Neutron contribution to the force on a proton vortex in superconducting neutron-star matter

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We investigate the forces acting on a proton vortex in superconducting neutron-star matter composed of neutrons, protons, and electrons, accounting for Fermi-liquid interactions in the neutron-proton subsystem. While the force arising from electron scattering by the vortex magnetic field is well known, we demonstrate that normal neutrons also exert a force on proton vortices due to Fermi-liquid coupling with superconducting protons. Using a kinetic approach based on Landau Fermi-liquid theory, we show that neutron quasiparticles scatter off a proton vortex through the spatially varying condensate momentum, giving rise to a longitudinal force proportional to the relative neutron-vortex velocity. In contrast with the electron contribution, this force has no transverse component and vanishes in the absence of neutron-proton Fermi-liquid interaction. Within a simple model, we analyze the main properties of this force and provide physically motivated estimates of its magnitude.

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I. INTRODUCTION

Studying neutron stars (NSs), the densest and most strongly magnetized objects known, requires a careful treatment of a wide range of physical phenomena, including nuclear interactions, superfluidity, superconductivity, and processes occurring in strong electromagnetic fields. Although the value of the proton pairing gap remains a topic of active theoretical research (see, e.g., Ref. [1]), it is widely believed that a substantial fraction of the NS core becomes superconducting once its temperature drops below a few times 10^9 K. While the properties of the deeper core regions remain more uncertain, the outer layers are expected to enter the type-II superconducting state [2]. In type-II superconductors, the magnetic field is confined within proton vortices (flux tubes). Therefore, understanding the forces acting on these vortices is required for accurate modeling of the NS magnetic-field evolution, as well as for calculating the damping rates of various stellar oscillation modes.

The forces acting on proton vortices (or similar objects) due to their motion relative to the surrounding matter have been the subject of numerous studies [2–12]. However, the analyses in most of these works were carried out under the assumption of vanishing temperature, with protons (neutrons) treated as entirely superconducting (superfluid). Under such conditions, the normal (nonsuperfluid) component consists solely of leptons, which scatter off the vortex magnetic field. Therefore, the force acting on the vortex arises as a reaction to the Lorentz force exerted on the leptons. In fact, in the zero-temperature limit, the lepton contribution may be regarded as the only velocity-dependent force acting on the proton vortex [10,12].

What would change if we allowed for the nucleonic thermal excitations? It is natural to expect that, if there exists a mechanism by which the thermal excitations can sense the presence of a vortex, an additional contribution to the total force may arise. To the best of our knowledge, Jones (2009) [6] is one of the very few works that attempt to account for nucleonic thermal excitations in NS proton-vortex dynamics, considering interactions with quasiparticle excitations bound to the vortex core. Thus, the role of the normal nucleon component remains underexplored.

It is well known from the theory of conventional superconductors that the energy of a Bogoliubov excitation depends explicitly on the condensate momentum $\mathbf{Q}$ [13]. To linear order in $\mathbf{Q}$, it takes the form

$$\varepsilon_{\mathbf{p}} = \frac{\mathbf{pQ}}{m} + \sqrt{(\epsilon_{\mathbf{p}} - E_F)^2 + \Delta^2}, \tag{1}$$

where m is the (quasi)particle mass, $\epsilon_{\mathbf{p}}$ is the (quasi)particle energy in the normal state, E_F is the Fermi energy, and Δ is the superfluid energy gap. The spatially varying condensate momentum can be treated as a vector field that can scatter the Bogoliubov excitations. This scattering gives rise to a force whose transverse component is often referred to as the Iordanskii force [14].

In nucleonic matter, the dependence of the excitation energy on the condensate momentum is more involved. The neutron-proton subsystem forms a mixture in which Fermi-liquid effects play a significant role. These effects imply that the energy of a quasiparticle of a given species α, say $\alpha = n$, depends on the distribution functions of quasiparticles of both species, $\alpha' = n, p$. In particular, one can expect that, in such a mixture, quasiparticles of both species interact with the condensate momentum associated with a proton vortex. In the present paper, we show that this interaction gives rise to a force exerted on a proton vortex by the neutron component, proportional to the incident flux of normal neutrons.

The paper is organized as follows: In Sec. II, we formulate the problem and discuss the main assumptions. Section III describes the interaction of the different constituents of the

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mixture with a proton vortex. This section is divided into three sections devoted, respectively, to protons (together with the electromagnetic field), electrons, and neutrons. In Sec. IV, we calculate the contributions of the different constituents to the total force. In Sec. V, we discuss the obtained results and present our conclusions. For clarity of presentation, most of the technical derivations are relegated to Appendixes A–D.

II. PROBLEM FORMULATION

We consider a mixture consisting of neutrons, protons, and electrons. The neutrons and protons are assumed to form a Fermi-liquid mixture. In the present paper, we consider a mixture at temperatures T well below the proton critical temperature $T_{\rm cp}$, which allows us to neglect proton thermal excitations. At the same time, we assume that $T > T_{\rm cn}$, where $T_{\rm cn}$ is the neutron critical temperature. Therefore, the neutron component remains nonsuperfluid.

We assume the following hierarchy of length scales:

$$\hbar/p_{F\alpha} \ll \xi \ll \lambda \ll d_v \ll \ell_\alpha, \tag{2}$$

where $p_{F\alpha}$ are the Fermi momenta of particle species α, ξ is the proton coherence length, λ is the London penetration depth, d_v is the intervortex distance, and, finally, ℓ_α are the characteristic mean-free paths of particle species α in the absence of vortices. These inequalities allow us to substantially simplify the problem. Owing to the large mean-free paths ℓ_α compared with all other characteristic length scales, we can neglect collisions between (quasi)particles. More rigorous conditions on ℓ_α that justify the ballistic regime for the problem considered are discussed in Secs. III B and III C. The large intervortex distance d_v compared with the London penetration depth λ justifies neglecting the influence of neighboring vortices, allowing us to consider an isolated proton vortex.

Among the conditions (2), the least well-justified is that relating ξ and λ. These two quantities are typically of comparable magnitude in NS interiors [2]. In the present paper, however, we adopt the simplifying assumption that the vortex core is treated as infinitely thin, while the London penetration depth remains finite. Although this assumption is not entirely realistic for NS interiors, it substantially simplifies the analysis and allows us to isolate the physical mechanisms underlying the forces under consideration. The limitations of this model are discussed in Sec. V.

The vortex is assumed to be straight, enabling us to introduce cylindrical coordinates (ρ, ϕ, z) with the z axis coinciding with the vortex axis and directed such that the integral (9) is positive, see below. Various particle species are allowed to flow through the vortex. The incident fluxes are taken to be uniform and stationary. We further assume that the electric field vanishes far from the vortex. Consequently, the quasineutrality condition

$$n_{p0} = n_{e0} \tag{3}$$

is satisfied for the unperturbed proton and electron densities n_{p0} and n_{e0}, respectively.

The main goal of the present paper is to analyze the forces acting on the proton vortex induced by the incident-particle fluxes. To define these forces, we utilize a momentum conservation equation, which, for our system, has the following form (see Appendix C and Refs. [10,12]):

$$\frac{\partial \mathfrak{P}^i}{\partial t} + \frac{\partial \Pi^{ik}}{\partial r^k} = \delta^{(2)}(\boldsymbol{\rho})\, F^i_{\rm ext}, \tag{4}$$

where the total momentum density $\mathfrak{P}^i$ and the total stress tensor Π^{ik} consist of contributions from superconducting *npe* matter as well as from the electromagnetic field. The exact expressions for these quantities are given by Eqs. (C24) and (C25), respectively. The right-hand side of Eq. (4) contains a term that we refer to as "external force." This term is added to ensure momentum conservation in the general case (as discussed shortly below). The two-dimensional delta function is explicitly written to emphasize that the external force is applied to the infinitely thin vortex core.

If all the incident fluxes are independent of time, the problem is stationary. In this case, the time derivative in Eq. (4) is zero. Let us choose a cylinder of unit length and radius R, whose axis coincides with the vortex axis. Integrating Eq. (4) over the volume of this cylinder, we obtain a force balance equation of the following form:

$$\mathbf{F}_{\rm m\to v} + \mathbf{F}_{\rm ext} = 0. \tag{5}$$

In this equation, the first term is defined as

$$F^i_{\rm m\to v} = -R \int_0^{2\pi} \Pi^{i\rho}\, d\phi. \tag{6}$$

In deriving $\mathbf{F}_{\rm m\to v}$, the volume integral was converted into the surface integral using Gauss's theorem. Accordingly, the integration over a cylindrical surface reduces to the integration over the azimuthal angle, multiplied by the radius R and the unit length of the cylinder. The quantity $\mathbf{F}_{\rm m\to v}$ describes the momentum flux entering the cylinder. It can be interpreted as the total force (per unit length) acting on the vortex from the surrounding matter. Therefore, the force given by Eq. (6) is the subject of our analysis.

The second term in the force balance equation (5) represents the external force applied to the vortex core, as previously stated. This external force counteracts the force $\mathbf{F}_{\rm m\to v}$. But why does such an external force appear in the force balance equation? Recall that the vortex is immersed in stationary incident fluxes and is prescribed to remain at rest. However, in the general case, the total force $\mathbf{F}_{\rm m\to v}$, which includes the contributions from the various particle species, does not vanish unless the particle fluxes are fine-tuned. Consequently, to maintain the stationarity, it must be counterbalanced by some force, which, in our model, is represented by the formally introduced term $\mathbf{F}_{\rm ext}$. In real NSs, such a force can be provided by buoyancy or tension effects [10,15], as well as by pinning to neutron vortices. A more detailed discussion of Eq. (5) can be found in Ref. [12].

III. INTERACTION OF *npe* MIXTURE WITH A PROTON VORTEX

The stress tensor appearing in Eqs. (4) and (6) includes contributions from protons, neutrons, electrons, and the electromagnetic field [see Eq. (C25)]. Therefore, before evaluating the force, we must first determine the behavior of each constituent in the vicinity of the vortex. The present section is devoted to this task.

A. Protons and the electromagnetic field

To describe the superconducting protons, we introduce the condensate momentum [16]

$$\mathbf{Q}_p = \frac{\hbar}{2}\nabla\chi_p - \frac{e}{c}\mathbf{A}, \tag{7}$$

and the nonequilibrium proton chemical potential [16–18]

$$\breve{\mu}_p = E_{Fp} - \frac{\hbar}{2}\frac{\partial\chi_p}{\partial t} - e\varphi, \tag{8}$$

where χ_p is the order parameter phase, φ and $\mathbf{A}$ are the electrostatic and vector potentials, $e = |e|$ is the elementary charge, and E_{Fp} is the proton Fermi energy calculated in the absence of the vortex as well as any currents. The vortex manifests itself through a nonzero circulation of the phase gradient:

$$\frac{\hbar}{2}\oint \nabla\chi_p d\mathbf{l} = m_p\varkappa, \tag{9}$$

where the integral is taken along an arbitrary contour encircling the z axis once in the counterclockwise direction as seen from $+z$ and

$$\varkappa = \frac{2\pi\hbar}{2m_p} \tag{10}$$

is the circulation quantum.

The vector potential can be represented as $\mathbf{A} = \mathbf{A}_v + \delta\mathbf{A}$, where $\mathbf{A}_v$ is the vector potential generated by the vortex while $\delta\mathbf{A}$ is a small correction to $\mathbf{A}_v$ arising due to the nonzero particle flows streaming through the vortex. The vortex vector potential can be represented as

$$\mathbf{A}_v = \frac{c}{e}\frac{m_p\varkappa}{2\pi\rho}\mathcal{P}(\rho)\mathbf{e}_\phi, \tag{11}$$

where $\mathcal{P}(\rho)$ is a function that will be specified below. The magnetic field, in turn, equals $\mathbf{B} = \mathrm{curl}\mathbf{A} = \mathbf{B}_v + \delta\mathbf{B}$, where

$$\mathbf{B}_v = \mathrm{curl}\mathbf{A}_v = \frac{c}{e}\frac{m_p\varkappa}{2\pi\rho}\mathcal{P}'(\rho)\mathbf{e}_z, \tag{12}$$

$\mathcal{P}'(\rho)$ denotes the derivative of the function $\mathcal{P}(\rho)$, and $\delta\mathbf{B} = \mathrm{curl}\delta\mathbf{A}$. In Eqs. (11) and (12), $\mathbf{e}_\phi$ and $\mathbf{e}_z$ denote the unit vectors in the ϕ and z directions. One can also calculate the magnetic flux enclosed within a circle of radius ρ:

$$\Phi_B(\rho) = \int_\rho d^2\boldsymbol{\rho}_1\, \mathbf{e}_z\mathbf{B}_v(\rho_1) = \Phi_0\mathcal{P}(\rho), \tag{13}$$

where $\Phi_0 = cm_p\varkappa/e$ is the total vortex magnetic flux. Hence, the function $\mathcal{P}(\rho)$ can be interpreted as the dimensionless magnetic flux enclosed by the circle of radius ρ. It varies from 0 at the vortex axis to 1 at distances much larger than λ. To determine the exact form of the function $\mathcal{P}(\rho)$, one must solve the microscopic quantum problem in conjunction with Maxwell's equations. However, for the case of extreme type-II superconductivity ($\xi \ll \lambda$), it can be easily obtained and equals [10,13]

$$\mathcal{P}(\rho) = 1 - \frac{\rho}{\lambda}K_1\left(\frac{\rho}{\lambda}\right). \tag{14}$$

Consequently,

$$\mathcal{P}'(\rho) = \frac{\rho}{\lambda^2}K_0\left(\frac{\rho}{\lambda}\right), \tag{15}$$

where K_n denotes the modified Bessel function of the second kind.

The total condensate momentum (7) can be represented as

$$\mathbf{Q}_p = \mathbf{Q}_{p0} + \mathbf{Q}_{pv} + \delta\mathbf{Q}_p. \tag{16}$$

Here, the first term, $\mathbf{Q}_{p0}$, is a constant vector describing the homogeneous flow measured far from the vortex. The second term,

$$\mathbf{Q}_{pv} = \frac{m_p\varkappa}{2\pi\rho}[1 - \mathcal{P}(\rho)]\mathbf{e}_\phi, \tag{17}$$

represents the vortex flow. Finally, the third term of Eq. (16) is the correction induced by the interaction of the incident fluxes with the vortex. A detailed analysis of this correction can be found in Ref. [12] for the case of vanishing temperature. For the purposes of the present study, the exact form of $\delta\mathbf{Q}_p$ is not required. However, in the following calculations, we use the fact that this correction is linear in the incident flux and decays as ρ^{-1} for $\rho \gg \lambda$.

The subsequent calculations require the curl of the condensate momentum, which equals

$$\mathrm{curl}\mathbf{Q}_p = \frac{m_p\varkappa}{2\pi\rho}\delta(\rho)\mathbf{e}_z - \frac{e}{c}\mathbf{B} = \mathrm{curl}\mathbf{Q}_{pv} - \frac{e}{c}\delta\mathbf{B}, \tag{18}$$

where

$$\mathrm{curl}\mathbf{Q}_{pv} = -\frac{m_p\varkappa}{2\pi\rho}\tilde{\mathcal{P}}'(\rho)\mathbf{e}_z, \tag{19}$$

and we introduced the function

$$\tilde{\mathcal{P}}'(\rho) = \mathcal{P}'(\rho) - \delta(\rho), \tag{20}$$

with $\delta(\rho)$ denoting the Dirac delta function. The appearance of the delta function in Eq. (20) requires some clarification. It formally arises from taking the curl of the superfluid phase gradient. It is well known from vector analysis that taking the curl of a gradient of any well-defined, single-valued function yields identically zero. However, the phase χ_p and, consequently, the condensate momentum $\mathbf{Q}_p$ are not defined (singular) at the vortex axis. This singularity arises due to the nonzero circulation of the phase gradient. Using the identity (9), we define the formal construction curl $\nabla\chi_p$ at $\rho = 0$ by means of Stokes' theorem:

$$\frac{\hbar}{2}\int \mathrm{curl}\,\nabla\chi_p\mathbf{n}dS = \frac{\hbar}{2}\oint \nabla\chi_p d\mathbf{l} = m_p\varkappa, \tag{21}$$

where the left-hand integration is performed over an arbitrary (oriented) surface punctured by the vortex ($\mathbf{n}$ is the unit vector normal to the surface), while the right-hand integration is taken over the oriented boundary of this surface. Thus, for a straight vortex, we have $\mathrm{curl}\nabla\chi_p = 2\pi\delta^{(2)}(\boldsymbol{\rho})\,\mathbf{e}_z$. To arrive at Eq. (18), we applied the identity

$$\delta^{(2)}(\boldsymbol{\rho}) = \frac{\delta(\rho)}{2\pi\rho}. \tag{22}$$

By analogy with the vortex magnetic field $\mathbf{B}_v$, we define the flux of the vector $\mathrm{curl}\mathbf{Q}_{pv}$ enclosed within a circle of radius ρ:

$$\Phi_Q(\rho) = \frac{c}{e}\int_\rho d^2\boldsymbol{\rho}_1\, \mathbf{e}_z \mathrm{curl}\mathbf{Q}_{pv}(\rho_1) = \frac{c}{e}\oint_\rho d\mathbf{l}\,\mathbf{Q}_{pv} = \Phi_0[1-\mathcal{P}(\rho)]. \tag{23}$$

The factor c/e is used to make the dimension of Φ_Q consistent with that of Φ_B. Since the function $\mathcal{P}(\rho)$ tends to unity at large distances, the total flux Φ_Q equals zero. The vanishing of the total flux Φ_Q can be regarded as a consequence of the Meissner effect, resulting in the exponential suppression of the vortex circular current.

We assume that the proton incident flux is small; specifically,

$$\iota_p = \frac{Q_{p0}}{p_{Fp}} \ll 1. \tag{24}$$

Separately, we assume the smallness of the following ratio:

$$\frac{Q_{pv}}{p_{Fp}} \ll 1. \tag{25}$$

The latter condition is clearly violated in the vicinity of the vortex axis since Q_{pv} diverges as $\rho \to 0$. As it was stated, we adopt a model where the vortex core is approximated by an infinitely thin line. However, in reality, the vortex core has a finite radius of the order of the coherence length ξ given by the following formula:

$$\xi = \frac{\hbar p_{Fp}}{\pi\Delta^{(p)} m_p^*}, \tag{26}$$

where $\Delta^{(p)}$ is the proton energy gap and m_p^* is the proton effective mass. Thus, in the region outside the core (where our model is valid), we can estimate:

$$\frac{Q_{pv}}{p_{Fp}} \lesssim \frac{m_p\varkappa}{2\pi\xi p_{Fp}}. \tag{27}$$

Introducing the small parameter

$$\epsilon = \frac{m_p^*\Delta^{(p)}}{p_{Fp}^2}, \tag{28}$$

and making use of definitions (26) and (10), from Eq. (27), we get:

$$\frac{Q_{pv}}{p_{Fp}} \lesssim \frac{\pi}{2}\epsilon. \tag{29}$$

In what follows, we expand the quantities of interest as a power series in the small parameter ϵ. In particular, the quantities $\delta\mathbf{A}$, $\delta\mathbf{B}$, and $\delta\mathbf{Q}_p$ can be expanded in ϵ starting from the linear term.

Strictly speaking, there is another dimensionless parameter in the problem:

$$\frac{Q_p p_{Fp}}{m_p\Delta^{(p)}}, \tag{30}$$

which cannot be considered small everywhere. Finite values of this parameter lead to the existence of bounded Bogoliubov excitations even in the limit of vanishing temperature. This complicates the expression for the proton current density, reduces the proton gap compared with its unperturbed value, and eventually results in the breakdown of the superconductivity [13]. In our model, we include the region where these effects become substantial in the vortex core. Thus, from the physical point of view, we effectively consider only the region $\rho > \xi$.

In the stationary case, the chemical potential $\breve{\mu}_p$ is related to the electric field via the following equation [cf. Eq. (C3)]:

$$\nabla\breve{\mu}_p = -e\nabla\varphi = e\mathbf{E}. \tag{31}$$

If there are no incident fluxes, the electric field is zero [10] and, therefore, the function $\breve{\mu}_p$ remains unperturbed. The same is true if there is no vortex. Thus, accounting for the definition (8), we can represent $\breve{\mu}_p$ as

$$\breve{\mu}_p = E_{Fp} + \delta\breve{\mu}_p, \tag{32}$$

where $\delta\breve{\mu}_p$ is a small correction caused by the interaction of the vortex with the incident flux. The electric field and the correction $\delta\breve{\mu}_p$, in principle, should be determined self-consistently. However, the exact form of these quantities is not required for the calculation of the force.

B. Electrons

The problem of electron scattering off a proton vortex has been previously studied in Refs. [10,12]. In this section, we provide a brief summary of the key results of those works.

The electrons are treated as an ideal relativistic gas. Since the typical electron wavelength is much smaller than the other length scales, their behavior can be described using a distribution function $\mathcal{N}_{\mathbf{p}}^{(e)}$ satisfying the Boltzmann-Vlasov equation:

$$\boldsymbol{v}_{\mathbf{p}}\frac{\partial\mathcal{N}_{\mathbf{p}}^{(e)}}{\partial\mathbf{r}} - e\left(\mathbf{E} + \frac{1}{c}\boldsymbol{v}_{\mathbf{p}}\times\mathbf{B}\right)\frac{\partial\mathcal{N}_{\mathbf{p}}^{(e)}}{\partial\mathbf{p}} = I^{(e)}, \tag{33}$$

where $\boldsymbol{v}_{\mathbf{p}} = \mathbf{p}c^2/\varepsilon_{\mathbf{p}}^{(e)}$ and $\varepsilon_{\mathbf{p}}^{(e)} = (m_e^2c^4 + p^2c^2)^{1/2}$.

We assume that collisions between (quasi)particles are negligible; hence, the collision integral $I^{(e)}$ in Eq. (33) can be neglected. We state the condition under which this approximation is justified at the end of this section.

We further assume that, in the absence of the vortex, the electron distribution function reduces to the standard Fermi distribution. Accordingly, in the presence of the vortex, we seek the electron distribution function in the following form:

$$\mathcal{N}_{\mathbf{p}}^{(e)} = f_F\left(\varepsilon_{\mathbf{p}}^{(e)} - E_{Fe} - e\varphi - \mathbf{p}\mathbf{V}_e + g_e\right), \tag{34}$$

where

$$f_F(x) = \frac{1}{e^{x/T}+1} \tag{35}$$

is the Fermi distribution function, E_{Fe} is the electron (unperturbed) Fermi energy, φ is the electrostatic potential, $\mathbf{V}_e$ is the velocity of the electron incident flux, and g_e is a function of $\mathbf{r}$ and $\mathbf{p}$ to be determined. The argument of the function in Eq. (34) is written to linear order in $\mathbf{V}_e$. Since we evaluate the force acting on the proton vortex in the linear approximation with respect to $\mathbf{V}_e$, this level of accuracy in Eq. (34) is sufficient. The function g_e describes the correction due to the presence of the vortex. Note that the electric field in the considered problem is induced solely by the interaction between the vortex and the incident charged fluxes. Therefore, the term $-e\varphi$ should, strictly speaking, be regarded as part of the function g_e. However, by explicitly isolating this contribution, the electric interaction can be fully taken into account within the accepted level of accuracy and, consequently, the function g_e in Eq. (34) describes only the magnetic interaction with the vortex [see Eq. (39) below]. Since a particle flow directed along the vortex cannot generate a force via the mechanisms considered in the present paper, we assume, without loss of generality, that $\mathbf{V}_e \perp \mathbf{e}_z$.

It is instructive to examine small parameters governing the electron-scattering problem. We assume that the velocity of the incident electron flux is small, specifically,

$$\iota_e = \frac{V_e}{v_{Fe}} \ll 1, \tag{36}$$

where v_{Fe} is the electron Fermi velocity. As was already noted, the electric field $\mathbf{E}$ and the correction to the magnetic field $\delta\mathbf{B}$ are generated by the interaction of the incident fluxes with the vortex. Consequently, these quantities are at least linear in the parameter ι_e or in the analogous small parameters characterizing the other incident fluxes [see Eqs. (24) and (53)].

Besides these ratios, Eq. (33) contains another small parameter,

$$\tilde{\epsilon} = \frac{\hbar}{p_{Fe}\lambda} \sim \frac{e}{c} B_v \frac{\lambda}{p_{Fe}} \ll 1, \tag{37}$$

where B_v is estimated using Eq. (12).[1] Utilizing Eq. (26) and bearing in mind that $p_{Fe} = p_{Fp}$, we can establish a relationship between this small parameter and the parameter ϵ introduced in Sec. III A [see Eq. (28)]:

$$\tilde{\epsilon} = \pi \frac{\xi}{\lambda} \epsilon. \tag{38}$$

Since $\xi < \lambda$, the smallness of ϵ ensures that $\tilde{\epsilon}$ is also small. The same parameter will appear a third time when considering neutrons. This allows us to treat ϵ as a universal small parameter governing the interaction of all the mixture constituents with the vortex.

Substituting the ansatz (34) into Eq. (33) and linearizing the result with respect to the incident fluxes, we obtain the following equation for the function g_e:

$$\mathbf{p}\frac{\partial g_e}{\partial \mathbf{r}} - \frac{e}{c}(\mathbf{p} \times \mathbf{B}_v)\frac{\partial g_e}{\partial \mathbf{p}} = -\frac{e}{c}\mathbf{V}_e(\mathbf{p} \times \mathbf{B}_v). \tag{39}$$

This equation can be integrated along the characteristics. We choose the boundary condition so that the function g_e vanishes far upstream of the scattering region. At the distances $\rho \gg \lambda$, the solution can be represented as

$$g_e = \left\{-[\mathbf{p}(\mathbf{e}_z \times \mathbf{V}_e)]\,\sigma_\perp^{(e)} + (\mathbf{p}_\perp \mathbf{V}_e)\,\sigma_{||}^{(e)}\right\}\frac{\delta(\phi - \phi_p)}{\rho}, \tag{40}$$

where we introduced, respectively, the transverse and transport cross sections:

$$\sigma_\perp^{(e)} = \frac{m_p \varkappa}{p_\perp} a_e, \tag{41}$$

$$\sigma_{||}^{(e)} = \frac{(m_p\varkappa)^2}{p_\perp^2} L_e^{-1}, \tag{42}$$

and the following notations:

$$a_e = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{\mathcal{P}'(\sqrt{x^2+y^2})}{2\pi\sqrt{x^2+y^2}}, \tag{43}$$

$$L_e^{-1} = \frac{1}{2}\int_{-\infty}^{\infty} dx \left(\int_{-\infty}^{\infty} dy \frac{\mathcal{P}'(\sqrt{x^2+y^2})}{2\pi\sqrt{x^2+y^2}}\right)^2. \tag{44}$$

In these expressions, $\mathbf{p}_\perp$ is the component of the vector $\mathbf{p}$ perpendicular to $\mathbf{e}_z$, φ_p is its azimuthal angle. The double integral in Eq. (43) is, in fact, an integration over the entire plane perpendicular to the vector $\mathbf{e}_z$. Therefore, the coefficient a_e can be represented as

$$a_e = \lim_{\rho\to\infty} \frac{\Phi_B(\rho)}{\Phi_0} = 1, \tag{45}$$

where the magnetic flux $\Phi_B(\rho)$ is given by Eq. (13). For the function $\mathcal{P}'$ given by Eq. (15), the parameter L_e^{-1} can be evaluated as well. It equals [10,12]

$$L_e^{-1} = \frac{1}{8\lambda}. \tag{46}$$

A solution to Eq. (39) valid at an arbitrary distance from the vortex axis can be found in Ref. [12], where Eq. (40) is obtained as a limiting case. A derivation of the solution (40) can also be found in Ref. [10]. Similar solutions for other superfluid or superconducting systems were obtained, e.g., in Refs. [16,19]. Note that we restrict our analysis to terms up to second order in the small parameter ϵ.[2] In particular, $\sigma_\perp^{(e)} \sim \epsilon$, while $\sigma_{||}^{(e)} \sim \epsilon^2$.

It remains to obtain a condition ensuring the validity of the ballistic approximation. The collision integral in Eq. (33) can be estimated as

$$I^{(e)} \sim \frac{v_\mathbf{p}}{\ell_e}\delta\mathcal{N}_\mathbf{p}^{(e)}, \tag{47}$$

where $\delta\mathcal{N}_\mathbf{p}^{(e)}$ is the departure of the function $\mathcal{N}_\mathbf{p}^{(e)}$ from the equilibrium distribution function. In our case, to the linear accuracy in the incident fluxes,

$$I^{(e)} \sim \frac{v_\mathbf{p}}{\ell_e}\frac{\partial\mathcal{N}_\mathbf{p}^{(e)}}{\partial\varepsilon_\mathbf{p}^{(e)}}(g_e - \mathbf{p}\mathbf{V}_e). \tag{48}$$

The collisions can be neglected when the collision integral is much less than the force term in Eq. (33):

$$\frac{p v_\mathbf{p}}{\ell_e} \ll \frac{e}{c} v_\mathbf{p} B_v. \tag{49}$$

[1]The smallness of the parameter $\tilde{\epsilon}$ means that the electrons pass through the vortex along trajectories that are almost rectilinear.

[2]One can say that the expansion in the small parameter ϵ is equivalent to an expansion in the dimensionful parameter $m_p\varkappa$.

Substituting Eqs. (10), (12), and noting that the considered region has a characteristic length scale $\rho \sim \lambda$, we arrive at the following condition [20]:

$$\ell_e \gg \frac{p_{Fe}\lambda^2}{\hbar}. \tag{50}$$

This inequality is well satisfied under typical NS conditions.

C. Neutrons

Let us finally consider the neutron constituent. To describe it, we employ Landau's theory of Fermi liquids [21]. Being electrically neutral, the neutron quasiparticles are unaffected by the vortex magnetic field.[3] However, they can still sense the presence of the vortex through Fermi-liquid interaction: the neutron quasiparticle energy $\varepsilon_{\mathbf{p}}^{(n)}$ depends on the proton distribution function $\mathcal{N}_{\mathbf{p}}^{(p)}$ [see Eq. (A5)], which, in turn, depends on $\mathbf{Q}_p$.

The effect of a proton vortex on the neutron quasiparticle distribution function $\mathcal{N}_{\mathbf{p}}^{(n)}$ can be studied by solving the stationary Landau-Boltzmann kinetic equation

$$\frac{\partial \varepsilon_{\mathbf{p}}^{(n)}}{\partial \mathbf{p}} \frac{\partial \mathcal{N}_{\mathbf{p}}^{(n)}}{\partial \mathbf{r}} - \frac{\partial \varepsilon_{\mathbf{p}}^{(n)}}{\partial \mathbf{r}} \frac{\partial \mathcal{N}_{\mathbf{p}}^{(n)}}{\partial \mathbf{p}} = I_n. \tag{51}$$

The right-hand side of this equation formally contains the collision integral I_n. However, similar to the case of electrons, this term can be neglected (see the condition at the end of the present section). By analogy with the electron case, we seek the neutron distribution function in the following form:

$$\mathcal{N}_{\mathbf{p}}^{(n)} = f_F\left(\varepsilon_{\mathbf{p}}^{(n)} - E_{Fn} - \mathbf{p}\mathbf{V}_n + g_n\right), \tag{52}$$

where E_{Fn} is the unperturbed neutron Fermi energy, and g_n is a function of $\mathbf{r}$ and $\mathbf{p}$ to be found. The constant vector $\mathbf{V}_n$ describes the neutron incident flux. Similar to the electron-scattering problem, we assume that $\mathbf{V}_n \perp \mathbf{e}_z$ and introduce the small parameter

$$\iota_n = \frac{m_n^* \mathbf{V}_n}{p_{Fn}} \ll 1, \tag{53}$$

where m_n^* is the neutron effective mass defined via Eq. (57), see below. Therefore, as in the case of electrons, we keep terms only to linear order in the velocity $\mathbf{V}_n$ in the argument of the function (52).

As was already stated, the neutron quasiparticle energy $\varepsilon_{\mathbf{p}}^{(n)}$ depends on the neutron and proton distribution functions. Therefore, strictly speaking, Eq. (52) together with the proton distribution function should be considered as a system of implicit equations. However, taking into account the smallness of the parameters defined in Eqs. (24), (28), (36), and (53), one can derive explicit expressions with the accuracy adopted in the present study. Let us consider the neutron quasiparticle energy. In the presence of neutron and proton currents, it can generally be expanded as a power series in the parameter ι_n and the ratio $\mathbf{Q}_p/p_{Fp}$:

$$\varepsilon_{\mathbf{p}}^{(n)} = \varepsilon_{\mathbf{p},0}^{(n)} + \Delta H_{\mathbf{p},1}^{(n)} + \Delta H_{\mathbf{p},2}^{(n)} + \cdots . \tag{54}$$

Note that, in Eq. (54), we treat the ratio $\mathbf{Q}_p/p_{Fp}$ as a single, unified small parameter. However, this expansion can, in principle, be rearranged as a double expansion in two separate small parameters given by Eqs. (24) and (25) [or Eq. (28)]. In Eq. (54), the unperturbed energy $\varepsilon_{\mathbf{p},0}^{(n)}$ depends only on the magnitude of the vector $\mathbf{p}$, whereas the first-order correction generally has the following form [18]:

$$\Delta H_{\mathbf{p},1}^{(n)} = \gamma_{\rm np}\, \mathbf{p}\mathbf{Q}_p + K_{nn}\, \mathbf{p}\mathbf{V}_n, \tag{55}$$

where the coefficients $\gamma_{\rm np}$ and K_{nn} are functions of Landau parameters and the effective masses. The second-order correction $\Delta H_{\mathbf{p},2}^{(n)}$ has a more complex structure. In particular, it includes the contribution from the function g_n, which describes the scattered neutron quasiparticles. Indeed, this function is proportional to the incident neutron flux $\sim\iota_n \sim \mathbf{V}_n$ and at least linear in the parameter $\epsilon \sim \mathbf{Q}_{pv}$ governing the interaction with the vortex [see the obtained solution below; cf. Eq. (40)]. Therefore, the contribution from the function g_n is at least quadratically small with respect to the expansion (54). Fortunately, the exact form of the correction $\Delta H_{\mathbf{p},2}^{(n)}$ will not be required.

As we will see, it is convenient to introduce a new momentum variable,

$$\mathbf{q} = \mathbf{p} + \gamma_{\rm np} m_n^* \mathbf{Q}_p, \tag{56}$$

where the neutron effective mass m_n^* is defined near the Fermi surface via the following equation:

$$\frac{\partial \varepsilon_{\mathbf{p},0}^{(n)}}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m_n^*}. \tag{57}$$

In terms of the new momentum variable, the neutron energy (54) becomes

$$\varepsilon_{\mathbf{q}-\gamma_{\rm np} m_n^* \mathbf{Q}_p}^{(n)} = \varepsilon_{\mathbf{q},0}^{(n)} + K_{nn}\, \mathbf{q}\mathbf{V}_n + \Delta \tilde{H}_{\mathbf{q},2}^{(n)} + \cdots , \tag{58}$$

where $\Delta \tilde{H}_{\mathbf{q},2}^{(n)}$ is the new second-order correction defined with respect to the momentum variable $\mathbf{q}$. Under the change of variables from $(\mathbf{r},\ \mathbf{p})$ to $(\mathbf{r},\ \mathbf{q})$, Eq. (51) takes the form

$$\boldsymbol{v}_{gn} \frac{\partial \mathcal{N}_{\mathbf{q}-\gamma_{\rm np} m_n^* \mathbf{Q}_p}^{(n)}}{\partial \mathbf{r}} - \left[\frac{\partial \varepsilon_{\mathbf{q}-\gamma_{\rm np} m_n^* \mathbf{Q}_p}^{(n)}}{\partial \mathbf{r}} + \boldsymbol{v}_{gn} \times \operatorname{curl}(m_n^* \gamma_{\rm np} \mathbf{Q}_p)\right] \cdot \frac{\partial \mathcal{N}_{\mathbf{q}-\gamma_{\rm np} m_n^* \mathbf{Q}_p}^{(n)}}{\partial \mathbf{q}} = 0, \tag{59}$$

while the distribution function (52) becomes

$$\mathcal{N}_{\mathbf{q}-\gamma_{\rm np} m_n^* \mathbf{Q}_p}^{(n)} = f_F\left(\varepsilon_{\mathbf{q}-\gamma_{\rm np} m_n^* \mathbf{Q}_p}^{(n)} - E_{Fn} - \mathbf{q}\mathbf{V}_n + \tilde{g}_n\right), \tag{60}$$

where

$$\tilde{g}_n = g_n + \gamma_{\rm np} m_n^* \mathbf{V}_n \mathbf{Q}_p. \tag{61}$$

In Eq. (59), we introduced the neutron group velocity

$$\boldsymbol{v}_{gn} = \frac{\partial \varepsilon_{\mathbf{q}-m_n^* \gamma_{\rm np} \mathbf{Q}_p}^{(n)}}{\partial \mathbf{q}} = \frac{\mathbf{q}}{m_n^*} + K_{nn} \mathbf{V}_n + \frac{\partial \Delta \tilde{H}_{\mathbf{q},2}^{(n)}}{\partial \mathbf{q}} + \cdots . \tag{62}$$

[3]The interaction with the magnetic field via the neutron magnetic moment is small and can be neglected.

One can see that the introduction of the new momentum variable $\mathbf{q}$ offers two advantages. First, it allows us to exclude the linear term $\propto \mathbf{Q}_p$ (and, consequently, $\propto \mathbf{Q}_{pv}$) from the neutron group velocity (62). Second, Eq. (59), written in the variables $(\mathbf{r},\ \mathbf{q})$, closely resembles Eq. (33) with $\mathrm{curl}(m_n^*\gamma_{\mathrm{np}}\mathbf{Q}_p)$ playing the role of a magnetic field.[4] Note, however, that Eq. (59) contains the gradient of the neutron energy $\varepsilon_{\mathbf{p}}^{(n)}$, which cannot be fully matched with the electric field that is present in Eq. (33). Indeed, unlike the electric field, this gradient (a) generally depends on $\mathbf{q}$, and (b) does not vanish even in the absence of any incident fluxes. However, these differences will not prevent us from obtaining a solution to Eq. (59) by exploiting its similarity to Eq. (33).

To obtain a solution in the same manner as in Sec. III B, we require a small parameter that characterizes the interaction with the vortex. A naturally arising dimensionless ratio is[5]

$$\frac{Q_{pv}}{p_{Fn}} \sim \frac{p_{Fp}}{p_{Fn}}\epsilon, \tag{63}$$

where the parameter ϵ is given by Eq. (28). Substituting Eq. (60) into Eq. (59) and retaining terms linear in the incident flux and up to quadratic order in the parameter ϵ, we obtain

$$\mathbf{q}\frac{\partial \tilde{g}_n}{\partial \mathbf{r}} - [\mathbf{q}\times \mathrm{curl}(m_n^*\gamma_{\mathrm{np}}\mathbf{Q}_{pv})]\frac{\partial \tilde{g}_n}{\partial \mathbf{q}} = -\mathbf{V}_n[\mathbf{q}\times \mathrm{curl}(m_n^*\gamma_{\mathrm{np}}\mathbf{Q}_{pv})] - m_n^*\mathbf{V}_n\frac{\partial \Delta\tilde{H}_{\mathbf{q},2}^{(n)}}{\partial \mathbf{r}}, \tag{64}$$

where Eqs. (58) and (62) were accounted for. Note that, since the function $\tilde{g}_n$ is required only to linear order in the incident fluxes, it is sufficient to retain only the part of $\Delta\tilde{H}_{\mathbf{q},2}^{(n)}$ that is independent of $\mathbf{V}_e$, $\mathbf{V}_n$, and $\mathbf{Q}_{p0}$. The general form of this part is

$$\Delta\tilde{H}_{\mathbf{q},2}^{(n)} = \mathcal{A}\mathbf{Q}_{pv}^2 + \mathcal{B}(\mathbf{q}\mathbf{Q}_{pv})^2, \tag{65}$$

where $\mathcal{A}$ and $\mathcal{B}$ are some functions of Landau parameters. Thus, the first term on the right-hand side of Eq. (64) is of order ϵ, whereas the second term is of order ϵ^2.

Equation (64) is a linear first-order differential equation. Similar to Eq. (39), it can be integrated along the characteristics. Therefore, its solution can generally be represented as a sum of two terms:

$$\tilde{g}_n = \tilde{g}_{n,\mathrm{scat.}} + \tilde{g}_{n,H_2}, \tag{66}$$

where $\tilde{g}_{n,\mathrm{scat.}}$ and $\tilde{g}_{n,H_2}$ are contributions to the total solution, arising from the first and second terms on the right-hand side, respectively. One can notice that Eq. (64), with the second term on the right-hand side omitted, is equivalent to Eq. (39), which describes the electron scattering. Hence, $\tilde{g}_{n,\mathrm{scat.}}$ coincides with the function g_e discussed in Sec. III B, with the magnetic field $\mathbf{B}_v$ replaced by $(c/e)\mathrm{curl}(m_n^*\gamma_{\mathrm{np}}\mathbf{Q}_{pv})$ and the momentum variable $\mathbf{p}$ replaced by $\mathbf{q}$. The explicit expression for the large-distance asymptotic of this term is given by Eq. (69) below. To obtain the function $\tilde{g}_{n,H_2}$, we recall that the second term on the right-hand side of Eq. (64) is already of order ϵ^2. Therefore, to the adopted level of accuracy, it is sufficient to integrate it along the unperturbed characteristics (i.e., along straight lines). Choosing again the boundary condition so that the function $\tilde{g}_{n,H_2}$ vanishes far upstream of the scattering region, we get

$$\tilde{g}_{n,H_2} = -\frac{m_n^*}{q_\perp}\mathbf{V}_n\int_{-\infty}^{y_q} dy_1 \frac{\partial \Delta\tilde{H}_{\mathbf{q},2}^{(n)}(\boldsymbol{\rho}_1)}{\partial \boldsymbol{\rho}_1}, \tag{67}$$

where $\mathbf{q}_\perp$ is the component of the vector $\mathbf{q}$ perpendicular to $\mathbf{e}_z$. Here, we introduced the spatial coordinates defined relative to the vector $\mathbf{q}$:

$$x_q = \rho\sin(\phi_q - \phi), \quad y_q = \rho\cos(\phi_q - \phi), \tag{68}$$

ϕ_q is the azimuthal angle of the vector $\mathbf{q}$, and $\boldsymbol{\rho}_1 = (x_q, y_1)$. As will be demonstrated in what follows (see Sec. IV and Appendix D), the function $\tilde{g}_{n,H_2}$ does not contribute to the force on the vortex. Hence, it is not of interest to us in this paper.

Returning to the first term of Eq (66), at distances $\rho \gg \lambda$, it has the form [cf. Eq. (40)]:

$$\tilde{g}_{n,\mathrm{scat.}} = [-\mathbf{q}(\mathbf{e}_z\times\mathbf{V}_n)\sigma_\perp^{(n)} + \mathbf{q}_\perp\mathbf{V}_n\sigma_{||}^{(n)}]\frac{\delta(\phi-\phi_{\mathbf{q}})}{\rho}, \tag{69}$$

where we introduced, respectively, the transverse and transport cross sections for the neutrons:

$$\sigma_\perp^{(n)} = \frac{m_p\varkappa}{q_\perp}a_n, \tag{70}$$

$$\sigma_{||}^{(n)} = \frac{(m_p\varkappa)^2}{q_\perp^2}L_n^{-1}, \tag{71}$$

and the following notation:

$$a_n = -m_n^*\gamma_{\mathrm{np}}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\frac{\tilde{\mathcal{P}}'(\sqrt{x^2+y^2})}{2\pi\sqrt{x^2+y^2}}, \tag{72}$$

$$L_n^{-1} = (m_n^*\gamma_{\mathrm{np}})^2\frac{1}{2}\int_{-\infty}^{\infty}dx\left(\int_{-\infty}^{\infty}dy\frac{\tilde{\mathcal{P}}'(\sqrt{x^2+y^2})}{2\pi\sqrt{x^2+y^2}}\right)^2. \tag{73}$$

Eqs. (72) and (73) are similar in form to Eqs. (43) and (44). Note, however, that the function $\mathcal{P}'(\rho)$, which determines the vortex magnetic field [see Eq. (12)], is replaced by the function - $m_n^*\gamma_{\mathrm{np}}\tilde{\mathcal{P}}'(\rho)$, arising from $\mathrm{curl}(m_n^*\gamma_{\mathrm{np}}\mathbf{Q}_p)$ [see Eq. (19)].

Let us first consider the transverse cross section. It is easy to see that the coefficient a_n is proportional to the total flux Φ_Q [see Eq. (23); cf. Eq. (45)]. However, as it was argued in Sec. III A, $\Phi_Q(\rho)$ tends to zero as $\rho\to\infty$. Therefore, the transverse cross section vanishes:

$$\sigma_\perp^{(n)} = 0. \tag{74}$$

Inserting the function $\tilde{\mathcal{P}}'(\rho)$ given by Eq. (20) into the function (73), which controls the transport cross section (71),

[4]The factor $m_n^*\gamma_{\mathrm{np}}$ is assumed to be constant outside the vortex core. However, placing it under the curl operator makes equations more concise and helps to clarify some subsequent reasoning.

[5]The interpretation of the smallness of this parameter within the framework of the infinitely thin vortex core model is discussed immediately following Eq. (25).

we find that it decomposes into three distinct terms:

$$L_n^{-1} = (m_n^*\gamma_{\rm np})^2 L_e^{-1} - (m_n^*\gamma_{\rm np})^2 \int_{-\infty}^{\infty} dy \frac{\mathcal{P}'(|y|)}{2\pi|y|} + \frac{(m_n^*\gamma_{\rm np})^2}{2}\int_{-\infty}^{\infty} dx \left(\int_{-\infty}^{\infty} dy \frac{\delta(\sqrt{x^2+y^2})}{2\pi\sqrt{x^2+y^2}}\right)^2. \quad (75)$$

Here, L_e^{-1} is given by Eq. (44). In evaluating the second term of Eq. (75), we made use of the identity (22). The third term formally diverges due to the presence of two delta functions. This divergence indicates that the assumption of an infinitely thin vortex core is an oversimplification that prevents a complete calculation of the transport cross section. Therefore, to calculate this cross section, one must determine the structure of the vortex core by solving a self-consistent problem. Nonetheless, it remains possible to make an estimate.

A physical vortex has a core of size $\sim\xi$, where the energy gap Δ_p drops to zero. On the other hand, the coefficient $\gamma_{\rm np}$ vanishes when the superconductivity is destroyed [22]. Therefore, somewhat loosely, the vortex core could be modeled by introducing a spatially dependent coefficient $\gamma_{\rm np} = \gamma_{\rm np}^{\infty} h(\rho)$, where $\gamma_{\rm np}^{\infty}$ denotes its bulk value and $h(\rho)$ is some function smoothly rising from zero at the vortex axis to unity over a characteristic distance $\sim\xi$.[6] In this case, we can still derive formally the same Eq. (59) and, consequently, Eq. (64), but now $\mathrm{curl}(m_n^*\gamma_{\rm np}\mathbf{Q}_p)$ is assumed to vanish at the vortex axis. As a result, we again arrive at Eq. (69) with the cross sections given by Eqs. (70) and (71), but with the corresponding coefficients now taking the form

$$a_n = -m_n^*\gamma_{\rm np}^{\infty} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{\mathcal{R}'(\sqrt{x^2+y^2})}{2\pi\sqrt{x^2+y^2}}, \quad (76)$$

$$L_n^{-1} = \left(m_n^*\gamma_{\rm np}^{\infty}\right)^2 \frac{1}{2}\int_{-\infty}^{\infty} dx \left(\int_{-\infty}^{\infty} dy \frac{\mathcal{R}'(\sqrt{x^2+y^2})}{2\pi\sqrt{x^2+y^2}}\right)^2, \quad (77)$$

where

$$\mathcal{R}' = \frac{d}{d\rho}\{h(\rho)[\mathcal{P}(\rho)-1]\} \approx \mathcal{P}'(\rho) - h'(\rho), \quad (78)$$

and $h'(\rho) = dh/d\rho$. The approximate equality in Eq. (78) was obtained assuming that the function $h(\rho)$ varies much faster than the function $\mathcal{P}(\rho)$ (i.e., $\xi \ll \lambda$). One can see that the approximate form of the function $\mathcal{R}'$ is similar to the function $\mathcal{P}'(\rho)$ given by Eq. (20), but with the smooth function $h'(\rho)$ instead of the delta function.

Being substituted into Eq. (76), the function (78) still gives $a_n = 0$. Considering the parameter L_n^{-1} now given by Eq. (77), one can verify that, under the assumption $\xi \ll \lambda$, the first two terms of the representation (75) remain approximately the same except that the coefficient $\gamma_{\rm np}$ should be replaced with $\gamma_{\rm np}^{\infty}$. To estimate the third term, we adopt the following model function:

$$h(\rho) = 1 - \exp\left(-\frac{\rho^2}{\xi^2}\right), \quad (79)$$

which yields, instead of the third term in Eq. (75),

$$L_{n,\rm core}^{-1} = \frac{\left(m_n^*\gamma_{\rm np}^{\infty}\right)^2}{2}\int_{-\infty}^{\infty} dx \left(\int_{-\infty}^{\infty} dy \frac{h'(\sqrt{x^2+y^2})}{2\pi\sqrt{x^2+y^2}}\right)^2 = \frac{1}{\sqrt{8\pi}}\frac{\left(m_n^*\gamma_{\rm np}^{\infty}\right)^2}{\xi}. \quad (80)$$

The numerical factor $1/\sqrt{8\pi}$ here depends on the specific choice of the model function $h(\rho)$ and can change by a factor of a few depending on the choice of $h(\rho)$. However, the coefficient $L_{n,\rm core}^{-1}$, representing the core contribution to the total parameter L_n^{-1}, is, in general, of order $(m_n^*\gamma_{\rm np})^2/\xi$. On the other hand, taking into account that the function $\mathcal{P}'(\rho)$ vanishes at characteristic distance $\sim\lambda$, the first two terms in Eq. (75) can be estimated as $(m_n^*\gamma_{\rm np})^2/\lambda$.[7] Thus, we can conclude that, in the limit $\xi \ll \lambda$, the third term produces the leading contribution to the neutron transport cross section.

Finally, let us return to the collision integral. Applying the same reasoning as in Sec. III B, we derive the following condition [cf. Eq. (50)]:

$$\ell_n \gg \frac{1}{m_n^*\gamma_{\rm np}}\frac{p_{Fn}\lambda^2}{\hbar}, \quad (81)$$

which ensures the smallness of I_n.

IV. FORCES ON THE VORTEX

We now turn to calculating the forces using Eq. (6). Assuming the smallness of the incident particle fluxes, we restrict the calculation to linear order in $\mathbf{Q}_{p0}$, $\mathbf{V}_e$, and $\mathbf{V}_n$. It is most convenient to choose the radius of the integration cylinder R such that $R \gg \lambda$.[8] The contributions from the electrons, proton-neutron mixture, and electromagnetic field can be calculated by substituting the corresponding part of the total stress tensor into Eq. (6). We begin with the electrons. Inserting the distribution function (34) into the general expression (C21) and extracting the contributions linear in the incident fluxes, we get

$$\Pi_e^{ik} = \sum_{\mathbf{p}\sigma} p^i {v_{\mathbf{p}}}^k (g_e - e\varphi)\frac{\partial \mathcal{N}_{\mathbf{p},0}^{(e)}}{\partial \varepsilon_{\mathbf{p}}^{(e)}}, \quad (82)$$

where $\mathcal{N}_{\mathbf{p},0}^{(e)} = f_F(\varepsilon_{\mathbf{p}}^{(e)} - E_{Fe})$ is the unperturbed electron distribution function. Here, we have taken into account that the electrostatic potential φ, generated by the scattering, is at least a linear function of $\mathbf{V}_e$, $\mathbf{V}_n$, and $\mathbf{Q}_{p0}$. At the radius $R \gg \lambda$, we can make use of the approximate expression (40). This allows

[6]Note that, in reality, the dependence of the energy $\Delta H_{\mathbf{p}}^{(n)}$ on $\mathbf{Q}_p$ is nonlocal at the length scale of ξ.

[7]For the function $\mathcal{P}'(\rho)$ given by Eq. (15), these two integrals can be evaluated explicitly: the first one is given by Eq. (46) and equals $1/8\lambda$, while the second one equals $1/2\lambda$.

[8]Simultaneously, this radius should be much smaller than the intervortex distance.

us to write down the electron contribution to the force as

$$\mathbf{F}^{(e)}_{\mathrm{m}\to\mathrm{v}} = -\frac{1}{2}\sum_{\mathbf{p}\sigma} v_{\mathbf{p}\perp} p_\perp^2 [-(\mathbf{e}_z\times\mathbf{V}_e)\sigma_\perp^{(e)} + \mathbf{V}_e\sigma_{||}^{(e)}]\frac{\partial\mathcal{N}^{(e)}_{\mathbf{p},0}}{\partial\varepsilon^{(e)}_{\mathbf{p}}} + e n_{e0}\int d^3\mathbf{r}\,\mathbf{E}, \tag{83}$$

where $v_{\mathbf{p}\perp} = p_\perp c^2/\varepsilon^{(e)}_{\mathbf{p}}$ and, in the last term, we made use of Gauss's theorem. Approximating the derivative of the electron distribution function by the delta function:

$$\frac{\partial\mathcal{N}^{(e)}_{\mathbf{p},0}}{\partial\varepsilon^{(e)}_{\mathbf{p}}} \approx -\delta\big(\varepsilon^{(e)}_{\mathbf{p}} - E_{Fe}\big), \tag{84}$$

and substituting Eqs. (41), (42), and (45), we obtain the electron contribution in the following form:

$$\mathbf{F}^{(e)}_{\mathrm{m}\to\mathrm{v}} = D_e'(\mathbf{e}_z\times\mathbf{V}_e) + D_e\,\mathbf{V}_e + e\,n_{e0}\int d^3\mathbf{r}\,\mathbf{E}, \tag{85}$$

where the following coefficients were introduced (cf. Refs. [10,12], see also Ref. [11]):

$$D_e' = -m_p\varkappa\, n_{e0} \text{ and } D_e = (m_p\varkappa)^2\frac{3\pi}{8}\frac{n_{e0}}{p_{Fe}}L_e^{-1}, \tag{86}$$

and $n_{e0} = p_{Fe}^3/3\pi^2\hbar^3$.

Let us now turn to the neutron-proton mixture. The general expression for the neutron-proton stress tensor is given by Eq. (C17). It can be rewritten as follows:

$$\Pi^{ik}_{\mathrm{np}} = Q_p^i j_p^k - \sum_{\mathbf{p}\sigma}\big(\varepsilon^{(n)}_{\mathbf{p}} - E_{Fn}\big)p^i\frac{\partial\mathcal{N}^{(n)}_{\mathbf{p}}}{\partial p^k} + \delta^{ik}\big(n_p\breve{\mu}_p + n_n E_{Fn} - \mathcal{E}_{\mathrm{np}}\big). \tag{87}$$

Here, $\mathbf{j}_p$ is the proton current given by Eq. (C6), and $\mathcal{E}_{\mathrm{np}}$ is the energy of the neutron-proton subsystem given by Eq. (A2). To obtain this expression from Eq. (C17), we added and subtracted the combination $\sum_{\mathbf{p}\sigma} E_{Fn}p^i(\partial\mathcal{N}^{(n)}_{\mathbf{p}}/\partial p^k)$ and integrated one of these terms by parts.

Our aim is to extract from Eq. (87) the contribution linear in the incident fluxes far from the vortex. At distances $\rho \gg \lambda$, the vortex azimuthal flow $\mathbf{Q}_{pv}$ is exponentially suppressed. Therefore, the proton current can depend only on the incident flux and, consequently, the first term of Eq. (87) is quadratically small. Next, we note that the neutron-proton energy depends on the incident fluxes through the neutron and proton distribution functions. Therefore, the linear contribution to the combination multiplied by δ^{ik} can be written as

$$\delta_{\mathrm{l}}(n_p\breve{\mu}_p + n_n E_{Fn} - \mathcal{E}_{\mathrm{np}}) \approx -\sum_{\mathbf{p}\sigma}\left\{\left[\left(\frac{\delta\mathcal{E}_{\mathrm{np}}}{\delta\mathcal{N}^{(n)}_{\mathbf{p}}}\right)_{\mathrm{eq},0} - E_{Fn}\right]\delta_{\mathrm{l}}\mathcal{N}^{(n)}_{\mathbf{p}} + \left[\left(\frac{\delta\mathcal{E}_{\mathrm{np}}}{\delta\mathcal{N}^{(p)}_{\mathbf{p}}}\right)_{\mathrm{eq},0} - E_{Fp}\right]\delta_{\mathrm{l}}\mathcal{N}^{(p)}_{\mathbf{p}}\right\} + n_{p0}\,\delta_{\mathrm{l}}\breve{\mu}_p. \tag{88}$$

Here, the symbol δ_{l} denotes the linear correction to a quantity, and the index "eq, 0" means that the corresponding variational derivative should be evaluated using equilibrium distribution functions with all currents set to zero ($\mathbf{V}_e = \mathbf{V}_n = \mathbf{Q}_p = 0$). In deriving Eq. (88), we also took into account Eq. (32). As follows from Eq. (A13), the proton contribution enclosed in the second square brackets vanishes. As for the neutrons, by definition [see Eq. (A5)],

$$\left(\frac{\delta\mathcal{E}_{\mathrm{np}}}{\delta\mathcal{N}^{(n)}_{\mathbf{p}}}\right)_{\mathrm{eq},0} = \varepsilon^{(n)}_{\mathbf{p},0}. \tag{89}$$

Therefore, the linear contribution to the stress tensor in Eq. (87) takes the following form:

$$\delta_{\mathrm{l}}\Pi^{ik}_{\mathrm{np}} = -\sum_{\mathbf{p}\sigma}\frac{p^i p^k}{m_n^*}\frac{\partial\mathcal{N}^{(n)}_{\mathbf{p},0}}{\partial\varepsilon^{(n)}_{\mathbf{p},0}}\delta_{\mathrm{l}}\varepsilon^{(n)}_{\mathbf{p}} - \sum_{\mathbf{p}\sigma}\big(\varepsilon^{(n)}_{\mathbf{p},0} - E_{Fn}\big)p^i\frac{\partial}{\partial p^k}\delta_{\mathrm{l}}\mathcal{N}^{(n)}_{\mathbf{p}} - \delta^{ik}\left(\sum_{\mathbf{p}\sigma}\big(\varepsilon^{(n)}_{\mathbf{p},0} - E_{Fn}\big)\delta_{\mathrm{l}}\mathcal{N}^{(n)}_{\mathbf{p}} - n_{p0}\,\delta_{\mathrm{l}}\breve{\mu}_p\right), \tag{90}$$

where $\mathcal{N}^{(n)}_{\mathbf{p},0} = f_F(\varepsilon^{(n)}_{\mathbf{p},0} - E_{Fn})$. To proceed further, we notice that [see Eq. (52)]

$$\delta_{\mathrm{l}}\mathcal{N}^{(n)}_{\mathbf{p}} = \frac{\partial\mathcal{N}^{(n)}_{\mathbf{p},0}}{\partial\varepsilon^{(n)}_{\mathbf{p},0}}\big(\delta_{\mathrm{l}}\varepsilon^{(n)}_{\mathbf{p}} - \mathbf{p}\mathbf{V}_n + g_n\big). \tag{91}$$

Substituting this representation into Eq. (90), we first observe that the terms $\propto\mathbf{p}\mathbf{V}_n$ vanish after averaging over the solid angle of the vector $\mathbf{p}$. We then note that, at distances $\rho \gg \lambda$ and to linear order in the incident fluxes, one may simply replace the variable $\mathbf{p}$ with $\mathbf{q}$ [see Eq. (56)] and the function g_n with $\tilde{g}_n$ [see Eq. (61)]. After these manipulations, we arrive at

$$\Pi^{ik}_{\mathrm{np}} = \sum_{\mathbf{q}\sigma}\tilde{g}_n\frac{q^i q^k}{m_n^*}\frac{\partial\mathcal{N}^{(n)}_{\mathbf{q},0}}{\partial\varepsilon^{(n)}_{\mathbf{q},0}} + \delta^{ik}\,n_{p0}\,\delta_{\mathrm{l}}\breve{\mu}_p. \tag{92}$$

To calculate the proton-neutron mixture contribution to the force, one needs to substitute the function $\tilde{g}_n$ given by Eq. (66). It is shown in Appendix D that there is no contribution to the force from the function $\tilde{g}_{n,H_2}$. As for the function $\tilde{g}_{n,\mathrm{scat.}}$, its contribution has the form similar to Eq. (83). Thus, the total neutron-proton contribution equals

$$\mathbf{F}^{(np)}_{\mathrm{m}\to\mathrm{v}} = -\frac{1}{2}\sum_{\mathbf{q}\sigma}\frac{q_\perp^3}{m_n^*}[-(\mathbf{e}_z\times\mathbf{V}_n)\sigma_\perp^{(n)} + \mathbf{V}_n\sigma_{||}^{(n)}]\frac{\partial\mathcal{N}^{(n)}_{\mathbf{q},0}}{\partial\varepsilon^{(n)}_{\mathbf{q},0}} - e n_{p0}\int d^3\mathbf{r}\,\mathbf{E}. \tag{93}$$

Here, in the last term, we substituted Eq. (31) and applied Gauss's theorem. As in the case of electrons, we can approximate:

$$\frac{\partial\mathcal{N}^{(n)}_{\mathbf{q},0}}{\partial\varepsilon^{(n)}_{\mathbf{q},0}} \approx -\delta\big(\varepsilon^{(n)}_{\mathbf{q},0} - E_{Fn}\big). \tag{94}$$

This allows us to obtain

$$\mathbf{F}^{(np)}_{\mathrm{m}\to\mathrm{v}} = D_n \mathbf{V}_n - e n_{p0} \int d^3\mathbf{r}\,\mathbf{E}, \tag{95}$$

where we introduced the coefficient

$$D_n = (m_p \varkappa)^2 \frac{3\pi}{8} \frac{n_{n0}}{p_{Fn}} L_n^{-1}, \tag{96}$$

and the unperturbed neutron density $n_{n0} = p_{Fn}^3/3\pi^2\hbar^3$, and substituted Eqs. (71) and (74).

All that remains is to consider the contribution from the electromagnetic field. The corresponding stress tensor is given by Eq. (C23). The electric field is linear in the incident fluxes; hence, the electric component of the tensor is quadratically small. As for the magnetic contribution, apart from terms quadratic in the incident fluxes, it contains terms proportional to $B_v^i B_v^k$ and $B_v^i \delta B^k$. However, these terms are exponentially suppressed at large distances and can therefore be neglected. Consequently, the electromagnetic contribution vanishes within the accepted level of accuracy.

Now we can write the total force acting on the vortex. To this end, we notice that the terms proportional to the electric field in Eqs. (85) and (95) cancel each other due to the quasineutrality condition (3). Therefore, the total force equals

$$\begin{aligned}\mathbf{F}_{\mathrm{m}\to\mathrm{v}} &= D_e'[\mathbf{e}_z \times (\mathbf{V}_e - \mathbf{V}_L)] \\ &\quad - D_e\,\mathbf{e}_z \times [\mathbf{e}_z \times (\mathbf{V}_e - \mathbf{V}_L)] \\ &\quad - D_n\,\mathbf{e}_z \times [\mathbf{e}_z \times (\mathbf{V}_n - \mathbf{V}_L)].\end{aligned} \tag{97}$$

To make this expression more suitable for practical use, we switched to the laboratory frame of reference, introducing the velocity of the vortex motion $\mathbf{V}_L$. We also substituted $\mathbf{V}_\alpha \to -\mathbf{e}_z \times (\mathbf{e}_z \times \mathbf{V}_\alpha)$ to allow for possible motion of the mixture constituents along the vortex lines.

V. DISCUSSION AND CONCLUSIONS

Equation (97), with the coefficient D_n given by Eq. (96), constitutes the main result of this work. The total force given by this equation consists of three components. The first two components are well known in the astrophysical literature. They arise from the scattering of the electrons by the vortex magnetic field. One can say that these components are electromagnetic in nature.[9] In contrast, the third term cannot be reduced to an electromagnetic interaction. It describes neutron scattering and represents a manifestation of the nuclear forces. To investigate this force, we employed the Landau Fermi-liquid formalism. This enabled us to treat the scattering of neutron quasiparticles in close analogy with the electron-scattering problem. Moreover, since the considered effect is a pure consequence of Fermi-liquid interaction, one expects the coefficient D_n to be expressible in terms of the Landau parameters. Indeed, it is proportional to the square of the coefficient γ_{np}, introduced via Eq. (55), which, in turn, in the case of strong proton superconductivity and normal neutrons, is given by [18][10]

$$\gamma_{\mathrm{np}} = \sqrt{\frac{1}{m_n^* m_p^*}} \sqrt{\frac{n_{p0}}{n_{n0}}} \frac{F_1^{np}/3}{\left(1 + F_1^{nn}/3\right)}, \tag{98}$$

where $F_1^{\alpha\alpha'}$ are the symmetric dimensionless Landau parameters defined for a Fermi-liquid mixture in the standard way [21–23]. Therefore, this effect vanishes if the neutron-proton Fermi-liquid interaction is switched off ($F_1^{np} = 0$).

An important difference between the electron and neutron scattering is that the latter does not produce a transverse component of the force. The coefficient D_e' governing the transverse electron force is proportional to the total magnetic flux generated by a proton vortex [see Eqs. (41) and (45)]. In the case of neutron scattering, the role of the magnetic field is played by the vector $\mathrm{curl}(m_n^* \gamma_{\mathrm{np}} \mathbf{Q}_{pv})$. The total flux of this vector equals zero. As argued in Sec. III A, this is a consequence of the Meissner effect, which is universal for superconductors for any value of λ/ξ. Therefore, the absence of the neutron-induced transverse force is expected to persist in the more realistic case of $\xi \sim \lambda$.

Considering the longitudinal components, as we saw, the infinitely thin vortex core model does not permit a complete calculation of the coefficient $D_n \sim L_n^{-1}$. To determine it, one must solve a self-consistent problem for the vortex-core structure. However, as was argued, it is natural to expect that $L_n \sim (m_n^* \gamma_{\mathrm{np}})^{-2} \xi$, and, consequently, one can make a physically motivated estimation:

$$\frac{D_n}{D_e} = \frac{n_{n0}}{n_{e0}} \frac{p_{Fe}}{p_{Fn}} \frac{L_e}{L_n} \approx \mathcal{K} (m_n^* \gamma_{\mathrm{np}})^2 \frac{n_{n0}}{n_{e0}} \frac{p_{Fe}}{p_{Fn}} \frac{\lambda}{\xi}, \tag{99}$$

where $\mathcal{K}$ is a dimensionless coefficient of the order of unity. The value of the ratio n_{n0}/n_{e0} (or p_{Fe}/p_{Fn}) as well as the Landau parameters entering the coefficient γ_{np} depend on the chosen equation of state. Figure 1 demonstrates the value of the combination $(m_n^* \gamma_{\mathrm{np}})^2 (n_{n0}/n_{e0})(p_{Fe}/p_{Fn})$ as a function of the baryon number density calculated for the BSk family of equations of state [24,25]. As for the ratio λ/ξ, it is large in the regime of strong type-II superconductivity. However, it remains bounded because the quasiclassical approximation adopted in the present work requires the quasiparticle wavelengths to be small compared with ξ. Therefore, to maintain the consistency of the model, one must assume that $\lambda/\xi < p_{Fn}\lambda/\hbar \approx 30$–$50$ [2]. Finally, to estimate the relative magnitude of the third term in Eq. (97), one needs to compare the velocities V_e and V_n measured in the vortex frame of reference. This cannot be done in general. The relationship between neutron and electron velocities can depend substantially on the specific process under consideration. It should be emphasized that, irrespective of its magnitude, the new force may qualitatively alter the dynamics in the *npe* mixture. Indeed, if one sets $D_n = 0$ in Eq. (97), the equation $\mathbf{F}_{\mathrm{m}\to\mathrm{v}} = 0$

[9]Strictly speaking, this conclusion is fully valid only at length scales much larger than the London penetration depth. At smaller scales, the mechanism of momentum transfer becomes more intricate [12].

[10]In the cited reference, the coefficient γ_{np} is expressed through less common parameters $G_{\alpha\alpha'}$, which are connected with the standard Landau parameters via the equation $F_1^{\alpha\alpha'}/3 = \sqrt{m_\alpha^* m_{\alpha'}^* / n_{\alpha 0} n_{\alpha' 0}}\, G_{\alpha\alpha'}$.

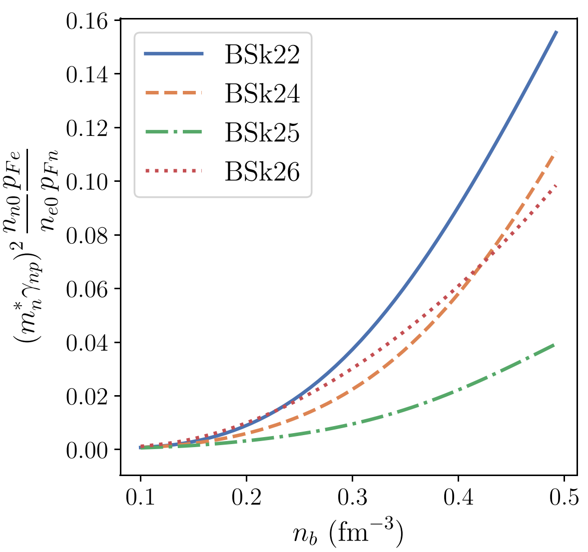

FIG. 1. The quantity $(m_n^*\gamma_{\rm np})^2(n_{n0}/n_{e0})(p_{Fe}/p_{Fn})$ plotted as a function of baryon number density n_b for the BSk family of equations of state. The Landau parameters are calculated as described in Ref. [23].

has only a trivial solution $\mathbf{V}_e = \mathbf{V}_L$. [11] Therefore, a nonzero electron velocity can be generated only by external forces [see Eq. (5)]. In contrast, Eq. (97) with a finite coefficient D_n allows us to maintain the difference $\mathbf{V}_e - \mathbf{V}_n$ without any velocity-independent external forces.

As was already mentioned in Sec. II, the strong type-II superconductivity approximation is not particularly well suited for NS interiors. Therefore, one can expect that the length scales L_e and L_n are of comparable magnitude. However, this expectation should be verified through suitable calculations. Another limitation of treating the vortex core as infinitely thin is the inability to consistently account for the scattering of (quasi)particles by bound excitations [14,26]. However, given the (quasi)particle large mean-free path compared with other length scales of the problem, one may expect the omitted processes to produce only a small correction. It is important to emphasize that the force discussed in the present work cannot be reduced to the interaction of neutrons with the excitations bound in the vortex core. This is most clearly seen in the case of a strong type-II superconductor. Inspection of Eq. (59) shows that a neutron quasiparticle can sense the presence of the proton vortex at distances $\sim\lambda \gg \xi$, where no bound core excitations exist.

In the present paper, we assumed that the temperature is sufficiently low to completely neglect proton continuum excitations. Scattering of the proton quasiparticles is expected to contribute additional force components to Eq. (97). The same applies if muons are present. On the other hand, we assumed that the neutrons are normal (nonsuperfluid). If this condition is not satisfied, the neutron Bogoliubov excitations must be considered instead of the standard Landau quasiparticles. Moreover, calculations indicate that the neutrons in the NS core form Cooper pairs in the triplet channel with an anisotropic energy gap [1]. Although this would complicate the analysis, it is unlikely to qualitatively affect the results in the vicinity of the neutron critical temperature. If the temperature is far below $T_{\rm cn}$, the number of neutron Bogoliubov excitations becomes exponentially suppressed, and the force arising from neutron quasiparticle scattering correspondingly becomes negligible. It should also be noted that our analysis is restricted to the scattering of Fermi quasiparticles. Collective excitations may likewise contribute to the effect under consideration, particularly at low temperatures. The primary aim of the present work is to demonstrate the existence of a force arising from neutron scattering, which has previously been overlooked. Therefore, we consider the simplest set of conditions that allows this effect to be isolated. Note, however, that the condition $T_{\rm cn} \lesssim T \ll T_{\rm cp}$ is not overly restrictive. In the outer layers of NS cores, where type-II superconductivity is expected to occur [2], the proton critical temperature $T_{\rm cp}$ is generally believed to exceed the neutron critical temperature $T_{\rm cn}$ by at least a factor of a few [1]. In turn, the regime $T_{\rm cn} \lesssim T$ may be relevant for several classes of neutron stars of observational interest. For example, the inferred internal temperatures of magnetars (see, e.g., Refs. [27–30]), young, isolated cooling neutron stars (see, e.g., Refs. [31–33]), and accreting neutron stars in low-mass x-ray binaries (LMXBs) (see, e.g., Refs. [34–37]) can be comparable to, or even exceed, the density-dependent critical temperature for triplet neutron pairing in some regions of the core.

In conclusion, we have argued that the neutron-proton Fermi-liquid interaction gives rise to a force proportional to the neutron velocity relative to the proton vortices. In this work, we employ a relatively simple model. Nevertheless, the main properties of the force, as well as physically motivated estimates, can already be extracted from the analysis presented here. The impact of this force on each specific process (magnetic field evolution, stellar oscillations, etc.) warrants separate investigation.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this paper. Requests for further information or data should be sent to the authors.

APPENDIX A: ENERGY

The total energy density of the system can be represented as

$$\mathcal{E} = \mathcal{E}_{\rm np} + \mathcal{E}_e + \mathcal{E}_{\rm EM}, \tag{A1}$$

where $\mathcal{E}_{\rm np}$ is the energy of the neutron-proton subsystem, $\mathcal{E}_e$ is the energy of the electrons, and $\mathcal{E}_{\rm EM}$ is the energy of the

[11] Strictly speaking, an electron velocity component parallel to the vortices is still allowed. However, a flow aligned with the vortices everywhere can hardly be realized in NS.

electromagnetic field. The energy of the neutron-proton subsystem, in turn, is assumed to be a sum of two terms [18,22]:

$$\mathcal{E}_{\rm np} = \mathcal{E}_{\rm LF} + \mathcal{E}_{\rm pair}. \tag{A2}$$

In this expression, the first term formally coincides with the energy of a normal (nonsuperfluid) Fermi-liquid mixture [21]:

$$\mathcal{E}_{\rm LF} = E_0 + \sum_{\mathbf{p}\sigma\alpha} \varepsilon_0^{(\alpha)}(\mathbf{p})\big(\mathcal{N}_{\mathbf{p}}^{(\alpha)} - \theta_{\mathbf{p}}^{(\alpha)}\big) + \frac{1}{2}\sum_{\mathbf{p}\mathbf{p}'\sigma\sigma'\alpha\alpha'} f^{\alpha\alpha'}(\mathbf{p},\mathbf{p}')\big(\mathcal{N}_{\mathbf{p}}^{(\alpha)} - \theta_{\mathbf{p}}^{(\alpha)}\big)\big(\mathcal{N}_{\mathbf{p}'}^{(\alpha')} - \theta_{\mathbf{p}'}^{(\alpha')}\big), \tag{A3}$$

where $\mathcal{N}_{\mathbf{p}}^{(\alpha)}$ are the distribution functions of Landau quasiparticles for species $\alpha = n, p$, $\theta_{\mathbf{p}}^{(\alpha)} = \theta(p_{F\alpha} - p)$, $\theta(x)$ is the Heaviside step function, $\varepsilon_0^{(\alpha)}(\mathbf{p})$ and $f^{\alpha\alpha'}(\mathbf{p},\mathbf{p}')$ are the first and second variations of the energy $\mathcal{E}_{\rm LF}$ with respect to the distribution function $\mathcal{N}_{\mathbf{p}}^{(\alpha)}$ evaluated at the ground (nonsuperfluid) state $\mathcal{N}_{\mathbf{p}}^{(\alpha)} = \theta_{\mathbf{p}}^{(\alpha)}$. We consider spin-saturated matter, and therefore omit spin indices from the distribution functions $\mathcal{N}_{\mathbf{p}}^{(\alpha)}$ while by $f^{\alpha\alpha'}(\mathbf{p},\mathbf{p}')$ we mean the spin-averaged interaction function. However, we retain formal summations over the spin indices σ, σ', etc., which essentially means multiplication by 2.

The pairing energy term in Eq. (A2) equals [13]

$$\mathcal{E}_{\rm pair} = \frac{\Delta^{(p)2}}{\nu}, \tag{A4}$$

where $\Delta^{(p)}$ denotes the proton energy gap and $\nu < 0$ is the effective coupling constant. We assume isotropic pairing for the protons. Following the BCS approximation, we ignore the dependence of the coupling constant and, consequently, of the gap on the absolute value of the momentum variable.

Equation (A3) allows us to define the Landau quasiparticle energy:

$$\varepsilon_{\mathbf{p}}^{(\alpha)} = \frac{\delta\mathcal{E}_{\rm LF}}{\delta\mathcal{N}_{\mathbf{p}}^{(\alpha)}} = \varepsilon_0^{(\alpha)}(\mathbf{p}) + \sum_{\mathbf{p}'\sigma'\alpha'} f^{\alpha\alpha'}(\mathbf{p},\mathbf{p}')\big(\mathcal{N}_{\mathbf{p}'}^{(\alpha')} - \theta_{\mathbf{p}'}^{(\alpha')}\big). \tag{A5}$$

For the superconducting protons, the following quantities can also be introduced:

$$\mathcal{H}_{\mathbf{q}}^{(p)} = \frac{\varepsilon_{\mathbf{q}+\mathbf{Q}_p}^{(p)} + \varepsilon_{-\mathbf{q}+\mathbf{Q}_p}^{(p)}}{2} - \breve{\mu}_p, \tag{A6}$$

$$E_{\mathbf{q}}^{(p)} = \sqrt{\mathcal{H}_{\mathbf{q}}^{(p)2} + \Delta^{(p)2}}. \tag{A7}$$

Note that, in the presence of the proton supercurrent $\propto\mathbf{Q}_p$, it is more convenient to describe protons in terms of the momentum variable $\mathbf{q}$ defined via Eq. (A6). This variable should not be confused with the shifted momentum variable $\mathbf{q}$ defined via Eq. (56) for neutrons. The function $E_{\mathbf{q}}^{(p)}$ represents the symmetric part ($E_{\mathbf{q}}^{(p)} = E_{-\mathbf{q}}^{(p)}$) of the proton Bogoliubov excitation energy [cf. Eq. (1)].

The following self-consistency relation should be satisfied for the energy gap:

$$1 = -\frac{\nu}{4}\sum_{\mathbf{q},\sigma}\frac{1}{E_{\mathbf{q}}^{(p)}}. \tag{A8}$$

On the other hand, the energy gap can be expressed as follows:

$$\Delta^{(p)} = 2E_{\mathbf{q}}^{(p)} u_{\mathbf{q}}^{(p)} v_{\mathbf{q}}^{(p)}, \tag{A9}$$

where

$$u_{\mathbf{q}}^{(p)} = \sqrt{\frac{1}{2}\left(1 + \frac{\mathcal{H}_{\mathbf{q}}^{(p)}}{E_{\mathbf{q}}^{(p)}}\right)}, \quad v_{\mathbf{q}}^{(p)} = \sqrt{\frac{1}{2}\left(1 - \frac{\mathcal{H}_{\mathbf{q}}^{(p)}}{E_{\mathbf{q}}^{(p)}}\right)} \tag{A10}$$

are the Bogoliubov coherence factors related by the normalization condition

$$u_{\mathbf{q}}^{(p)2} + v_{\mathbf{q}}^{(p)2} = 1. \tag{A11}$$

In the limit of vanishing temperature, the proton distribution function can be expressed through the following equation [13,22]:

$$\mathcal{N}_{\mathbf{q}+\mathbf{Q}_p}^{(p)} = \mathcal{N}_{-\mathbf{q}+\mathbf{Q}_p}^{(p)} = v_{\mathbf{q}}^{(p)2}. \tag{A12}$$

It is easy to check (see Appendix B) that, in this limit, the following variational derivative vanishes,

$$\frac{\delta(\mathcal{E}_{\rm np} - \breve{\mu}_p n_p)}{\delta\mathcal{N}_{\mathbf{q}+\mathbf{Q}_p}^{(p)}} = \frac{\delta(\mathcal{E}_{\rm np} - \breve{\mu}_p n_p)}{\delta v_{\mathbf{q}}^{(p)2}} = 0, \tag{A13}$$

thereby ensuring that $v_{\mathbf{q}}^{(p)}$ is related to the energy gap by Eq. (A10). Note that, when varying (A13), the chemical potential $\breve{\mu}_p$ must be treated as constant.

The energy of the electrons can be represented as

$$\mathcal{E}_e = \sum_{\mathbf{p}\sigma} \varepsilon_{\mathbf{p}}^{(e)}\mathcal{N}_{\mathbf{p}}^{(e)}. \tag{A14}$$

The energy of the electromagnetic field is given by the standard expression [38]:

$$\mathcal{E}_{\rm EM} = \frac{1}{8\pi}(E^2 + B^2). \tag{A15}$$

APPENDIX B: PROOF OF EQ. (A13)

In this Appendix, we prove Eq. (A13). The variation $\delta(\mathcal{E}_{\rm np} - \breve{\mu}_p n_p)$ with respect to the proton distribution function $\mathcal{N}_{\mathbf{p}}^{(p)}$, taken under the assumption that $\breve{\mu}_p$ is fixed, can be decomposed into two parts [see Eq. (A2)]:

$$\delta(\mathcal{E}_{\rm np} - \breve{\mu}_p n_p) = \delta(\mathcal{E}_{\rm LF} - \breve{\mu}_p n_p) + \delta\mathcal{E}_{\rm pair}. \tag{B1}$$

Taking into account Eq. (A5) and substituting $\mathbf{p} = \mathbf{q} + \mathbf{Q}_p$, the first term in Eq. (B1) can be represented as

$$\delta(\mathcal{E}_{\rm LF} - \breve{\mu}_p n_p) = \sum_{\mathbf{q}\sigma}\big(\varepsilon_{\mathbf{q}+\mathbf{Q}_p}^{(p)} - \breve{\mu}_p\big)\delta\mathcal{N}_{\mathbf{q}+\mathbf{Q}_p}^{(p)}. \tag{B2}$$

We restrict our consideration to functions $\delta\mathcal{N}_{\mathbf{q}+\mathbf{Q}_p}^{(p)}$ that are invariant under the transformation $\mathbf{q} \to -\mathbf{q}$. This reflects the fact that Cooper pairing occurs between particles with momenta $\mathbf{q}+\mathbf{Q}_p$ and $-\mathbf{q}+\mathbf{Q}_p$. Accounting for this symmetry, Eq. (B2) can be rewritten as

$$\delta(\mathcal{E}_{\rm LF}-\breve{\mu}_p n_p)=\sum_{\mathbf{q}\sigma}\mathcal{H}^{(p)}_{\mathbf{q}}\delta\mathcal{N}^{(p)}_{\mathbf{q}+\mathbf{Q}_p}, \tag{B3}$$

where the function $\mathcal{H}^{(p)}_{\mathbf{q}}$ is defined according to Eq. (A6).

Let us now turn to the pairing energy term in Eq. (B1). In view of Eqs. (A8), (A9), and (A11), we have

$$\begin{aligned}\delta\mathcal{E}_{\rm pair}&=-\frac{1}{2}\sum_{\mathbf{q}\sigma}\Delta^{(p)}\,\delta\left(\frac{\Delta^{(p)}}{E^{(p)}_{\mathbf{q}}}\right)\\&=-\sum_{\mathbf{q}\sigma}\Delta^{(p)}\,\frac{u^{(p)2}_{\mathbf{q}}-v^{(p)2}_{\mathbf{q}}}{u^{(p)}_{\mathbf{q}}}\,\delta v^{(p)}_{\mathbf{q}},\end{aligned} \tag{B4}$$

where in the first equality we used the fact that the variation of the right-hand side of Eq. (A8) vanishes. The function $v^{(p)}_{\mathbf{q}}$ is related to the proton distribution function via Eq. (A12). Taking this into account and substituting Eqs. (A10), we obtain

$$\delta\mathcal{E}_{\rm pair}=-\sum_{\mathbf{q}\sigma}\mathcal{H}^{(p)}_{\mathbf{q}}\delta\mathcal{N}^{(p)}_{\mathbf{q}+\mathbf{Q}_p}. \tag{B5}$$

Thus, the total variation in Eq. (B1) vanishes. Note that, in deriving this result, we did not assume that the whole system is in thermodynamic equilibrium with all currents set to zero. If the variation is evaluated in such an equilibrium, one should replace $\varepsilon^{(p)}_{\mathbf{q}+\mathbf{Q}_p}\to\varepsilon^{(p)}_{\mathbf{q},0}$ and $\breve{\mu}_p\to E_{Fp}$.

APPENDIX C: MOMENTUM CONSERVATION

To derive the momentum-conservation equation, let us write down the time-dependent kinetic equations for neutrons and electrons:

$$\frac{\partial\mathcal{N}^{(n)}_{\mathbf{p}}}{\partial t}+\frac{\partial\varepsilon^{(n)}_{\mathbf{p}}}{\partial\mathbf{p}}\frac{\partial\mathcal{N}^{(n)}_{\mathbf{p}}}{\partial\mathbf{r}}-\frac{\partial\varepsilon^{(n)}_{\mathbf{p}}}{\partial\mathbf{r}}\frac{\partial\mathcal{N}^{(n)}_{\mathbf{p}}}{\partial\mathbf{p}}=I_n, \tag{C1}$$

$$\frac{\partial\mathcal{N}^{(e)}_{\mathbf{p}}}{\partial t}+\boldsymbol{v}_{\mathbf{p}}\frac{\partial\mathcal{N}^{(e)}_{\mathbf{p}}}{\partial\mathbf{r}}-e\left(\mathbf{E}+\frac{1}{c}\boldsymbol{v}_{\mathbf{p}}\times\mathbf{B}\right)\frac{\partial\mathcal{N}^{(e)}_{\mathbf{p}}}{\partial\mathbf{p}}=I_e. \tag{C2}$$

The proton condensate satisfies the "superfluid" equation [16,17]

$$\frac{\partial\mathbf{Q}_p}{\partial t}+\nabla\breve{\mu}_p=e\mathbf{E} \tag{C3}$$

and the continuity equation

$$\frac{\partial n_p}{\partial t}+{\rm div}\mathbf{j}_p=0. \tag{C4}$$

Here, the proton particle number density equals [13,22]

$$n_p=\sum_{\mathbf{q}\sigma}\mathcal{N}^{(p)}_{\mathbf{q}+\mathbf{Q}_p}, \tag{C5}$$

where the distribution function is given by Eq. (A12), while the proton current density can be expressed as [39,40]

$$\mathbf{j}_p=\sum_{\mathbf{p}\sigma}\frac{\partial\varepsilon^{(p)}_{\mathbf{p}}}{\partial\mathbf{p}}\mathcal{N}^{(p)}_{\mathbf{p}}=\sum_{\mathbf{q}\sigma}\frac{\partial\varepsilon^{(p)}_{\mathbf{q}+\mathbf{Q}_p}}{\partial\mathbf{q}}\mathcal{N}^{(p)}_{\mathbf{q}+\mathbf{Q}_p}. \tag{C6}$$

Let us multiply Eq. (C1) by $\mathbf{p}$ and sum the result over all momenta $\mathbf{p}$ and spin states σ. This yields, after some integration by parts:

$$\begin{aligned}&\frac{\partial}{\partial t}\sum_{\mathbf{p}\sigma}p^i\mathcal{N}^{(n)}_{\mathbf{p}}+\frac{\partial}{\partial r^k}\sum_{\mathbf{p}\sigma}p^i\frac{\partial\varepsilon^{(n)}_{\mathbf{p}}}{\partial p^k}\mathcal{N}^{(n)}_{\mathbf{p}}+\sum_{\mathbf{p}\sigma}\frac{\partial\varepsilon^{(n)}_{\mathbf{p}}}{\partial r^i}\mathcal{N}^{(n)}_{\mathbf{p}}\\&\quad=\sum_{\mathbf{p}\sigma}p^iI_n.\end{aligned} \tag{C7}$$

The third term on the left-hand side does not generally vanish due to the Fermi-liquid interaction. To proceed further, let us calculate the gradient of the energy $\mathcal{E}_{LF}$. Using its definition given by Eq. (A3), one can show that

$$\nabla\mathcal{E}_{\rm LF}=\sum_{\mathbf{p}\sigma\alpha=n,p}\left[\nabla\left(\varepsilon^{(\alpha)}_{\mathbf{p}}\mathcal{N}^{(\alpha)}_{\mathbf{p}}\right)-\mathcal{N}^{(\alpha)}_{\mathbf{p}}\nabla\varepsilon^{(\alpha)}_{\mathbf{p}}\right]. \tag{C8}$$

Therefore, Eq. (C7) transforms to

$$\begin{aligned}&\frac{\partial}{\partial t}\sum_{\mathbf{p}\sigma}p^i\mathcal{N}^{(n)}_{\mathbf{p}}+\frac{\partial}{\partial r^k}\left[\sum_{\mathbf{p}\sigma}p^i\frac{\partial\varepsilon^{(n)}_{\mathbf{p}}}{\partial p^k}\mathcal{N}^{(n)}_{\mathbf{p}}\right.\\&\qquad\left.+\,\delta^{ik}\left(\sum_{\mathbf{p}\sigma\alpha=n,p}\varepsilon^{(\alpha)}_{\mathbf{p}}\mathcal{N}^{(\alpha)}_{\mathbf{p}}-\mathcal{E}_{\rm LF}\right)\right]-\sum_{\mathbf{p}\sigma}\frac{\partial\varepsilon^{(p)}_{\mathbf{p}}}{\partial r^i}\mathcal{N}^{(p)}_{\mathbf{p}}\\&\quad=\sum_{\mathbf{p}\sigma}p^iI_n.\end{aligned} \tag{C9}$$

The last term on the left-hand side can be rewritten as

$$\begin{aligned}\sum_{\mathbf{p}\sigma}\mathcal{N}^{(p)}_{\mathbf{p}}\nabla\varepsilon^{(p)}_{\mathbf{p}}&=\sum_{\mathbf{q}\sigma}\mathcal{N}^{(p)}_{\mathbf{q}+\mathbf{Q}_p}\nabla\left(\varepsilon^{(p)}_{\mathbf{q}+\mathbf{Q}_p}-\breve{\mu}_p\right)\\&\quad-j^k_p\nabla Q^k_p+n_p\nabla\breve{\mu}_p.\end{aligned} \tag{C10}$$

To obtain this identity, we shifted the running momentum variable via the substitution $\mathbf{p}=\mathbf{q}+\mathbf{Q}_p$ and then applied Eqs. (C5) and (C6). To proceed further, one can show that (cf. Appendix B)

$$\sum_{\mathbf{q}\sigma}\left(\varepsilon^{(p)}_{\mathbf{q}+\mathbf{Q}_p}-\breve{\mu}_p\right)\nabla\mathcal{N}^{(p)}_{\mathbf{q}+\mathbf{Q}_p}=\sum_{\mathbf{q}\sigma}\mathcal{H}^{(p)}_{\mathbf{q}}\nabla v^{(p)2}_{\mathbf{q}}=-\nabla\mathcal{E}_{\rm pair}. \tag{C11}$$

Using Eq. (C11), one can obtain the following expression for the first term on the right-hand side of Eq. (C10):

$$\begin{aligned}&\sum_{\mathbf{q}\sigma}\mathcal{N}^{(p)}_{\mathbf{q}+\mathbf{Q}_p}\nabla\left(\varepsilon^{(p)}_{\mathbf{q}+\mathbf{Q}_p}-\breve{\mu}_p\right)\\&\quad=\nabla\left[\sum_{\mathbf{q}\sigma}\left(\varepsilon^{(p)}_{\mathbf{q}+\mathbf{Q}_p}-\breve{\mu}_p\right)\mathcal{N}^{(p)}_{\mathbf{q}+\mathbf{Q}_p}+\mathcal{E}_{\rm pair}\right]\\&\quad=\nabla\left[\sum_{\mathbf{p}\sigma}\varepsilon^{(p)}_{\mathbf{p}}\mathcal{N}^{(p)}_{\mathbf{p}}-n_p\breve{\mu}_p+\mathcal{E}_{\rm pair}\right].\end{aligned} \tag{C12}$$

The gradient of the chemical potential in the third term of Eq. (C10) can be expressed with the help of Eq. (C3). Substituting the resulting expression for the term $\sum_{\mathbf{p}\sigma}\mathcal{N}^{(p)}_{\mathbf{p}}\nabla\varepsilon^{(p)}_{\mathbf{p}}$

back into Eq. (C9) and making use of the continuity equation (C4), we get

$$\frac{\partial}{\partial t}\left(\sum_{\mathbf{p}\sigma} p^i \mathcal{N}_{\mathbf{p}}^{(n)} + n_p Q_p^i\right) + Q_p^i \frac{\partial j_p^k}{\partial r^k} + j_p^k \frac{\partial Q_p^k}{\partial r^i}$$
$$+ \frac{\partial}{\partial r^k}\left[\sum_{\mathbf{p}\sigma} p^i \frac{\partial \varepsilon_{\mathbf{p}}^{(n)}}{\partial p^k}\mathcal{N}_{\mathbf{p}}^{(n)} + \delta^{ik}\left(\sum_{\mathbf{p}\sigma}\varepsilon_{\mathbf{p}}^{(n)}\mathcal{N}_{\mathbf{p}}^{(n)} + n_p\breve{\mu}_p - \mathcal{E}_{\mathrm{np}}\right)\right]$$
$$- e n_p E^i = \sum_{\mathbf{p}\sigma} p^i I_n. \tag{C13}$$

Next, we can add and subtract the combination $j_p^k \frac{\partial Q_p^i}{\partial r^k}$ and then substitute Eq. (18). This leads us to the following equation:

$$\frac{\partial \mathfrak{P}_{\mathrm{np}}^i}{\partial t} + \frac{\partial \Pi_{\mathrm{np}}^{ik}}{\partial r^k} = e\left(n_p E^i + \frac{1}{c}[\mathbf{j}_p \times \mathbf{B}]^i\right) + \delta^{(2)}(\boldsymbol{\rho}) F_{\mathrm{ext}}^i$$
$$+ \sum_{\mathbf{p}\sigma} p^i I_n, \tag{C14}$$

where we defined the neutron-proton momentum density

$$\mathfrak{P}_{\mathrm{np}}^i = \sum_{\mathbf{p}\sigma} p^i \mathcal{N}_{\mathbf{p}}^{(n)} + n_p Q_p^i \tag{C15}$$

and the neutron-proton stress tensor

$$\Pi_{\mathrm{np}}^{ik} = Q_p^i j_p^k + \sum_{\mathbf{p}\sigma} p^i \frac{\partial \varepsilon_{\mathbf{p}}^{(n)}}{\partial p^k}\mathcal{N}_{\mathbf{p}}^{(n)}$$
$$+ \delta^{ik}\left(\sum_{\mathbf{p}\sigma}\varepsilon_{\mathbf{p}}^{(n)}\mathcal{N}_{\mathbf{p}}^{(n)} + n_p\breve{\mu}_p - \mathcal{E}_{\mathrm{np}}\right). \tag{C16}$$

Performing some integrations by parts, the stress tensor can be rewritten in a form that is more in line with the Fermi-liquid framework:

$$\Pi_{\mathrm{np}}^{ik} = Q_p^i j_p^k - \sum_{\mathbf{p}\sigma}\varepsilon_{\mathbf{p}}^{(n)} p^i \frac{\partial \mathcal{N}_{\mathbf{p}}^{(n)}}{\partial p^k} + \delta^{ik}(n_p\breve{\mu}_p - \mathcal{E}_{\mathrm{np}}). \tag{C17}$$

The first term on the right-hand side of Eq. (C14) is the standard Lorentz force. In the second term, we introduced the quantity $\mathbf{F}_{\mathrm{ext}}$ via the equation

$$\delta^{(2)}(\boldsymbol{\rho})\,\mathbf{F}_{\mathrm{ext}} = \frac{m_p \varkappa}{2\pi\rho}\,\delta(\rho)\,\mathbf{e}_z \times \mathbf{j}_p, \tag{C18}$$

in which the identity (22) was taken into account. This delta-function-containing term arises from taking the curl of the condensate momentum $\mathbf{Q}_p$ [see Eq. (18)]. As shown in Ref. [12], the quantity $\mathbf{F}_{\mathrm{ext}}$ defined in Eq. (C18) can be interpreted as the external force acting on an infinitely thin vortex core. We do not discuss this force further here and refer the reader to Ref. [12] for a detailed consideration. The third term on the right-hand side of Eq. (C14) describes the exchange of momentum between neutrons and electrons via collisions.

Let us now turn to the electrons. Multiplying Eq. (C2) by $\mathbf{p}$ and summing it over $\mathbf{p}$ and σ, we obtain

$$\frac{\partial \mathfrak{P}_e^i}{\partial t} + \frac{\partial \Pi_e^{ik}}{\partial r^k} = -e\left(n_e \mathbf{E} + \frac{1}{c}\mathbf{j}_e \times \mathbf{B}\right) + \sum_{\mathbf{p}\sigma} p^i I_e, \tag{C19}$$

where

$$\mathfrak{P}_e^i = \sum_{\mathbf{p}\sigma} p^i \mathcal{N}_{\mathbf{p}}^{(e)} \tag{C20}$$

is the electron momentum density,

$$\Pi_e^{ik} = \sum_{\mathbf{p}\sigma} p^i \boldsymbol{v}_{\mathbf{p}}{}^k \mathcal{N}_{\mathbf{p}}^{(e)} \tag{C21}$$

is the electron stress tensor, and $\mathbf{j}_e = \sum_{\mathbf{p}\sigma} \boldsymbol{v}_{\mathbf{p}} \mathcal{N}_{\mathbf{p}}^{(e)}$ is the electron current density. Finally, we sum Eqs. (C14) and (C19) and rewrite the total Lorentz force in terms of the derivatives of the electromagnetic momentum density

$$\mathfrak{P}_{\mathrm{EM}}^i = \frac{(\mathbf{E}\times\mathbf{B})^i}{4\pi c} \tag{C22}$$

and the electromagnetic stress tensor

$$\Pi_{\mathrm{EM}}^{ik} = -\frac{1}{4\pi}\left[E^i E^k + B^i B^k - \frac{\delta^{ik}}{2}(E^2 + B^2)\right] \tag{C23}$$

in the standard manner [38]. This leads us to Eq. (4) with the total momentum density

$$\mathfrak{P}^i = \mathfrak{P}_{\mathrm{np}}^i + \mathfrak{P}_e^i + \mathfrak{P}_{\mathrm{EM}}^i \tag{C24}$$

and total stress tensor

$$\Pi^{ik} = \Pi_{\mathrm{np}}^{ik} + \Pi_e^{ik} + \Pi_{\mathrm{EM}}^{ik}. \tag{C25}$$

APPENDIX D: CONTRIBUTION TO THE FORCE FROM THE FUNCTION $\tilde{g}_{n,H_2}$

Substituting Eq. (67) into the stress tensor (92) and, then, the stress tensor into Eq. (6), we can represent the result in the following form:

$$F_{\mathrm{m}\to\mathrm{v}}^{(np)i} = \frac{2}{(2\pi\hbar)^3}\int_0^\infty dq\, q^2 \frac{\partial \mathcal{N}_{\mathbf{q},0}^{(n)}}{\partial \varepsilon_{\mathbf{q}}^{(n)}}\int_{4\pi} d\Omega_q\, q^i$$
$$\int_0^{2\pi} d\phi\, y_q\left[\frac{\mathbf{V}_n \mathbf{q}_\perp}{q_\perp}\Delta\tilde{H}_{\mathbf{q},2}^{(n)}(x_q, y_q) - \mathbf{V}_n \frac{\mathbf{e}_z \times \mathbf{q}_\perp}{q_\perp}\right.$$
$$\left.\cdot \int_{-\infty}^{y_q} dy_1 \frac{\partial \Delta\tilde{H}_{\mathbf{q},2}^{(n)}(x_q, y_1)}{\partial x_q}\right]. \tag{D1}$$

In this equation, we replaced the summation over quantum states $(\mathbf{q}, \sigma)$ by the integration over the $\mathbf{q}$ variable multiplied by 2, $d\Omega_q$ denotes a solid-angle element of the vector $\mathbf{q}$. We also employed the coordinates x_q and y_q defined in Eq. (68) with ρ set equal to R, the radius of the cylindrical integration surface. The quantity $\Delta\tilde{H}_{\mathbf{q},2}^{(n)}$ given by Eq. (65) can be represented as

$$\Delta\tilde{H}_{\mathbf{q},2}^{(n)}(x_q, y_q) = \left(\mathcal{A} + q_\perp^2 \mathcal{B}\frac{x_q^2}{x_q^2 + y_q^2}\right) Q_{pv}^2\left(\sqrt{x_q^2 + y_q^2}\right). \tag{D2}$$

Let us consider the integral

$$\mathcal{I} = \int_0^{2\pi} d\phi\, y_q \int_{-\infty}^{y_q} dy_1 \frac{\partial \Delta\tilde{H}_{\mathbf{q},2}^{(n)}(x_q, y_1)}{\partial x_q}. \tag{D3}$$

It can be represented as $\mathcal{I} = (1/2)\mathcal{I} + (1/2)\mathcal{I}$. In the second term, we perform the following substitutions sequentially: first, we replace $\phi \to \phi + \pi$ during the integration over ϕ and then we replace $y_1 = -y_1$ during the integration over y_1. After applying these transformations, the two terms can be combined, yielding

$$\mathcal{I} = \frac{1}{2}\int_0^{2\pi} d\phi\, y_q \int_{-\infty}^{\infty} dy_1 \frac{\partial \Delta\tilde{H}^{(n)}_{\mathbf{q},2}(x_q, y_1)}{\partial x_q}. \tag{D4}$$

Thus, instead of (D1), we can write

$$F^{(np)i}_{\mathrm{m}\to\mathrm{v}} = \frac{2}{(2\pi\hbar)^3}\int_0^{\infty} dq\, q^2 \frac{\partial \mathcal{N}^{(n)}_{\mathbf{q},0}}{\partial \varepsilon^{(n)}_{\mathbf{q}}} \int_{4\pi} d\Omega_q\, q^i \int_0^{2\pi} d\phi\, y_q \left[\frac{\mathbf{V}_n \mathbf{q}_\perp}{q_\perp}\Delta\tilde{H}^{(n)}_{\mathbf{q},2}(x_q, y_q) - \mathbf{V}_n \frac{\mathbf{e}_z \times \mathbf{q}_\perp}{q_\perp}\frac{1}{2}\int_{-\infty}^{\infty} dy_1 \frac{\partial \Delta\tilde{H}^{(n)}_{\mathbf{q},2}(x_q, y_1)}{\partial x_q}\right]. \tag{D5}$$

One can see that the first term in the square brackets vanishes upon integration over ϕ, since it changes sign under the transformation $\phi \to \phi + \pi$. Likewise, the second term in the square brackets vanishes, because for any function $F(s)$ integrable on the interval $[-1, 1]$, one has $\int_0^{2\pi} d\phi \cos\phi\, F(\sin\phi) = 0$.

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