|\mathbf{Q}_{\alpha'} + \mathbf{p}_4,\sigma_4,\alpha'\rangle_b|^2\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_4-\mathbf{p}_3}, \tag{E6}$$

$$W_{\text{coal.}}(4_\alpha 3_{\alpha'}2_{\alpha'}|1_\alpha) = \frac{1}{2}\sum_{\sigma_4,\sigma_3,\sigma_2,\sigma_1} 2\pi\,|\langle \mathbf{Q}_\alpha + \mathbf{p}_1,\sigma_1,\alpha|V|\mathbf{Q}_{\alpha'} + \mathbf{p}_2,\sigma_2,\alpha';\ \mathbf{Q}_{\alpha'} + \mathbf{p}_3,\sigma_3,\alpha';\ \mathbf{Q}_\alpha + \mathbf{p}_4,\sigma_4,\alpha\rangle_b|^2\delta_{\mathbf{p}_1-\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}, \tag{E7}$$

where $\langle f|V|i\rangle_b$ are the matrix elements of the effective interaction Hamiltonian in the basis of Bogoliubov excitations given in Appendix D, and the factor $1/2$ arises due to averaging over the spin index σ_1. The energy delta functions are already taken into account in the expression (E2). The functions $W_q(i|f)$ are symmetric with respect to permutations of quantum numbers of the Bogoliubov excitations in the initial state, as well as in the final states [e.g., $W_{\text{scat.}}(4_\alpha 3_\alpha|2_\alpha 1_\alpha) = W_{\text{scat.}}(3_\alpha 4_\alpha|2_\alpha 1_\alpha) = W_{\text{scat.}}(4_\alpha 3_\alpha|1_\alpha 2_\alpha)$ and so on] since the matrix elements $\langle f|V|i\rangle_b$ are antisymmetric with respect to such transformations (see Appendix D). The function $W_q(i|f)$ is also symmetric with respect to interchange of initial and final states [e.g., $W_{\text{scat.}}(4_\alpha 3_\alpha|2_\alpha 1_\alpha) = W_{\text{scat.}}(2_\alpha 1_\alpha|4_\alpha 3_\alpha)$]. This symmetry follows from the Hermiticity of the matrix $\langle f|V|i\rangle_b$.

Considering integrals $I_{\alpha\alpha'}$, let us analyze, for example, the expression in the first square brackets in (E2). It can be represented as

$$\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'}}\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}}\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha}\left(\frac{1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'}}}{\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'}}}\frac{1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}}{\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha}} - \frac{1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}}}{\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'}}}\frac{1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha}}{\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha}}\right). \tag{E8}$$

Using Eqs. (77), (79), and (98), one can write

$$\frac{1-\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}}{\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}} \approx \frac{1-\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}}{\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}}\left(1-\frac{\phi_\alpha}{T}\right) = \exp\left(\frac{\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha} - \mathbf{p}\,\mathbf{u}}{T}\right)\left(1-\frac{\phi_\alpha}{T}\right), \tag{E9}$$

where we linearized this expression with respect to explicitly written function ϕ_α but keep it untouched inside the distribution functions $\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$.[16] Plugging Eq. (E9) into (E8) and taking into account the energy delta function from the integral (E2) and the Kronecker delta from the expression (E3), one obtains[17]

$$\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\overline{\mathcal{F}}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'},0}\big(1-\overline{\mathcal{F}}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}\big)\big(1-\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}\big)\frac{\phi_{4\alpha}+\phi_{3\alpha'}-\phi_{2\alpha'}-\phi_{1\alpha}}{T}. \tag{E10}$$

Now one can complete the linearization with respect to the Knudsen number $\mathcal{K}$ by replacing the distribution functions $\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}$ with the equilibrium distributions $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}$. Repeating similar procedure for all the expressions in the square brackets in Eq. (E2), one gets

$$\begin{aligned}
I_{\alpha\alpha'} = \frac{1}{T}\sum_{\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4}\Bigg\{&\frac{1}{1+\delta_{\alpha,\alpha'}}W_{\text{scat}.1}(4_\alpha 3_{\alpha'}|2_{\alpha'}1_\alpha)\delta\big(\mathfrak{E}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}+\mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\\
&\times\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'},0}\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}\big)\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}\big)[\phi_{4\alpha}+\phi_{3\alpha'}-\phi_{2\alpha'}-\phi_{1\alpha}]\\
&+\frac{1-\delta_{\alpha,\alpha'}}{2}W_{\text{scat}.2}(4_{\alpha'}3_{\alpha'}|2_\alpha 1_\alpha)\delta\big(\mathfrak{E}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'},0}+\mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha,0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\\
&\times\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha,0}\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}\big)\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'},0}\big)[\phi_{4\alpha'}+\phi_{3\alpha'}-\phi_{2\alpha}-\phi_{1\alpha}]\\
&+\frac{1-\delta_{\alpha,\alpha'}}{2}W_{\text{dec}.1}(4_\alpha|3_{\alpha'}2_{\alpha'}1_\alpha)\delta\big(\mathfrak{E}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}-\mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)
\end{aligned}$$

[16]Recall that the distribution function $\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ depends on the local Bogoliubov excitation energy $\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$, which, in turn, depends on the distribution functions $\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$.

[17]It should be emphasized that the true energy $\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ is conserved during the collision event, not the equilibrium energy $\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}$.

$$
\begin{aligned}
&\times \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_4-\mathbf{p}_3}\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'},0}\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}\big)[\phi_{4\alpha}-\phi_{3\alpha'}-\phi_{2\alpha'}-\phi_{1\alpha}]\\
&+\frac{1}{1+\delta_{\alpha,\alpha'}}W_{\mathrm{dec.2}}(4_{\alpha'}|3_{\alpha'}2_\alpha 1_\alpha)\delta\big(\mathfrak{E}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha,0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\\
&\times \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_4-\mathbf{p}_3}\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'},0}\big)[\phi_{4\alpha'}-\phi_{3\alpha'}-\phi_{2\alpha}-\phi_{1\alpha}]\\
&+\frac{1}{2+4\delta_{\alpha,\alpha'}}W_{\mathrm{coal.}}(4_\alpha 3_{\alpha'}2_{\alpha'}|1_\alpha)\delta\big(\mathfrak{E}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}+\mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}+\mathfrak{E}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\\
&\times \delta_{\mathbf{p}_1-\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'},0}\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}[\phi_{4\alpha}+\phi_{3\alpha'}+\phi_{2\alpha'}-\phi_{1\alpha}]\Big\}.
\end{aligned}
\tag{E11}
$$

Here we replaced $\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}$ with $\mathfrak{E}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}$ in the energy delta functions, which is justifiable in the linear approximation, and extracted the momentum Kronecker deltas from the transition probabilities (E3)–(E7). One can see that, if $\phi_\alpha=0$ ($\mathcal{F}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha}=\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}$), the collision integral vanishes. Thus, the distribution functions $\overline{\mathcal{F}}^{(\alpha)}_{\mathbf{p}+\mathbf{Q}_\alpha,0}$ can be considered as solutions to Eq. (78).

Our next aim will be to find the expression for the integral $\sum_{\mathbf{p}_1,\sigma_1}\mathbf{p}_1 I_{\alpha\alpha'}$. To that end, we multiply Eq. (E11) by $\mathbf{p}_1$, sum the result over the quantum states 1_α, and substitute the expression (99). The result is

$$
\begin{aligned}
\sum_{\mathbf{p}_1,\sigma_1}\mathbf{p}_1 I_{\alpha\alpha'}=&-\frac{2}{T}\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4}\Big\{\frac{1}{1+\delta_{\alpha,\alpha'}}W_{\mathrm{scat.1}}(4_\alpha 3_{\alpha'}|2_{\alpha'}1_\alpha)\delta\big(\mathfrak{E}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}+\mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\\
&\times \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'},0}\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}\big)\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}\big)\mathbf{p}_1[(\mathbf{p}_1-\mathbf{p}_4)(\mathbf{V}_{i\alpha}-\mathbf{V}_{i\alpha'})]\\
&+\frac{1-\delta_{\alpha,\alpha'}}{2}W_{\mathrm{scat.2}}(4_{\alpha'}3_{\alpha'}|2_\alpha 1_\alpha)\delta\big(\mathfrak{E}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'},0}+\mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha,0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\\
&\times \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha,0}\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}\big)\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'},0}\big)\mathbf{p}_1[(\mathbf{p}_1+\mathbf{p}_2)(\mathbf{V}_{i\alpha}-\mathbf{V}_{i\alpha'})]\\
&+\frac{1-\delta_{\alpha\alpha'}}{2}W_{\mathrm{dec.1}}(4_\alpha|3_{\alpha'}2_{\alpha'}1_\alpha)\delta\big(\mathfrak{E}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}-\mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\\
&\times \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_4-\mathbf{p}_3}\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'},0}\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}\big)\mathbf{p}_1[(\mathbf{p}_1-\mathbf{p}_4)(\mathbf{V}_{i\alpha}-\mathbf{V}_{i\alpha'})]\\
&+\frac{1}{1+\delta_{\alpha,\alpha'}}W_{\mathrm{dec.2}}(4_{\alpha'}|3_{\alpha'}2_\alpha 1_\alpha)\delta\big(\mathfrak{E}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha,0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\\
&\times \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_4-\mathbf{p}_3}\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}\big(1-\mathcal{F}^{(\alpha')}_{\mathbf{p}_4+\mathbf{Q}_{\alpha'},0}\big)\mathbf{p}_1[(\mathbf{p}_1+\mathbf{p}_2)(\mathbf{V}_{i\alpha}-\mathbf{V}_{i\alpha'})]\\
&+\frac{1}{2+4\delta_{\alpha,\alpha'}}W_{\mathrm{coal.}}(4_\alpha 3_{\alpha'}2_{\alpha'}|1_\alpha)\delta\big(\mathfrak{E}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}+\mathfrak{E}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}+\mathfrak{E}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'},0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\\
&\times \delta_{\mathbf{p}_1-\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\mathcal{F}^{(\alpha')}_{\mathbf{p}_2+\mathbf{Q}_{\alpha'},0}\mathcal{F}^{(\alpha')}_{\mathbf{p}_3+\mathbf{Q}_{\alpha'},0}\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}\mathbf{p}_1[(\mathbf{p}_1-\mathbf{p}_4)(\mathbf{V}_{i\alpha}-\mathbf{V}_{i\alpha'})]\Big\},
\end{aligned}
\tag{E12}
$$

where we make use of the momentum conservation in particle collisions. The factor 2 in Eq. (E12) arises from the summation over the spin index σ_1. Note that the sum (E12) vanishes identically if $\alpha'=\alpha$ (i.e., when, $\mathbf{V}_{i\alpha'}=\mathbf{V}_{i\alpha}$). This is a consequence of the fact that a given particle species cannot lose or gain momentum through interaction with itself. In what follows, we only consider a nontrivial case when $\alpha'=\beta\neq\alpha$. Note that an arbitrary vector $\mathbf{a}$ can be added to the velocities $\mathbf{V}_{i\alpha}$ without affecting the collision integrals. This is a direct consequence of the ambiguity related to the definition of the velocity $\mathbf{u}$, see Sec. IV.

Before proceeding further let us notice that, in view of the properties of the matrix $\langle f|V|i\rangle_b$ discussed at the end of Appendix D, the transition probabilities $W_{\mathrm{coal.}}(4_\alpha 3_\beta 2_\beta|1_\alpha)$ and $W_{\mathrm{dec.1}}(1_\alpha|2_\beta 3_\beta 4_\alpha)$ coincide, $W_{\mathrm{coal.}}(4_\alpha 3_\beta 2_\beta|1_\alpha)=W_{\mathrm{dec.1}}(1_\alpha|2_\beta 3_\beta 4_\alpha)$. Accounting for this fact and using other symmetries of the transition probabilities $W_q(i|f)$ [see a passage after Eq. (E7)] one gets, after some redefinitions of running variables,

$$
\begin{aligned}
\sum_{\mathbf{p}_1\sigma_1}\mathbf{p}_1 I_{\alpha\beta}=&-\frac{1}{T}\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4}\Big\{W_{\mathrm{scat.1}}(4_\alpha 3_\beta|2_\beta 1_\alpha)\delta\big(\mathfrak{E}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}+\mathfrak{E}^{(\beta)}_{\mathbf{p}_3+\mathbf{Q}_\beta,0}-\mathfrak{E}^{(\beta)}_{\mathbf{p}_2+\mathbf{Q}_\beta,0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\\
&\times \mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\beta)}_{\mathbf{p}_2+\mathbf{Q}_\beta,0}\big(1-\mathcal{F}^{(\beta)}_{\mathbf{p}_3+\mathbf{Q}_\beta,0}\big)\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}\big)(\mathbf{p}_1-\mathbf{p}_4)\big[(\mathbf{p}_1-\mathbf{p}_4)(\mathbf{V}_{i\alpha}-\mathbf{V}_{i\beta})\big]\\
&+\frac{1}{2}W_{\mathrm{scat.2}}(4_\beta 3_\beta|2_\alpha 1_\alpha)\delta\big(\mathfrak{E}^{(\beta)}_{\mathbf{p}_4+\mathbf{Q}_\beta,0}+\mathfrak{E}^{(\beta)}_{\mathbf{p}_3+\mathbf{Q}_\beta,0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha,0}-\mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\\
&\times \mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha,0}\big(1-\mathcal{F}^{(\beta)}_{\mathbf{p}_3+\mathbf{Q}_\beta,0}\big)\big(1-\mathcal{F}^{(\beta)}_{\mathbf{p}_4+\mathbf{Q}_\beta,0}\big)(\mathbf{p}_1+\mathbf{p}_2)[(\mathbf{p}_1+\mathbf{p}_2)(\mathbf{V}_{i\alpha}-\mathbf{V}_{i\beta})]
\end{aligned}
$$

$$
\begin{aligned}
&+ W_{\mathrm{dec.1}}(4_\alpha|3_\beta 2_\beta 1_\alpha)\delta\big(\mathfrak{E}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0} - \mathfrak{E}^{(\beta)}_{\mathbf{p}_3+\mathbf{Q}_\beta,0} - \mathfrak{E}^{(\beta)}_{\mathbf{p}_2+\mathbf{Q}_\beta,0} - \mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_4-\mathbf{p}_3}\\
&\times \mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\beta)}_{\mathbf{p}_2+\mathbf{Q}_\beta,0}\mathcal{F}^{(\beta)}_{\mathbf{p}_3+\mathbf{Q}_\beta,0}\big(1-\mathcal{F}^{(\alpha)}_{\mathbf{p}_4+\mathbf{Q}_\alpha,0}\big)(\mathbf{p}_1-\mathbf{p}_4)[(\mathbf{p}_1-\mathbf{p}_4)(\mathbf{V}_{i\alpha}-\mathbf{V}_{i\beta})]\\
&+ W_{\mathrm{dec.2}}(4_\beta|3_\beta 2_\alpha 1_\alpha)\delta\big(\mathfrak{E}^{(\beta)}_{\mathbf{p}_4+\mathbf{Q}_\beta,0} - \mathfrak{E}^{(\beta)}_{\mathbf{p}_3+\mathbf{Q}_\beta,0} - \mathfrak{E}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha,0} - \mathfrak{E}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\big)\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_4-\mathbf{p}_3}\\
&\times \mathcal{F}^{(\alpha)}_{\mathbf{p}_1+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\alpha)}_{\mathbf{p}_2+\mathbf{Q}_\alpha,0}\mathcal{F}^{(\beta)}_{\mathbf{p}_3+\mathbf{Q}_\beta,0}\big(1-\mathcal{F}^{(\beta)}_{\mathbf{p}_4+\mathbf{Q}_\beta,0}\big)(\mathbf{p}_1+\mathbf{p}_2)[(\mathbf{p}_1+\mathbf{p}_2)(\mathbf{V}_{i\alpha}-\mathbf{V}_{i\beta})]\Big\}.
\end{aligned}
\tag{E13}
$$

This formula can be further simplified provided that the hydrodynamic velocities are small. Since Eq. (E13) already contains a small difference $(\mathbf{V}_{i\alpha}-\mathbf{V}_{i\beta})$, one just needs to set $\mathbf{u}=\mathbf{Q}_\alpha=\mathbf{Q}_\beta=0$ in all other functions in this expression. The final result is

$$
\sum_{\mathbf{p}_1\sigma_1}\mathbf{p}_1 I_{\alpha\beta} = -J_{\alpha\beta}(\mathbf{V}_{i\alpha}-\mathbf{V}_{i\beta}),
\tag{E14}
$$

where

$$
\begin{aligned}
J_{\alpha\beta} = \frac{1}{3T}\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4}\Big[&W_{\mathrm{scat.1}}(4_\alpha 3_\beta|2_\beta 1_\alpha)\delta\big(E^{(\alpha)}_{\mathbf{p}_4}+E^{(\beta)}_{\mathbf{p}_3}-E^{(\beta)}_{\mathbf{p}_2}-E^{(\alpha)}_{\mathbf{p}_1}\big)\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\mathfrak{f}^{(\alpha)}_{\mathbf{p}_1}\mathfrak{f}^{(\beta)}_{\mathbf{p}_2}\big(1-\mathfrak{f}^{(\beta)}_{\mathbf{p}_3}\big)\big(1-\mathfrak{f}^{(\alpha)}_{\mathbf{p}_4}\big)(\mathbf{p}_1-\mathbf{p}_4)^2\\
&+\frac{1}{2}W_{\mathrm{scat.2}}(4_\beta 3_\beta|2_\alpha 1_\alpha)\delta\big(E^{(\beta)}_{\mathbf{p}_4}+E^{(\beta)}_{\mathbf{p}_3}-E^{(\alpha)}_{\mathbf{p}_2}-E^{(\alpha)}_{\mathbf{p}_1}\big)\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\mathfrak{f}^{(\alpha)}_{\mathbf{p}_1}\mathfrak{f}^{(\alpha)}_{\mathbf{p}_2}\big(1-\mathfrak{f}^{(\beta)}_{\mathbf{p}_3}\big)\big(1-\mathfrak{f}^{(\beta)}_{\mathbf{p}_4}\big)(\mathbf{p}_1+\mathbf{p}_2)^2\\
&+W_{\mathrm{dec.1}}(4_\alpha|3_\beta 2_\beta 1_\alpha)\delta\big(E^{(\alpha)}_{\mathbf{p}_4}-E^{(\beta)}_{\mathbf{p}_3}-E^{(\beta)}_{\mathbf{p}_2}-E^{(\alpha)}_{\mathbf{p}_1}\big)\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_4-\mathbf{p}_3}\mathfrak{f}^{(\alpha)}_{\mathbf{p}_1}\mathfrak{f}^{(\beta)}_{\mathbf{p}_2}\mathfrak{f}^{(\beta)}_{\mathbf{p}_3}\big(1-\mathfrak{f}^{(\alpha)}_{\mathbf{p}_4}\big)(\mathbf{p}_1-\mathbf{p}_4)^2\\
&+W_{\mathrm{dec.2}}(4_\beta|3_\beta 2_\alpha 1_\alpha)\delta\big(E^{(\beta)}_{\mathbf{p}_4}-E^{(\beta)}_{\mathbf{p}_3}-E^{(\alpha)}_{\mathbf{p}_2}-E^{(\alpha)}_{\mathbf{p}_1}\big)\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_4-\mathbf{p}_3}\mathfrak{f}^{(\alpha)}_{\mathbf{p}_1}\mathfrak{f}^{(\alpha)}_{\mathbf{p}_2}\mathfrak{f}^{(\beta)}_{\mathbf{p}_3}\big(1-\mathfrak{f}^{(\beta)}_{\mathbf{p}_4}\big)(\mathbf{p}_1+\mathbf{p}_2)^2\Big]
\end{aligned}
\tag{E15}
$$

is the momentum transfer rate.

Since the total momentum of the mixture is conserved in collisions, the momentum transfer rate should be symmetric with respect to interchanging of particle species indices: $J_{\alpha\beta}=J_{\beta\alpha}$. It is easy to verify that Eq. (E15) satisfies this property. Indeed, the scattering terms in Eq. (E15) are obviously symmetric, while the "dec.1" term turns into the "dec.2" term (and vice versa) after the replacement $\alpha\leftrightarrow\beta$. To see this, one should compare the expression (E5) with (E6) after interchanging the indices $(\alpha\leftrightarrow\beta)$ and running variables $(\mathbf{p}_1\leftrightarrow\mathbf{p}_3)$.

The expression for the momentum transfer rate was obtained under assumption that both particle species are superfluid. If one of them (say, the species α) is normal, the result should be modified in two ways. First, the corresponding distribution function $\mathfrak{f}^{(\alpha)}_{\mathbf{p}}$ and energy $E^{(\alpha)}_{\mathbf{p}}$ of Bogoliubov excitations should be replaced with the (quasi)particle distribution function $n^{(\alpha)}_{\mathbf{p}}$ and the energy $\varepsilon^{(\alpha)}_{\mathbf{p}}$, respectively. Second, the collisions that do not conserve the number of (quasi)particles of species α should be disregarded, i.e., one should set $W_{\mathrm{scat.2}}(4_\beta 3_\beta|2_\alpha 1_\alpha)=W_{\mathrm{dec.2}}(4_\beta|3_\beta 2_\alpha 1_\alpha)=0$.

---

[1] P. Haensel, A. Y. Potekhin, and D. G. Yakovlev, *Neutron Stars 1: Equation of State and Structure*, Astrophysics and Space Science Library Vol. 326 (Springer, Berlin, 2007).

[2] U. Lombardo and H.-J. Schulze, in *Physics of Neutron Star Interiors*, Lecture Notes in Physics Vol. 578, edited by D. Blaschke, N. K. Glendenning, and A. Sedrakian (Springer, Berlin, 2001), p. 30.

[3] A. Sedrakian and J. W. Clark, Eur. Phys. J. A **55**, 167 (2019).

[4] I. Khalatnikov, *An Introduction To The Theory Of Superfluidity*, Advanced Book Classics (Westview, Boulder, CO, 2000).

[5] I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **32**, 653 (1957) [Sov. Phys. JETP **5**, 542 (1957)].

[6] G. Mendell and L. Lindblom, Ann. Phys. (NY) **205**, 110 (1991).

[7] K. Glampedakis, N. Andersson, and L. Samuelsson, Mon. Not. R. Astron. Soc. **410**, 805 (2011).

[8] M. E. Gusakov and V. A. Dommes, Phys. Rev. D **94**, 083006 (2016).

[9] P. B. Rau and I. Wasserman, Phys. Rev. D **102**, 063011 (2020).

[10] V. A. Dommes and M. E. Gusakov, Phys. Rev. D **104**, 123008 (2021).

[11] A. Schmitt and P. Shternin, in *The Physics and Astrophysics of Neutron Stars*, Astrophysics and Space Science Library Vol. 457, edited by L. Rezzolla, P. Pizzochero, D. I. Jones, N. Rea, and I. Vidaña (Springer, Berlin, 2018), p. 455.

[12] L. Landau and E. Lifshitz, *Fluid Mechanics: Volume 6* (Elsevier Science, Amsterdam, 2013).

[13] R. H. Anderson, C. J. Pethick, and K. F. Quader, Phys. Rev. B **35**, 1620 (1987).

[14] D. G. Yakovlev and D. A. Shalybkov, Pis'ma v Astronomicheskii Zhurnal, **16**, 202 (1990) [Sov. Astron. Lett. **16**, 86 (1990)].

[15] D. G. Yakovlev and D. A. Shalybkov, Astrophys. Space. Sci. **176**, 171 (1991).

[16] D. G. Yakovlev and D. A. Shalybkov, Astrophys. Space. Sci. **176**, 191 (1991).

[17] P. S. Shternin, Sov. J. Exp. Theor. Phys. **107**, 212 (2008).

[18] P. Shternin and I. Vidaña, Universe **7**, 203 (2021).

[19] P. S. Shternin and M. Baldo, Phys. Rev. D **102**, 063010 (2020).

[20] H. Heiselberg and C. J. Pethick, Phys. Rev. D **48**, 2916 (1993).

[21] M. G. Alford, H. Nishimura, and A. Sedrakian, Phys. Rev. C **90**, 055205 (2014).

[22] P. Goldreich and A. Reisenegger, Astrophys. J. **395**, 250 (1992).
[23] D. A. Shalybkov and V. A. Urpin, Mon. Not. R. Astron. Soc. **273**, 643 (1995).
[24] V. Urpin and D. Shalybkov, Mon. Not. R. Astron. Soc. **304**, 451 (1999).
[25] J. Hoyos, A. Reisenegger, and J. A. Valdivia, Astron. Astrophys. **487**, 789 (2008).
[26] J. H. Hoyos, A. Reisenegger, and J. A. Valdivia, Mon. Not. R. Astron. Soc. **408**, 1730 (2010).
[27] K. Glampedakis, D. I. Jones, and L. Samuelsson, Mon. Not. R. Astron. Soc. **413**, 2021 (2011).
[28] A. M. Beloborodov and X. Li, Astrophys. J. **833**, 261 (2016).
[29] F. Castillo, A. Reisenegger, and J. A. Valdivia, Mon. Not. R. Astron. Soc. **471**, 507 (2017).
[30] A. Passamonti, T. Akgün, J. A. Pons, and J. A. Miralles, Mon. Not. R. Astron. Soc. **465**, 3416 (2017).
[31] M. E. Gusakov, E. M. Kantor, and D. D. Ofengeim, Phys. Rev. D **96**, 103012 (2017).
[32] E. M. Kantor and M. E. Gusakov, Mon. Not. R. Astron. Soc. **473**, 4272 (2018).
[33] M. E. Gusakov, E. M. Kantor, and D. D. Ofengeim, Mon. Not. R. Astron. Soc. **499**, 4561 (2020).
[34] K. Y. Kraav, M. E. Gusakov, and E. M. Kantor, Mon. Not. R. Astron. Soc. **506**, L74 (2021).
[35] E. Østgaard and D. G. Yakovlev, Nucl. Phys. A **540**, 211 (1992).
[36] P. Bhattacharyya, C. J. Pethick, and H. Smith, Phys. Rev. B **15**, 3367 (1977).
[37] P. S. Shternin and D. G. Yakovlev, Phys. Rev. D **75**, 103004 (2007).
[38] P. S. Shternin and D. G. Yakovlev, Phys. Rev. D **78**, 063006 (2008).
[39] P. S. Shternin, Phys. Rev. D **98**, 063015 (2018).
[40] M. E. Gusakov, E. M. Kantor, and P. Haensel, Phys. Rev. C **80**, 015803 (2009).
[41] A. F. Andreev and E. P. Bashkin, Zh. Eksp. Teor. Fiz. **69**, 319 (1975) [Sov. Phys. JETP **42**, 164 (1976)].
[42] M. Borumand, R. Joynt, and W. Kluźniak, Phys. Rev. C **54**, 2745 (1996).
[43] M. E. Gusakov and P. Haensel, Nucl. Phys. A **761**, 333 (2005).
[44] N. Chamel and P. Haensel, Phys. Rev. C **73**, 045802 (2006).
[45] M. E. Gusakov, E. M. Kantor, and P. Haensel, Phys. Rev. C **79**, 055806 (2009).
[46] A. Sourie, M. Oertel, and J. Novak, Phys. Rev. D **93**, 083004 (2016).
[47] L. B. Leinson, Mon. Not. R. Astron. Soc. **479**, 3778 (2018).
[48] N. Chamel and V. Allard, Phys. Rev. C **100**, 065801 (2019).
[49] V. Allard and N. Chamel, Phys. Rev. C **103**, 025804 (2021).
[50] G. Baym and S. A. Chin, Nucl. Phys. A **262**, 527 (1976).
[51] M. E. Gusakov, Phys. Rev. C **81**, 025804 (2010).
[52] D. Pines and P. Nozières, *Theory of Quantum Liquids. Volume I: Normal Fermi liquids* (Benjamin, New York, 1966).
[53] E. Krotscheck, P. Papakonstantinou, and J. Wang, arXiv:2305.07096.
[54] D. G. Yakovlev, K. P. Levenfish, and Y. A. Shibanov, Phys. Usp. **42**, 737 (1999).
[55] D. A. Baiko, P. Haensel, and D. G. Yakovlev, Astron. Astrophys. **374**, 151 (2001).
[56] P. Haensel, K. P. Levenfish, and D. G. Yakovlev, Astron. Astrophys. **381**, 1080 (2002).
[57] C. Manuel, S. Sarkar, and L. Tolos, Phys. Rev. C **90**, 055803 (2014).
[58] L. B. Leinson, Mon. Not. R. Astron. Soc. **470**, 3374 (2017).
[59] D. J. Dean and M. Hjorth-Jensen, Rev. Mod. Phys. **75**, 607 (2003).
[60] D. Ding, A. Rios, H. Dussan, W. H. Dickhoff, S. J. Witte, A. Carbone, and A. Polls, Phys. Rev. C **94**, 025802 (2016).
[61] A. J. Leggett, Phys. Rev. **140**, A1869 (1965).
[62] S. Goriely, N. Chamel, and J. M. Pearson, Phys. Rev. C **88**, 061302(R) (2013).
[63] E. M. Kantor and M. E. Gusakov, J. Phys. Conf. Ser. **1697**, 012005 (2020).
[64] A. G. Aronov, Y. M. Gal'Perin, V. L. Gurevich, and V. I. Kozub, Adv. Phys. **30**, 539 (1981).
[65] O. Betbeder-Matibet and P. Nozieres, Ann. Phys. (NY) **51**, 392 (1969).
[66] L. Landau, E. Lifshitz, and L. Pitaevskii, *Statistical Physics*, Course of Theoretical Physics No. 2 (Butterworth-Heinemann, Oxford, 1980).
[67] J. Bardeen, Rev. Mod. Phys. **34**, 667 (1962).
[68] M. E. Gusakov and E. M. Kantor, Mon. Not. R. Astron. Soc. **428**, L26 (2013).
[69] S. Chapman, T. Cowling, and C. Cercignani, *The Mathematical Theory of Non-uniform Gases: An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases*, Cambridge Mathematical Library (Cambridge University Press, Cambridge, 1970).
[70] M. E. Gusakov, Phys. Rev. D **93**, 064033 (2016).
[71] M. E. Gusakov, Phys. Rev. D **76**, 083001 (2007).
[72] S. Putterman, *Superfluid Hydrodynamics*, Low Temperature Physics Series (North-Holland, Amsterdam, 1974).
[73] J. Ziman, *Electrons and Phonons: The Theory of Transport Phenomena in Solids*, International Series of Monographs on Physics (Clarendon, Oxford, 1960).
[74] V. A. Dommes, M. E. Gusakov, and P. S. Shternin, Phys. Rev. D **101**, 103020 (2020).
[75] L. Pitaevskii and E. Lifshitz, *Physical Kinetics*, Course of Theoretical Physics (Elsevier Science, Amsterdam, 1981).
[76] N. Andersson and G. L. Comer, Living Rev. Relativity **24**, 3 (2021).
[77] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Course of Theoretical Physics (Butterworth Heinemann, Oxford, 1975).
[78] J. Blaizot and G. Ripka, *Quantum Theory of Finite Systems* (MIT Press, Cambridge, 1986).