

Beyond the Tayler instability: A new global instability of toroidal magnetic fields in stars

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Stellar toroidal magnetic fields are known to be unstable to the Tayler instability. Here we demonstrate the existence of a complementary current-driven instability of essentially *arbitrary* toroidal-field configurations in stably stratified nonrotating stars with the following properties: (i) in ideal magneto-hydrodynamics, it grows on the Alfvén timescale τ_A ; (ii) under certain conditions, it may reveal itself by driving shellular differential rotation about an arbitrary axis perpendicular to the magnetic-field symmetry axis; (iii) it is large scale in the angular directions θ and φ , and develops at radial wave numbers $k \lesssim \mathcal{N}\tau_A/R$, where $\mathcal{N}$ is the Brunt-Väisälä frequency and R is the stellar radius. Thus, unlike the Tayler instability, the proposed instability is intrinsically global. Consequently, it may be less susceptible to dissipative suppression than the Tayler instability and can prevail over it in some regimes. This instability may have broad implications for magnetic field generation in stars and could modify scenarios of magnetic field amplification within the Tayler-Spruit dynamo, contributing to models of efficient angular-momentum transport and chemical mixing in stellar interiors.

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I. INTRODUCTION

Magnetic fields play an important role in the dynamics and evolution of stars [1]. The nature of these fields and the processes responsible for their generation are of central interest in astrophysics. In particular, large-scale toroidal magnetic fields are expected to arise naturally due to stellar differential rotation. The instability of such fields was demonstrated by Tayler (1973) [2] (see also the earlier work by Vandakurov [3]). In his study, Tayler identified a specific class of unstable displacements in toroidal magnetic fields; the resulting instability is now referred to as the Tayler instability.¹

Over the years, this instability has been studied using analytical and semianalytical methods (e.g., [4,5,7–17]), as well as through three-dimensional numerical simulations (e.g., [18–26]).

Since the pioneering works of Spruit [27,28], who argued that the instability of toroidal magnetic fields could drive a dynamo process in stars, now known as the Tayler-Spruit dynamo, interest in the Tayler instability has increased substantially. In particular, the Tayler-Spruit dynamo is used to explain the magnetic field observed in magnetars [29–31] and low-field magnetars [32]; its influence is taken into account when modeling chemical mixing in nondegenerate stars (see [28,33,34] and, for illustration, two recent studies [35,36] and references therein). Most importantly for stellar evolution theory, the magnetic stresses arising in the course of such a dynamo give rise to an effective viscosity (e.g., [28,33,34,37]) and to an additional, efficient transport of angular momentum in stars.

At the same time, asteroseismic data indicate that existing angular-momentum transport mechanisms appear to be insufficient to account for the observed level of internal (core-to-envelope) differential rotation in certain

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¹Some authors (e.g., [4–6]) use the term “Tayler instability” to denote, by definition, any $m = 1$ current-driven (pinch-type) instability of toroidal fields. In this work, we do not adopt this terminology and, by the Tayler instability in stars, we mean the current-driven $m = 1$ instability whose most characteristic feature is that the corresponding perturbations are small scale (at least) in the radial direction and, in the limit of ideal magneto-hydrodynamics, remain unstable even as this scale tends to zero (see Sec. VIII for more details).

classes of stars (in particular, in low-mass evolved stars and the Sun). In this context, the mechanism of angular-momentum transport due to the Tayler-Spruit dynamo acquires special importance and is regarded as one of the most promising candidates (see [37,38] and references therein).

It should be noted that the original dynamo scenario proposed by Spruit [28] was criticized in [39,40] and was later modified by Fuller *et al.* in Ref. [37]. Studies aimed at detecting the Tayler-Spruit dynamo in numerical simulations have appeared only relatively recently [24,30,31,41–43] (but see [18]). A discussion of these results and their consistency with linear theory can be found in Ref. [16]. Currently, there is no clear consensus on which version of the dynamo, as described in Ref. [28] or [37], is reproduced in simulations (cf. Refs. [42] and [24,30,31]).

In addition, a number of studies have shown that neither version of the Tayler-Spruit dynamo satisfactorily reproduces the observational constraints on the internal differential-rotation profiles and the surface abundances of certain chemical elements in stars (e.g., [35,36,44–49]).

In light of the above, our goal in this paper is to revisit the stability of toroidal magnetic fields in stars. Using the energy principle of Bernstein *et al.* [50], we show analytically that, in addition to the well-known Tayler-type unstable displacements, there exists another class of unstable modes that differ qualitatively from those originally proposed by Tayler. Both types of displacements have comparable growth rates, typically of the order of the Alfvén frequency. Depending on stellar parameters and magnetic-field configuration, either the Tayler instability or the newly found instability may dominate. Our theoretical predictions are consistent with numerical simulations performed with the PENCIL Code [51], a high-order finite-difference code for compressible magnetohydrodynamics (MHD) simulations.

The newly identified instability suggests that the traditional Tayler-Spruit dynamo mechanism may need to be reconsidered. This, in turn, could have implications for the angular-momentum transport and the accompanying chemical mixing in stellar interiors, as well as for the observational properties of magnetic stars.

The structure of this paper is as follows. Section II presents the governing equations and the basic assumptions for a stably stratified, nonrotating magnetized star in equilibrium. In Sec. III, we set up the energy-principle framework and notation, state the assumptions on the equilibrium toroidal field, and introduce the displacement parametrization used in the rest of the paper. Section IV further transforms the energy functional and develops a perturbative framework that identifies the expected structure and key properties of the most unstable displacements. In Sec. V, we use this framework to derive an upper bound for the characteristic growth time of the new instability and its scaling with the relevant small parameters. Section VI

applies the formalism to a concrete stellar model and a specified toroidal-field configuration, performs the minimization over trial displacements, and obtains numerical estimates for the growth time. Section VII estimates the strength of a (weak) poloidal magnetic field required to suppress the instability. In Sec. VIII, we compare the Tayler instability with the global instability identified here, emphasizing the differences in their preferred length scales and their sensitivity to dissipation. Section IX presents numerical simulations that illustrate and support the theoretical results. Finally, Sec. X summarizes our conclusions.

The paper also includes four appendixes. Appendix A tests the accuracy of the minimization procedure discussed in the main text, in particular that of Eq. (37). Appendix B presents integral expressions for the dissipation timescale associated with shear viscosity, which affects the instability found in this work. In Appendix C, we discuss a formula quantifying the effect of viscous dissipation on the instability timescale. Finally, Appendix D examines the sensitivity of the new instability to various simplifying approximations that are often adopted in the literature when studying MHD instabilities in stars.

II. PRELIMINARY COMMENTS AND GOVERNING EQUATIONS

In this work, for simplicity, we consider a nonrelativistic magnetized star, neglecting the effects of general relativity. It is assumed that the stellar matter has a nonbarotropic equation of state, in which the pressure p depends not only on the density ρ , but also on at least one additional parameter x that is conserved along the motion of each fluid element: $p = p(\rho, x)$.² This parameter may be, for example, the ratio of the entropy per unit volume s to ρ , $x = s/\rho$, or, if the star contains two species of particles, the ratio of the density of one species ρ_1 to the total density, $x = \rho_1/\rho$. For neutron stars, it is convenient to use the baryon number density n_b instead of ρ , so that the parameter can be written as $x = n_e/n_b$ (n_e is the electron number density).

The complete system of magnetohydrodynamic equations describing a nonrotating magnetized star in the nondissipative limit has the form

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p - \rho \nabla \Phi + \mathbf{F}_L, \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (2)$$

²There may be several such additional parameters; this will not affect the results of the work. The only modification required in the case of multiple parameters concerns the definition of the frozen adiabatic index γ_{fr} . See the explanation following Eq. (18) below.

$$\frac{\partial x}{\partial t} + \mathbf{u} \cdot \nabla x = 0, \quad (3)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (4)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j}, \quad (5)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (6)$$

$$\Delta \Phi = 4\pi G \rho. \quad (7)$$

Here, $\mathbf{u}$ is the fluid velocity; $D\mathbf{u}/Dt \equiv \partial\mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\mathbf{u}$; $\mathbf{B}$ is the magnetic field of the star; $\mathbf{j}$ is the electric current density; $\mathbf{F}_L = \mathbf{j} \times \mathbf{B}/c$ is the Lorentz force per unit volume; G is the gravitational constant; c is the speed of light; and Φ is the gravitational potential.

A. Equilibrium and small deviations from equilibrium

In an equilibrium star, Eqs. (5)–(7) formally remain unchanged, while the Euler equation (1) takes the form

$$\nabla p + \rho \nabla \Phi = \mathbf{F}_L. \quad (8)$$

Here and below, the quantities p , Φ , ρ , $\mathbf{F}_L$, etc. refer to their equilibrium values, and Eulerian perturbations from these equilibrium values will be denoted by δp , $\delta \Phi$, $\delta \rho$, $\delta \mathbf{F}_L$, etc. (with the exception of the fluid velocity, which will be denoted by $\mathbf{u}$).

Let us consider a small perturbation of the system (1)–(7) relative to equilibrium. Introducing the Lagrangian displacement ξ according to the relation

$$\mathbf{u} = \frac{\partial \xi}{\partial t}, \quad (9)$$

the linearized system of equations takes the form

$$\rho \frac{\partial^2 \xi}{\partial t^2} = -\nabla \delta p - \delta \rho \nabla \Phi - \rho \nabla \delta \Phi + \delta \mathbf{F}_L, \quad (10)$$

$$\delta \rho + \nabla \cdot (\rho \xi) = 0, \quad (11)$$

$$\delta x + \xi \cdot \nabla x = 0, \quad (12)$$

$$\delta \mathbf{B} = \nabla \times (\xi \times \mathbf{B}), \quad (13)$$

$$\nabla \times \delta \mathbf{B} = \frac{4\pi}{c} \delta \mathbf{j}, \quad (14)$$

$$\Delta \delta \Phi = 4\pi G \delta \rho, \quad (15)$$

where

$$\delta \mathbf{F}_L = (\delta \mathbf{j} \times \mathbf{B} + \mathbf{j} \times \delta \mathbf{B})/c. \quad (16)$$

Equation $\nabla \cdot \delta \mathbf{B} = 0$ is not presented here since it is automatically satisfied in view of Eq. (13). Keeping in mind that perturbations occur at frozen parameter x , we can write

$$\delta p + \xi \cdot \nabla p = \frac{\gamma_{\text{fr}} p}{\rho} (\delta \rho + \xi \cdot \nabla \rho), \quad (17)$$

where γ_{fr} is the frozen adiabatic index,

$$\gamma_{\text{fr}} = \frac{\rho}{p} \frac{\partial p(\rho, x)}{\partial \rho}. \quad (18)$$

In the case where the pressure, in addition to the density, is described by N parameters $x_1, \dots, x_N$ instead of a single parameter x , the partial derivative in Eq. (18) should be taken at constant $x_1, \dots, x_N$.

III. BERNSTEIN *ET AL.* ENERGY PRINCIPLE AND OUR ASSUMPTIONS

A. Bernstein *et al.* energy principle

The stability of the equilibrium state of a magnetized star can be conveniently studied using the energy principle [50], described in detail in the book [52] (see also, e.g., Refs. [15,53], where applications of this principle to stars are considered). Below, we will focus on the stability of the toroidal magnetic field, which must be completely localized inside the star (see Sec. III B below for the boundary conditions that we assume such a field satisfies). In this case, the energy functional $W[\xi]$, describing the “potential” energy of the system, takes the form

$$W[\xi] = W^{\text{hydro}}[\xi] + W^{\text{mag}}[\xi], \quad (19)$$

and consists of two parts—the hydrodynamic part, $W^{\text{hydro}}[\xi]$, and the magnetic part, $W^{\text{mag}}[\xi]$,

$$W^{\text{hydro}}[\xi] = \frac{1}{2} \int [\gamma_{\text{fr}} p (\nabla \cdot \xi)^2 + (\xi \cdot \nabla p) (\nabla \cdot \xi) - (\xi \cdot \nabla \Phi) \nabla \cdot (\rho \xi) + \rho \xi \cdot \nabla \delta \Phi] dV, \quad (20)$$

$$W^{\text{mag}}[\xi] = \frac{1}{2} \int \left[\frac{\delta \mathbf{B} \cdot \delta \mathbf{B}}{4\pi} + \frac{1}{c} \mathbf{j} \cdot (\xi \times \delta \mathbf{B}) \right] dV. \quad (21)$$

In expressions (20) and (21), the integration is performed over the volume of the star V . Furthermore, in our analysis, we will not use the complex formalism with respect to the function ξ ; instead we will assume that the Lagrangian displacement ξ is a real function.

The Bernstein *et al.* energy principle for stability states that [52] “An equilibrium is stable if (sufficient) and only if (necessary) $W[\xi] > 0$ for all displacements $\xi(\mathbf{r})$ that are bound in norm and satisfy the boundary conditions.” In the case of a toroidal magnetic field localized inside the star,

the boundary conditions for $\xi(r)$ are irrelevant and will not be discussed further.

According to the energy principle, if we can find a Lagrangian displacement ξ such that $W[\xi] < 0$, then the system is unstable. In fact, this energy principle follows from a more general statement known as the spectral variational principle. To formulate it, note that the system of linearized magnetohydrodynamic equations (10)–(15) is self-adjoint (e.g., [52]). This means that any Lagrangian displacement can be expanded in terms of the eigenmodes of the magnetized star $\xi_n(r)$, with eigenfrequencies ω_n , such that ω_n^2 is real. The spectral variational principle states that the eigenfunctions $\xi_n(r)$ make the Rayleigh quotient

$$\Lambda \equiv \frac{W[\xi]}{I[\xi]} \quad (22)$$

stationary, i.e. the functional derivative $\delta(W[\xi]/I[\xi])/\delta\xi = 0$ for $\xi = \xi_n$. The corresponding stationary values of Λ are the eigenvalues ω_n^2 [52]. The positive “kinetic energy” functional $I[\xi]$ in Eq. (22) is defined as

$$I[\xi] \equiv \frac{1}{2} \int \rho \xi \cdot \xi dV. \quad (23)$$

Thus, solving the linearized hydrodynamic equations is fully equivalent to finding the functions that make the ratio (22) stationary. The energy principle of Bernstein *et al.* clearly follows from the spectral variational principle. In particular, if there exists at least one unstable mode with $\omega_n^2 < 0$ (which corresponds to an exponential growth of this mode $\propto e^{|\omega_n|t}$), then one can always choose a Lagrangian displacement ξ such that $W[\xi] < 0$. However, the spectral variational principle allows one to go further and obtain an estimate for the characteristic growth time of the instability. Indeed, the absolute minimum of the functional (22) is equal to the square of the frequency of the most unstable mode, $\omega_{\min}^2$ (if the system is unstable, then $\omega_{\min}^2 < 0$). Therefore, for any trial function ξ one can say that

$$\omega_{\min}^2 \leq \frac{W[\xi]}{I[\xi]}. \quad (24)$$

By choosing increasingly accurate trial functions, one can improve the upper bound on $\omega_{\min}^2$.

From the estimate of the (negative) value of $\omega_{\min}^2$, it is straightforward to obtain an upper bound on the characteristic growth time of the instability, $\tau_{\min}$. Defining $\tau_{\min}^2 \equiv -1/\omega_{\min}^2$, we obtain from (24)

$$\tau_{\min} \leq \sqrt{\frac{I[\xi]}{|W[\xi]|}}. \quad (25)$$

B. Our assumptions

In the present paper, we assume that the unperturbed, axially symmetric star hosts a toroidal magnetic field, $\mathbf{B} = B(r, \theta) \mathbf{e}_\varphi$, where r , θ , and φ are the usual spherical coordinates with the origin at the center of the star in the absence of the magnetic field; $\mathbf{e}_i$ denotes the corresponding unit vector ($i = r, \theta, \varphi$). We emphasize that the same nonmagnetized star is assumed to be stable. We define the radius of such a star as R .

Requiring the charge current density $\mathbf{j}$ to remain finite at the poles and at the center of the star, we conclude that the toroidal field $B(r, \theta) \mathbf{e}_\varphi$ must vanish as $\theta \rightarrow 0$ and $\theta \rightarrow \pi$ at least as $\sin \theta$, and as $r \rightarrow 0$ at least as r .

Furthermore, in order to avoid (unphysical) surface currents, we must require that $B = 0$ at the stellar surface. In addition, since we are mainly interested in stellar models with vanishing density at the surface, we also assume that not only the toroidal magnetic field but also the charge current density $\mathbf{j}$ vanishes at the surface. This leads to additional conditions at the stellar surface: $\partial_r B(r, \theta) = 0$, $\partial_\theta B(r, \theta) = 0$. Here and in what follows, we employ the notation $\partial_i \equiv \partial/\partial x_i$, where x_i stands for r , θ , or φ .

To demonstrate the instability of the toroidal field, we need to find a Lagrangian displacement ξ such that $W[\xi] < 0$. In what follows, we shall always ignore the last term in expression (20), which depends on the gravitational field perturbation $\delta\Phi$. It is well known [54] that this term is always negative, so if we prove that the system is unstable without this term, it will remain unstable when this term is taken into account.

Our final important assumption concerns the strength of the magnetic field. We assume that the magnetic energy density is much smaller than the pressure everywhere in the region where the magnetic field is localized. Introducing the (constant) ratio κ^3 of some typical magnetic energy density $B^2/8\pi$ to the pressure p ,

$$\kappa \equiv \frac{B^2}{8\pi p} \ll 1, \quad (26)$$

we assume that any quantity X , characterizing the unperturbed star [e.g., the gravitational potential, $\Phi(r, \theta)$, or the “frozen” adiabatic index, γ_{fr}] can be expanded in a series in κ :

$$X(r, \theta) = X_0(r) + \kappa X_1(r, \theta) + \kappa^2 X_2(r, \theta) + \dots, \quad (27)$$

where $X_0(r)$ is the value of this quantity in the star without a magnetic field, while the remaining terms represent small perturbations caused by the field. Note that, with this definition of the functions $X_i(r, \theta)$ ($i = 1, 2, \dots$), the ratio $X_i/X_0 \sim \mathcal{O}(1)$.

³Usually, in the literature, instead of κ one uses the parameter $\beta \equiv 1/\kappa$, but for our purposes it is more convenient to work with κ .

The decomposition (27) can be used to relate the partial derivatives $\partial_\theta \Phi_1(r, \theta)$ and $\partial_\theta \rho_1(r, \theta)$ using the force balance equation (8). Taking the curl of this equation and using (27), one obtains, to linear accuracy in κ ,

$$-\partial_r \Phi_0 \partial_\theta \rho_1 + \partial_r \rho_0 \partial_\theta \Phi_1 = \frac{r(\nabla \times \mathbf{F}_L)_\varphi}{\kappa}, \quad (28)$$

where $(\nabla \times \mathbf{F}_L)_\varphi$ is the φ component of the vector $\nabla \times \mathbf{F}_L$. Note that the quantity $(\nabla \times \mathbf{F}_L)_\varphi$ on the right-hand side of (28) is quadratic in the small magnetic field. It is convenient to explicitly account for this smallness by introducing a function $\mathcal{B}$ [such that $\mathcal{B}^2/p \sim \mathcal{O}(1)$] according to the relation

$$\mathbf{B}(r, \theta) = \sqrt{8\pi\kappa} \mathcal{B}(r, \theta). \quad (29)$$

In this case, Eq. (28) can be rewritten as

$$-\partial_r \Phi_0 \partial_\theta \rho_1 + \partial_r \rho_0 \partial_\theta \Phi_1 = 2r\{\nabla \times [(\nabla \times \mathcal{B}) \times \mathcal{B}]\}_\varphi. \quad (30)$$

The relation (30) appears to be helpful in the subsequent derivations.

C. General analysis

Let us assume that the given magnetic field configuration renders our star unstable. Particular unstable modes develop at their imaginary characteristic frequencies ω such that $i\omega < 0$. Substituting $\xi(\mathbf{r}, t) = \xi(\mathbf{r})e^{-i\omega t}$, the Euler equation (10) can be written as

$$-\rho\omega^2 \xi = -\nabla \delta p - \delta\rho \nabla \Phi + \delta \mathbf{F}_L, \quad (31)$$

where, as already discussed in Sec. III B, we omit the term involving the perturbation of the gravitational potential, $\delta\Phi$.

The quantity $\delta \mathbf{F}_L$ can be estimated from Eq. (16) as $|\delta \mathbf{F}_L| \sim B\delta B/L \sim B^2|\xi|/L^2$, where L is the typical spatial scale of the problem (of the order of some fraction of the stellar radius R). Each of the remaining two terms on the right-hand side of Eq. (31) should not exceed $\delta \mathbf{F}_L$ on the order of magnitude, since, by assumption, the instability is caused by the magnetic field. Indeed, if either of these terms were much larger than $\delta \mathbf{F}_L$, magnetic effects (i.e., dependence on the magnetic field) in Eq. (31) could be neglected, and this equation would reduce to the usual eigenmode problem for a nonrotating nonmagnetized star, which, by assumption, is stable (and thus uninteresting for us).⁴

Assuming that all terms on the right-hand side of (31) are of the same order as $\delta \mathbf{F}_L$ or less, we obtain from (31)

⁴We would like to emphasize that a situation in which each of the terms $\nabla \delta p$ and $\delta\rho \nabla \Phi$ is large, but they nearly cancel when summed, is not realistic. Indeed, $\nabla \Phi$ is nearly radial. If we imagine a situation in which these two terms compensate each other with high precision, then $\nabla \delta p$ would also have to be nearly radial. This corresponds to radial perturbations, which are characterized by high frequencies and are unlikely to develop as a result of magnetic instability.

an estimate for the frequency ω : $|\omega| \lesssim 1/\tau_A$, where $\tau_A \sim L\sqrt{4\pi\rho}/B$ is the Alfvén timescale. We see that, as expected, the characteristic timescale of instability development turns out to be of the order of τ_A or greater.

From the fact that $|\nabla \delta p| \lesssim |\delta \mathbf{F}_L|$ and $|\delta\rho \nabla \Phi| \lesssim |\delta \mathbf{F}_L|$ we obtain the estimates $|\delta\rho/\rho| \lesssim |\kappa\xi|/L$ and $|\delta p/p| \lesssim |\kappa\xi|/L$ [see Eq. (26) for the definition of κ]. Recalling Eqs. (11) and (17), we then obtain that

$$|\nabla \cdot (\rho\xi)| \lesssim \rho\kappa \frac{|\xi|}{L}, \quad (32)$$

$$|\xi_r| \lesssim \frac{1}{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}} \kappa |\xi|. \quad (33)$$

The second estimate here follows from Eq. (17) if one notices that, up to (small) terms of order κ , one can write $\gamma_{\text{fr}} \approx \gamma_{\text{fr},0}(r)$, $\nabla\rho \approx \nabla\rho_0(r)$, and $\nabla p \approx \nabla p_0(r)$, where $\rho_0(r)$ and $p_0(r)$ are the equilibrium density and pressure profiles in the star without a magnetic field, and introduces the equilibrium index $\gamma_{\text{eq},0}$ according to

$$\gamma_{\text{eq},0} = \frac{\rho_0}{p_0} \frac{dp_0[\rho_0, x_0(\rho_0)]}{d\rho_0}, \quad (34)$$

where all derivatives in Eq. (34) are taken in the equilibrium spherically symmetric star in the absence of a magnetic field (see Sec. III B); $x_0(\rho_0)$ is the dependence of x on the density in the nonmagnetized star. In a nonbarotropic star, the frozen adiabatic index, $\gamma_{\text{fr},0}$, is not equal to the equilibrium index, $\gamma_{\text{eq},0}$ (see, e.g., Ref. [55] for details). Since we assume that the nonmagnetized star is stable, the difference $\gamma_{\text{fr},0} - \gamma_{\text{eq},0}$ between these indices must be positive for the stellar matter to be stably stratified (see, e.g., Refs. [15,55]). This condition is equivalent to requiring that the squared Brunt-Väisälä frequency $\mathcal{N}^2$ be positive throughout the star, where

$$\mathcal{N} = g \sqrt{\frac{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}}{\gamma_{\text{fr},0} \gamma_{\text{eq},0}}} \frac{\rho_0}{p_0} \quad (35)$$

and g is the gravitational acceleration.

From estimates (32) and (33), it follows that $\nabla \cdot (\rho\xi)$ and ξ_r in a nonbarotropic ($\gamma_{\text{fr},0} \neq \gamma_{\text{eq},0}$) star are suppressed and generally proportional to κ . This result was obtained under the assumption that the problem has a single length scale L over which all quantities vary. As will become clear in what follows, an additional length scale associated with the perturbation may arise; in that case, the estimates for $\nabla \cdot (\rho\xi)$ and ξ_r will be modified. However, the main conclusion, that $\nabla \cdot (\rho\xi)$ and ξ_r are suppressed, remains intact.

Note that in a barotropic star, where $\gamma_{\text{fr},0} = \gamma_{\text{eq},0}$, estimate (33) does not hold, as is formally evident from the vanishing denominator in this estimate. In this case,

ξ_r is expected to be comparable to the other components, ξ_θ and ξ_φ , of the Lagrangian displacement ξ .

D. Result of exact minimization in the case of a toroidal magnetic field

As shown by Tayler [2], in the case of a toroidal field and nonaxisymmetric Lagrangian displacements (i.e., when the azimuthal number $m \neq 0$), the minimum of $W[\xi]$ is achieved when ξ satisfies a certain condition.⁵ In the notation of Ref. [7], this condition is written as [see their equation (17)]

$$\delta p = -\xi \cdot F_L, \quad (36)$$

where δp is given by (17). Using Eqs. (8), (11), and (17), Eq. (36) reduces to the form (see, e.g., Ref. [15])

$$\nabla \cdot \xi = \frac{\rho}{\gamma_{\text{fr}} P} \nabla \Phi \cdot \xi. \quad (37)$$

This equation is valid for a toroidal magnetic field of arbitrary strength.

E. The function ξ

In the general case, the Lagrangian displacement ξ can be expanded into a Fourier series in φ :

$$\begin{aligned} \xi(r, \theta, \varphi) = \sum_{m \geq 0} \{ & [\tilde{\xi}_r^{(m)}(r, \theta) \sin(m\varphi), \tilde{\xi}_\theta^{(m)}(r, \theta) \sin(m\varphi), \tilde{\xi}_\varphi^{(m)}(r, \theta) \cos(m\varphi)] \\ & + [\hat{\xi}_r^{(m)}(r, \theta) \cos(m\varphi), \hat{\xi}_\theta^{(m)}(r, \theta) \cos(m\varphi), \hat{\xi}_\varphi^{(m)}(r, \theta) \sin(m\varphi)] \}, \end{aligned} \quad (38)$$

where $\xi_{r,\theta,\varphi}^{(m)}$ are unknown functions. It can be shown that each individual term in this expansion does not interfere with the others when substituted into the functional (19) and integrated over φ . In this work, we consider only nonaxisymmetric Lagrangian displacements ($m \neq 0$), for which formula (37) holds. Theory and numerical results (see, e.g., [2,7,18,56]) hint that such displacements are the most unstable for our problem. For $m \neq 0$, the first and second terms in the sum (38), which differ only by the substitution $\cos(m\varphi) \rightleftharpoons \sin(m\varphi)$, give identical contributions to (19) after integration in φ . Therefore, without loss of generality, we can search for ξ in the form

$$\begin{aligned} \xi(r, \theta, \varphi) = & [\tilde{\xi}_r^{(m)}(r, \theta) \sin(m\varphi), \tilde{\xi}_\theta^{(m)}(r, \theta) \sin(m\varphi), \\ & \tilde{\xi}_\varphi^{(m)}(r, \theta) \cos(m\varphi)]. \end{aligned} \quad (39)$$

As the main independent parameters, we choose the r and θ components of this Lagrangian displacement. The φ component is not independent and can be expressed in terms of the other components using Eq. (37). We now

introduce new notations, whose physical meaning will become clearer below:

$$\tilde{\xi}_r^{(m)} = \varepsilon g(r, \theta), \quad (40)$$

$$\tilde{\xi}_\theta^{(m)} = -m r a(r, \theta). \quad (41)$$

Here, $g(r, \theta)$ and $ra(r, \theta)$ are functions of the same order of smallness [$g(r, \theta)/a(r, \theta) \sim \mathcal{O}(r)$], and the small constant $\varepsilon \ll 1$ is introduced to reflect the result of Sec. III C, which states that the radial component of the Lagrangian displacement should be suppressed compared to the other components during the development of magnetic field instability in a nonbarotropic star. A specific estimate for the constant ε will be obtained below; it is clear, however, that ε becomes smaller as the ratio $\kappa = B^2/8\pi p$ decreases [see estimate (33) and the discussion following it].

Using the notations (40), (41), the function $\tilde{\xi}_\varphi^{(m)}$ in Eq. (39) can be expressed in terms of $\nabla \cdot \xi$, $g(r, \theta)$, and $a(r, \theta)$ as

$$\begin{aligned} \tilde{\xi}_\varphi^{(m)} = & \frac{\varepsilon \sin \theta}{m r} \partial_r [r^2 g(r, \theta)] - r \partial_\theta [a(r, \theta) \sin \theta] \\ & - \frac{r \sin \theta}{m} \mathcal{D}(r, \theta), \end{aligned} \quad (42)$$

where we have introduced $\mathcal{D}(r, \theta) \equiv \nabla \cdot \xi / \sin(m\varphi)$. As follows from Eq. (37), the function $\mathcal{D}(r, \theta)$ equals

$$\mathcal{D}(r, \theta) = \frac{\rho}{\gamma_{\text{fr}} P} [\varepsilon \partial_r \Phi g(r, \theta) - m \partial_\theta \Phi a(r, \theta)]. \quad (43)$$

Because $\partial_\theta \Phi = \mathcal{O}(\kappa)$, we have $\mathcal{D}(r, \theta) = \mathcal{O}(\varepsilon + \kappa)$.

⁵The minimization is carried out with respect to the component ξ_φ or, equivalently, with respect to the function $\nabla \cdot \xi$. We stress that it is the potential energy functional $W[\xi]$ that is being minimized here by the discussed condition, not the ratio $W[\xi]/I[\xi]$. As discussed in Sec. III A, this approach is fully justified if one is only interested in proving the very existence of the instability ($W[\xi] < 0$). However, when determining the characteristic growth time of the instability and the optimal unstable Lagrangian displacement, one should, strictly speaking, minimize the ratio $W[\xi]/I[\xi]$ [see Eq. (22)]. The results of this more accurate procedure are presented in Appendix A, where it is shown that for weak magnetic fields ($\kappa \ll 1$) the result of minimizing $W[\xi]/I[\xi]$ does not differ noticeably from the expressions (36) and (37) used below.

IV. FURTHER TRANSFORMING THE ENERGY FUNCTIONAL: DEVELOPMENT OF THE PERTURBATION THEORY

A. Expansion of $W[\xi]$ in powers of ε and κ

Now, knowing how ξ depends on the functions $\mathbf{g}(r, \theta)$ and $\mathbf{a}(r, \theta)$, we can express the functional (19) in terms of these functions. Expressing the equilibrium pressure gradient ∇p in (19) using the relation $\nabla p = F_L - \rho \nabla \Phi$ [see Eq. (8)], and neglecting the perturbation of the gravitational potential $\delta \Phi$ as discussed in Sec. III A, we obtain an expression for $W[\xi]$. After integration over the angle φ , this expression will depend on the magnetic field $B(r, \theta)$ and its first derivatives with respect to r and θ , on the functions $\mathbf{g}(r, \theta)$ and $\mathbf{a}(r, \theta)$ and their first derivatives, on the equilibrium density $\rho(r, \theta)$ and its first derivatives, on the equilibrium gradient of the gravitational potential $\Phi(r, \theta)$, and on the equilibrium functions $\gamma_{\text{fr}}(r, \theta)$ and $p(r, \theta)$.

We will not write out this rather cumbersome expression explicitly here, as its analysis would not provide us with much useful information. Instead, recall that we have not yet used the assumption of Sec. III B about the smallness of the magnetic field. This assumption allows us to expand all equilibrium functions p , ρ , Φ , and γ_{fr} in $W[\xi]$ in a series over the small parameter κ , representing them in the form (27). Introducing, instead of the small magnetic field $\mathbf{B}(r, \theta)$, the function $\mathcal{B}(r, \theta)$ according to Eq. (29), the functional $W[\xi]$ can be schematically written as

$$\begin{aligned} W[\xi] = & \varepsilon^0 (\underline{\kappa \alpha_{01}} + \underline{\kappa^2 \alpha_{02}} + \kappa^3 \alpha_{03} + \dots) \\ & + \varepsilon (\underline{\kappa \alpha_{11}} + \kappa^2 \alpha_{12} + \kappa^3 \alpha_{13} + \dots) \\ & + \varepsilon^2 (\underline{\alpha_{20}} + \underline{\kappa \alpha_{21}} + \kappa^2 \alpha_{22} + \dots), \end{aligned} \quad (44)$$

where the coefficients α_{ij} with $i = 0, 1, 2$ and $j = 0, 1, 2, \dots$ are ordinary numbers for the given specific stellar model, toroidal field, and the functions $\mathbf{g}(r, \theta)$ and $\mathbf{a}(r, \theta)$.⁶ If these quantities are not specified explicitly, the α_{ij} represent integrals over $dr d\theta$ of certain functions, which will be discussed below. Note that always $\alpha_{00} = \alpha_{10} = 0$.

We will be primarily interested in the largest terms (underlined once and twice) in the expansion (44).⁷ These terms, in particular, depend on the functions $\partial_\theta \rho_1(r, \theta)$, $\Phi'_0(r)$, and $\rho'_0(r)$ (here the prime ' denotes differentiation with respect to r). Expressing these functions using, respectively, the relations (30), $\Phi'_0(r) = -p'_0(r)/\rho_0$, and $\rho'_0(r) = \rho_0 p'_0(r)/(p_0 \gamma_{\text{eq},0})$, we arrive at the final

⁶Note that the expansion (44) does not contain terms $\sim \varepsilon^3$ and higher powers, since the functional $W[\xi]$ is quadratic in ξ .

⁷A single underline denotes the first order of perturbation theory. Terms doubly underlined are combined by us into the second order of perturbation theory. The term underlined with a wavy line is discussed in Sec. IV D.

expressions for the coefficients α_{01} , α_{02} , α_{11} , and α_{20} , which we will analyze below.

B. First order of perturbation theory

From Eq. (44) it follows that the dominant contribution to $W[\xi]$ is provided by the term $\kappa \alpha_{01}$, which originates from the magnetic contribution $W^{\text{mag}}[\xi]$ to the energy functional (19). The coefficient α_{01} is given by⁸

$$\alpha_{01} = 4\pi \int r^2 \sin \theta [C_1 \mathbf{a}^2 + C_2 \partial_\theta (\mathbf{a}^2) + C_3 (\partial_\theta \mathbf{a})^2] dr d\theta, \quad (45)$$

where

$$C_1 = \frac{1}{4} m^2 \left[\mathcal{B}^2 \left(1 + \frac{m^2 - 1}{\sin^2 \theta} \right) - \cot \theta \partial_\theta (\mathcal{B}^2) \right], \quad (46)$$

$$C_2 = -\frac{1}{4} m^2 \mathcal{B}^2 \cot \theta, \quad (47)$$

$$C_3 = \frac{1}{4} m^2 \mathcal{B}^2. \quad (48)$$

Here and below, the integration over the angle θ is performed from 0 to π , and over r from 0 to the stellar radius R , i.e., we integrate over the unperturbed volume of a spherically symmetric star without a magnetic field. Taking into account the nonsphericity of the star's equilibrium configuration caused by the magnetic field does not contribute to integrals of the type (45), due to our assumptions that both $B(r, \theta)$ and the matter density vanish at the stellar surface (see Sec. III B).

Integrating by parts the second term in the expression for C_1 depending on $\partial_\theta (\mathcal{B}^2)$, one arrives at the following expression:

$$\alpha_{01} = 4\pi \int r^2 \sin \theta \frac{m^2 \mathcal{B}^2}{4} \left[(\partial_\theta \mathbf{a})^2 + \frac{(m^2 - 1)}{\sin^2 \theta} \mathbf{a}^2 \right] dr d\theta. \quad (49)$$

To obtain this formula we make use of the boundary conditions discussed in Sec. III B. It is easy to see from Eq. (49) that to maximally destabilize the system (i.e., to make $W[\xi] \simeq \kappa \alpha_{01}$ as small as possible) one needs to choose

⁸Treating $\kappa \alpha_{01}$ as the leading term implicitly assumes the ordering $\kappa \gg \varepsilon^2$, which allows us to regard $\varepsilon^2 \alpha_{20}$ as a higher-order term. Under the opposite assumption, $\varepsilon^2 \gg \kappa$, the leading-order term of $W[\xi]$ would coincide with that of a nonmagnetized fluid, which is assumed to be stable. It can be shown that constructing the perturbation theory under the assumption $\varepsilon^2 \gg \kappa$ leads to $\mathbf{g}(r, \theta) = 0$ (no displacement in the r direction) and precludes $W[\xi]$ from being negative [see Eqs. (57) and (60) below].

the lowest possible value of m , in our case $m = 1$.⁹ In the remainder of the paper, we assume $m = 1$ and omit the superscript (m). Finally, one obtains

$$\alpha_{01} = \pi \int r^2 \sin \theta \mathcal{B}^2 (\partial_\theta \mathbf{a})^2 dr d\theta. \quad (50)$$

From Eq. (50), it follows that, to first order in perturbation theory, the lowest possible value of $W[\xi]$ for the $m = 1$ mode is zero. This value is attained when $\partial_\theta \mathbf{a} = 0$, i.e.

$$\mathbf{a}(r, \theta) = a(r) \text{ in the first order of perturbation theory,} \quad (51)$$

where $a(r)$ is some function of r . Regarding Eq. (51), we emphasize the following: (i) strictly speaking, it needs to hold only in the region where the magnetic field is localized; and (ii) Eq. (51) needs to be satisfied only to leading order in the expansion in κ and ε . At higher orders $\partial_\theta \mathbf{a} \neq 0$ and the function $\mathbf{a}$ may depend on θ . One can check that allowing for the θ dependence of $\mathbf{a}$ and substituting the ansatz (39) into (19) introduces only two terms in the energy functional $W[\xi]$ that depend on $\partial_\theta \mathbf{a}(r, \theta)$, namely

$$\pi \kappa \int r^2 \sin \theta \mathcal{B}^2 [(\partial_\theta \mathbf{a})^2 - 2\varepsilon \partial_\theta \mathbf{a} \partial_r \mathbf{g}] dr d\theta. \quad (52)$$

Here, the first term is non-negative, while the second one, which originates from α_{11} , is not sign definite. Thus, the contribution of $\partial_\theta \mathbf{a}$ to $W[\xi]$ can be profitable only if

$$|\partial_\theta \mathbf{a}| \lesssim |\varepsilon \partial_r \mathbf{g}| \quad (53)$$

with $\partial_\theta \mathbf{a}$ and $\varepsilon \partial_r \mathbf{g}$ having the same sign. In this case, the terms in the energy functional $W[\xi]$ that contain $\partial_\theta \mathbf{a}(r, \theta)$ are of order $\mathcal{O}(\kappa \varepsilon^2)$; that is, they contribute to the (small) third-order term underlined with a wavy line in Eq. (44).

What does the optimal Lagrangian displacement with $\mathbf{a}(\mathbf{r}, \theta)$ given by Eq. (51) correspond to in the first-order perturbation theory? First, from Eq. (40) it follows that $\tilde{\xi}_r = 0$ (taking into account the term depending on ε in this formula would lead to terms in $W[\xi]$ that depend on ε , which we neglect at this order of perturbation theory). Similarly, from Eq. (41) we have $\tilde{\xi}_\theta = -r\mathbf{a}(r, \theta) = -r[a(r) + \mathcal{O}(\kappa \varepsilon^2)] \approx -ra(r)$. Furthermore, from Eq. (42) it follows that $\tilde{\xi}_\varphi \approx -r\partial_\theta [\mathbf{a}(r, \theta) \sin \theta] = -r\partial_\theta \{[a(r) + \mathcal{O}(\kappa \varepsilon^2)] \sin \theta\} \approx -ra(r) \cos \theta$ [to show the first equality here, it is necessary to note that the function $\mathcal{D}$, defined by Eq. (43), is of order $\mathcal{O}(\varepsilon + \kappa)$, i.e., it can be neglected in the first approximation]. Thus, as follows from Eq. (39), the Lagrangian displacement in this order of perturbation

theory is $\xi \equiv \xi_{\text{rot}} = [0, -ra(r) \sin \varphi, -ra(r) \cos \theta \cos \varphi]$. As is easy to see, this Lagrangian displacement corresponds to a differential (in r) *shellular* rotation about the Cartesian x axis, given by the formula

$$\xi_{\text{rot}}(r, \theta, \varphi) = a(r) \hat{\mathbf{x}} \times \mathbf{r}, \quad (54)$$

where $\hat{\mathbf{x}} = (\sin \theta \cos \varphi, \cos \theta \cos \varphi, -\sin \varphi)$ and $\mathbf{r} = r\mathbf{e}_r$. Solid-body rotation corresponds to the case $a(r) = \text{const}$.

One can verify that if we looked for the Lagrangian displacement $\xi(r, \theta, \varphi)$ in the form of Eq. (39), but with $\sin \varphi$ replaced with $\cos \varphi$ and vice versa (as argued in Sec. III E, this is an alternative description), and repeated all the derivations in this section we would find that the corresponding optimal Lagrangian displacement ξ_{rot} in the first approximation is a shellular differential rotation about the y axis. This means that a shellular differential rotation with respect to an *arbitrary* axis $\hat{\mathbf{O}}$ in the $x - y$ plane,

$$\xi_{\text{rot}}(r, \theta) = a(r) \hat{\mathbf{O}} \times \mathbf{r}, \quad (55)$$

is optimal in the first order of the perturbation theory in κ and ε .¹⁰ Note in this context that differential rotation about the symmetry axis z is trivial [57] since it does not perturb the system and, in particular, does not change the energy $W[\xi]$. Thus, in principle, we could also add such a displacement to the displacement of the form (55). However, any trivial displacement increases the kinetic-energy functional (23), thereby increasing the characteristic growth time of the potential instability; see Eq. (25). In other words, trivial displacements are always unfavorable,

¹⁰In fact, the Lagrangian displacement of the form (55) does not change the magnetic part (21) of the energy functional (19), i.e., $W^{\text{mag}}[\xi_{\text{rot}}] = 0$, and this result is exact (it does not depend on whether the toroidal magnetic field is small or not). To demonstrate this more directly, we integrate Eq. (21) by parts using the identity $\nabla \cdot (\mathbf{A}_1 \times \mathbf{A}_2) = (\nabla \times \mathbf{A}_1) \cdot \mathbf{A}_2 - \mathbf{A}_1 \cdot (\nabla \times \mathbf{A}_2)$ to obtain

$$W^{\text{mag}}[\xi] = \frac{1}{2} \int dV (\xi \times \mathbf{B}) \cdot \nabla \times \left(\frac{\delta \mathbf{B}}{4\pi} + \frac{1}{c} \mathbf{j} \times \xi \right). \quad (56)$$

One may verify that, upon setting $\xi = \xi_{\text{rot}}$, the vector $(\xi \times \mathbf{B})$ in (56) is directed along r , while the r component of the underlined vector vanishes. Hence, the integrand in (56) vanishes at each point inside the star, implying that $W^{\text{mag}}[\xi_{\text{rot}}] = 0$.

To explain this result qualitatively, consider an arbitrary spherical shell. Since the background field is purely toroidal, the magnetic field lines are everywhere tangent to the shell and do not cross it. Now suppose that this shell is rotated as a rigid body about an axis perpendicular to the symmetry axis of the magnetic field. Because the field is frozen into the fluid, each fluid element carries its magnetic field with it. Under such a rigid rotation, the field associated with a given fluid element is merely reoriented, but its magnitude is unchanged. In other words, such displacement changes only the direction of the magnetic field, not its absolute value. Since this argument applies independently to every spherical shell, the magnetic energy is unchanged by the displacement $\xi = \xi_{\text{rot}}$.

⁹It might be shown that the case $m = 0$ is also correctly described by this expression, but we do not consider it in this work because of the reasons stated in Sec. III E.

and the optimal rotation should occur about an axis in the $x - y$ plane. Note also that the displacements (54) and (55) do not apply to the nonmagnetized region of the star, which need not sustain shellular differential rotation and may instead respond differently to the rotation of the magnetized regions. The optimal displacement for the nonmagnetized stellar region cannot be determined from the first few orders of perturbation theory considered in this paper. In what follows, whenever needed in specific examples, we will assume that the magnetic field permeates the entire star or a substantial part of it, so that the approximation $\xi \approx \xi_{\text{rot}}$ remains valid throughout the star to leading order.

Summarizing this section, we have found that for a toroidal field, the most unstable mode is the one with azimuthal number $m = 1$. The optimal Lagrangian displacement for this mode in the first order of the perturbation theory corresponds to a shellular differential rotation of the magnetized regions of the star about an arbitrary axis in the $x - y$ plane, for example, without loss of generality, about the x axis [see Eq. (54)]. Moreover, this Lagrangian displacement is “neutral” in the sense that the change it induces in the energy functional $W[\xi_{\text{rot}}]$ in the first-order perturbation theory is zero, $W[\xi_{\text{rot}}] \simeq \kappa \alpha_{01} = 0$.

C. Second-order perturbation theory

Let us now consider the second-order perturbation theory, which includes terms proportional to κ^2 , $\epsilon \kappa$, and ϵ^2 in the expansion (44) of the energy functional $W[\xi]$. Although, according to the initial assumption (and as we will verify below), the quantities κ and ϵ may differ significantly from each other, we group these terms together, keeping in mind that all higher-order contributions in the expansion (44) (for example, terms $\propto \kappa^3$, $\epsilon \kappa^2$, and $\epsilon^2 \kappa$) are much smaller than at least *some* of the terms we retain. We will return to the discussion of this assumption in Sec. IV D.

The coefficients α_{ij} for the second-order terms in the expansion (44) are given by

$$\alpha_{20} = \frac{1}{2} \int r^2 \sin \theta W_1 \mathbf{g}(r, \theta)^2 dr d\theta, \quad (57)$$

$$\begin{aligned} \alpha_{11} = & \frac{1}{2} \int r^2 \sin \theta W_2 \mathbf{g}(r, \theta) dr d\theta \\ & + \frac{1}{2} \int \partial_r [4\pi r^2 \cos \theta \mathbf{a}(r, \theta) \mathcal{B}^2 \mathbf{g}(r, \theta)] dr d\theta, \end{aligned} \quad (58)$$

$$\alpha_{02} = \frac{1}{2} \int r^2 \sin \theta W_3 dr d\theta, \quad (59)$$

where the functions W_1 , W_2 , and W_3 are defined as

$$W_1 = \frac{\pi(\gamma_{\text{fr},0} - \gamma_{\text{eq},0})[p'_0(r)]^2}{p_0 \gamma_{\text{fr},0} \gamma_{\text{eq},0}}, \quad (60)$$

$$\begin{aligned} W_2 = & \frac{2\pi}{p_0 \gamma_{\text{fr},0} \gamma_{\text{eq},0}} [-2\mathcal{B}^2 \cot \theta p_0 \gamma_{\text{fr},0} \gamma_{\text{eq},0} \partial_r \mathbf{a}(r, \theta) \\ & + p'_0(r) \partial_\theta \Phi_1 (\gamma_{\text{fr},0} - \gamma_{\text{eq},0}) p_0 \mathbf{a}(r, \theta)], \end{aligned} \quad (61)$$

$$\begin{aligned} W_3 = & \frac{\pi \partial_\theta \Phi_1 p_0 \mathbf{a}(r, \theta)^2}{r p'_0(r) p_0 \gamma_{\text{fr},0} \gamma_{\text{eq},0}} [r p'_0(r) \partial_\theta \Phi_1 (\gamma_{\text{fr},0} - \gamma_{\text{eq},0}) p_0 \\ & + 4\mathcal{B} p_0 \gamma_{\text{fr},0} \gamma_{\text{eq},0} (-\partial_\theta \mathcal{B} + r \cot \theta \partial_r \mathcal{B})]. \end{aligned} \quad (62)$$

In these equations, it is legitimate to replace $\mathbf{a}(r, \theta)$ with the function $a(r)$, since $\mathbf{a}(r, \theta) - a(r) \sim \mathcal{O}(\kappa \epsilon^2)$. We nevertheless prefer not to do so, thereby effectively retaining some higher-order terms within the second-order treatment considered here. Note that the underlined term in Eq. (58) is a total derivative and vanishes upon integration over r due to the boundary conditions (see Sec. III B). The corresponding functional $W[\xi]$, taken up to (and including) second-order perturbation theory, is given by [see Eq. (44)]

$$\begin{aligned} W[\xi] \simeq & \kappa^2 \alpha_{02} + \epsilon \kappa \alpha_{11} + \epsilon^2 \alpha_{20} \\ = & \frac{1}{2} \int r^2 \sin \theta [\epsilon^2 W_1 \mathbf{g}(r, \theta)^2 + \epsilon \kappa W_2 \mathbf{g}(r, \theta) + \kappa^2 W_3] \\ & \times dr d\theta. \end{aligned} \quad (63)$$

As can be seen, this functional depends, among other contributions, on terms that are quadratic in the radial function $\mathbf{g}(r, \theta)$ and does not depend on its derivatives. Furthermore, note that the coefficient W_1 given by Eq. (60) is always positive for stably stratified matter, in which $\gamma_{\text{fr},0} - \gamma_{\text{eq},0} > 0$ (see Sec. III C). This means that there exists a function $\mathbf{g}(r, \theta)$ which *minimizes* $W[\xi]$. Performing the minimization, one obtains

$$\mathbf{g}(r, \theta) = -\frac{\kappa}{\epsilon} \frac{W_2}{2W_1}. \quad (64)$$

Given the asymptotic behavior of the magnetic field at the poles and the center, it is straightforward to verify that $\mathbf{g}(r, \theta)$ is always finite within the star.

At this point, let us make a comment on barotropic matter, i.e., when $\gamma_{\text{fr},0} = \gamma_{\text{eq},0}$. In this case, one cannot minimize the functional (63) and arrive at Eq. (64), because $W_1 = 0$ and the functional (63) is linear in $\mathbf{g}(r, \theta)$. Still, one can always make the functional (63) negative by choosing an appropriate sign and amplitude of $\mathbf{g}(r, \theta)$. Thus, a Lagrangian displacement of the type considered in this section (a shellular differential rotation plus a small radial component) destabilizes a barotropic star as well. Nevertheless, numerical simulations (e.g., [56]) indicate that, in barotropic stars, this type of instability is not the dominant one. Instead, the system typically develops an instability in which the radial displacement ξ_r is comparable to the θ and φ components, ξ_θ and ξ_φ . This is not surprising, because the results of Sec. III C imply that, in

barotropic matter, the displacement ξ_r is not suppressed. We will mostly focus on stratified (nonbarotropic) matter in what follows.

Returning to nonbarotropic matter, note that the function W_2 in Eq. (64) depends on the function $\mathbf{a}(r, \theta)$ and its derivative $\partial_r \mathbf{a}(r, \theta)$ [see Eq. (61)]. Accordingly, using (60) and (61), the right-hand side of (64) can be estimated as

$$\mathbf{g}(r, \theta) \sim \frac{\kappa}{\varepsilon} R \left[\mathbf{a}(r, \theta) + \frac{\gamma_{\text{fr},0} \gamma_{\text{eq},0}}{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}} \partial_r \mathbf{a}(r, \theta) R \right], \quad (65)$$

where R is the characteristic length scale over which the pressure $p_0(r)$ varies (we identify it with the stellar radius). If we introduce the characteristic scale $1/k \lesssim R$ for the variation of the function $\mathbf{a}(r, \theta)$ in the radial direction, we arrive at the estimate

$$\mathbf{g}(r, \theta) \sim \frac{\kappa n}{\varepsilon} \frac{\gamma_{\text{fr},0} \gamma_{\text{eq},0}}{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}} R \mathbf{a}(r, \theta), \quad (66)$$

where n is a (possibly large) numerical coefficient of order kR , i.e. $n \sim kR$.

Let us now recall that, according to the initial assumption, the functions $\mathbf{g}(r, \theta)$ and $\mathbf{r}\mathbf{a}(r, \theta)$ should be of the same order of magnitude [see Eqs. (40) and (41) and the text following them]. This means that $\mathbf{g}(r, \theta)/\mathbf{a}(r, \theta) \sim \mathcal{O}(r)$, and hence ε must be

$$\varepsilon \sim \frac{n\kappa\gamma_{\text{fr},0}\gamma_{\text{eq},0}}{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}} \sim \frac{n\kappa}{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}}, \quad (67)$$

where we set $\gamma_{\text{fr},0}\gamma_{\text{eq},0} \sim 1$ in the second estimate. We see that ε will be of the order of κ only if the characteristic radial scale of variation of the function $\mathbf{a}(r, \theta)$ matches R , that is, is of the order of the stellar radius, and, additionally, the difference $\gamma_{\text{fr},0} - \gamma_{\text{eq},0}$ is not too small. Otherwise, ε can be much larger (but must remain much less than unity, $\varepsilon \ll 1$, by assumption). A precise upper bound estimate for ε [and, consequently, for n ; see Eq. (67)] will be obtained in Sec. IV D.

The parameter ε in Eq. (67) is expressed in terms of the quantities κ and $\gamma_{\text{fr},0} - \gamma_{\text{eq},0}$. Alternatively, one can express ε in terms of the typical Alfvén frequency $\omega_A \sim B/[(4\pi\rho_0)^{1/2}R]$ and the Brunt-Väisälä frequency $\mathcal{N}$ defined in Eq. (35):

$$\varepsilon \sim n \frac{\omega_A^2}{\mathcal{N}^2}. \quad (68)$$

Plugging Eq. (64) into (63), one gets

$$W[\xi] \simeq \frac{\kappa^2}{2} \int r^2 \sin \theta \left(-\frac{W_2^2}{4W_1} + W_3 \right) dr d\theta. \quad (69)$$

Substituting the expressions (60)–(62) for W_1 , W_2 , and W_3 , it is easy to see that the integrand in this expression is

nonzero only in the region where the magnetic field is localized, in which $\mathbf{a}(r, \theta) = a(r) + \mathcal{O}(\kappa\varepsilon^2) \approx a(r)$; see Eq. (51).

It is evident that the expression (69) for $W[\xi]$ consists of two terms, the first of which is always nonpositive, while the second can have an arbitrary sign. The entire expression is formally proportional to the small quantity κ^2 . However, such smallness of the functional will occur only if the function $\mathbf{a}(r, \theta)$ does not vary significantly on the scale of the stellar radius, i.e., $R\partial_r \mathbf{a}(r, \theta) \sim \mathbf{a}(r, \theta)$ (and, consequently, $n \sim 1$). In this case, both the first and second terms inside the parentheses of the integrand in Eq. (69) will be on the order of $p_0 r^2 \mathbf{a}(r, \theta)^2 / R^2$. Of course, it is not excluded that even in this case, for some models of the toroidal field, one can choose a function $\mathbf{a}(r, \theta)$ such that $W[\xi]$ becomes negative (and thus prove the instability of the field). However, such a result will not constitute a general proof of the instability of *any* toroidal field.¹¹

To prove instability in the most general case, let us assume that the function $\mathbf{a}(r, \theta)$ varies on small spatial scales, that is, $n = kR \gg 1$. In this case, the first term in (69), which depends on W_2 , becomes much larger than the second (underlined) term. Indeed, as we have already established above, W_2 contains a term proportional to $\partial_r \mathbf{a}(r, \theta) \sim n \mathbf{a}(r, \theta) / R$ ¹²; therefore, the first term in (69) is larger than the underlined term by a factor of n^2 , and the latter can be neglected. As a result, we obtain an expression for $W[\xi]$ that is manifestly negative,

$$W[\xi] \simeq -\frac{\kappa^2}{8} \int r^2 \sin \theta \frac{W_2^2}{W_1} dr d\theta, \quad (70)$$

guaranteeing the instability of any toroidal field. Note that the true order of smallness of the functional $W[\xi]$ will be $W[\xi] \sim \kappa^2 n^2 / (\gamma_{\text{fr},0} - \gamma_{\text{eq},0}) \sim (\gamma_{\text{fr},0} - \gamma_{\text{eq},0}) \varepsilon^2$, rather than κ^2 .

D. Estimate of the permissible magnitude of n

In principle, by choosing n arbitrarily large, one can make the functional (70) or (69) arbitrarily large in magnitude and negative. However, let us recall that n and ε are related by the estimate (67), and ε , by assumption, must satisfy $\varepsilon \ll 1$. From this inequality, we immediately find $n \ll (\gamma_{\text{fr},0} - \gamma_{\text{eq},0}) / \kappa$. In fact, there is a stricter bound on n arising from the requirement that the terms of the next (third) order in the expansion (44) of the functional $W[\xi]$ do not affect the results obtained above in the second-order perturbation theory.

¹¹Note that, in the commonly used approximation $\Phi_1 = 0$, the function W_3 vanishes [see Eq. (62)], and the instability sets in already for $n \sim 1$ (in fact, it sets in for any shellular differential rotation; see Appendix D for details).

¹²For example, if one chooses $\mathbf{a}(r, \theta)$ in the form $\mathbf{a}(r, \theta) = \sin(rn/R)$, then $R\partial_r \mathbf{a}(r, \theta) = n \cos(rn/R)$.

Before proceeding to the analysis of the third-order terms, let us briefly return to the second-order terms and analyze them in light of the results obtained in Sec. IV C. As we found in Sec. IV C, $\varepsilon \sim n\kappa/(\gamma_{\text{fr},0} - \gamma_{\text{eq},0})$, so the second-order terms $\kappa^2\alpha_{02}$, $\kappa\varepsilon\alpha_{11}$, and $\varepsilon^2\alpha_{20}$ actually have different orders of smallness in the limit of interest $n \gg 1$. It is also important to note that the coefficient α_{11} depends on the large derivative $\partial_r \mathbf{a}(r, \theta) \sim n\mathbf{a}(r, \theta)/R$ [this dependence arises through the function W_2 ; see Eqs. (58) and (61)]. This means that the term $\kappa\varepsilon\alpha_{11}$ actually has the order of smallness $\kappa\varepsilon\alpha_{11} \sim \kappa\varepsilon n \sim \varepsilon^2(\gamma_{\text{fr},0} - \gamma_{\text{eq},0})$, the same as the term $\varepsilon^2\alpha_{20} \sim \varepsilon^2(\gamma_{\text{fr},0} - \gamma_{\text{eq},0})$, while the term $\kappa^2\alpha_{02} \sim \kappa^2$ can be neglected [it corresponds to the small and subsequently discarded underlined term in Eq. (69)].¹³

This example shows that the terms formally grouped into the same order of smallness may, in fact, differ significantly in magnitude from each other. Another important observation, relevant also to the analysis of higher-order terms, is that one must carefully track the radial derivatives $\partial_r \mathbf{a}(r, \theta)$ and $\partial_r \mathbf{g}(r, \theta)$ in the integrands for the expansion coefficients α_{ij} , since these derivatives generate an additional large multiplicative factor n .¹⁴

Let us now apply the considerations discussed above to the analysis of the third-order contributions in (44), which have the form $\kappa^3\alpha_{03}$, $\varepsilon\kappa^2\alpha_{12}$, and $\varepsilon^2\kappa\alpha_{21}$. Formally, the largest among them is the last term $\propto \varepsilon^2\kappa$; it is underlined with a wavy line in Eq. (44). The other contributions could become comparable to it only if they depended on derivatives like $\partial_r \mathbf{a}(r, \theta) \sim n\mathbf{a}(r, \theta)/R$ or $\partial_r \mathbf{g}(r, \theta) \sim n\mathbf{g}(r, \theta)/R$. It turns out, however, that the only terms of this kind are indeed contained in $\varepsilon^2\kappa\alpha_{21}$. Thus, the dominant third-order contribution to $W[\xi]$ is

3rd order contribution into $W[\xi]$

$$\approx \pi\kappa \int r^2 \sin \theta \mathcal{B}^2 [\partial_\theta \mathbf{a}(r, \theta) - \varepsilon \partial_r \mathbf{g}(r, \theta)]^2 dr d\theta. \quad (71)$$

Note that this contribution is always non-negative, i.e., it stabilizes the instability. Using Eqs. (66), (67), and (53), one can estimate $[\partial_\theta \mathbf{a}(r, \theta) - \varepsilon \partial_r \mathbf{g}(r, \theta)]^2 \sim n^2 \varepsilon^2 \mathbf{a}(r, \theta)^2$. Accordingly, the expression (71) is of the order $\sim \kappa \varepsilon^2 n^2 \int \mathcal{B}^2 \mathbf{a}(r, \theta)^2 dV$. In order for the arguments of Sec. IV C to remain valid, this third-order contribution must be much smaller than the second-order one. The latter can be estimated as $\kappa\varepsilon\alpha_{11} \sim \kappa\varepsilon n \int \mathcal{B}^2 \mathbf{a}(r, \theta)^2 dV$ (see footnote 13). Combining these two estimates, we obtain

¹³Note that more accurately $\kappa\varepsilon\alpha_{11}$ can be estimated as $\kappa\varepsilon\alpha_{11} \sim \kappa\varepsilon n \int \mathcal{B}^2 \mathbf{a}(r, \theta)^2 dV$.

¹⁴The fact that the derivative $\partial_r \mathbf{g}(r, \theta)$ leads to the appearance of an additional factor n , i.e., $\partial_r \mathbf{g}(r, \theta) \sim n\mathbf{g}(r, \theta)$, follows from the form of the solution (64), which depends on $\mathbf{a}(r, \theta)$, or from the estimate (65).

$$n \ll \frac{1}{\varepsilon} \Rightarrow n \ll \frac{\sqrt{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}}}{\kappa^{1/2}}, \text{ or, equivalently, } n \ll \frac{\mathcal{N}}{\omega_A}, \quad (72)$$

where we made use of Eqs. (67) and (68). As can be seen, this condition turns out to be much more stringent than the naive criterion obtained at the beginning of this section. Note also that such a restriction on n ensures that the second-order terms remain much smaller than the first-order terms. Indeed, $\varepsilon^2(\gamma_{\text{fr},0} - \gamma_{\text{eq},0}) \sim n^2 \kappa^2 / (\gamma_{\text{fr},0} - \gamma_{\text{eq},0}) \ll \kappa$.

E. A more accurate estimate of the optimal n

In the previous section, we found that in order to guarantee $W[\xi] < 0$, and thus demonstrate the instability of the toroidal field, it is necessary to take $n \gg 1$, which, at the same time, must not be too large: $n \ll \sqrt{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}} / \kappa^{1/2} \sim \mathcal{N} / \omega_A$. In reality, however, one should expect that n will grow up to values of the order $\sim \sqrt{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}} / \kappa^{1/2} \sim \mathcal{N} / \omega_A$, after which the contribution from the third-order terms (71) will begin to stabilize the system. Therefore, to obtain a more accurate estimate for n , it is necessary to minimize the functional (63) with respect to both $\mathbf{g}(r, \theta)$ and $\mathbf{a}(r, \theta)$, taking into account the third-order term (71)¹⁵:

$$W[\xi] \simeq \frac{1}{2} \int r^2 \sin \theta [\varepsilon^2 W_1 \mathbf{g}(r, \theta)^2 + \varepsilon \kappa W_2 \mathbf{g}(r, \theta) + \kappa^2 W_3 + \kappa 2\pi \mathcal{B}^2 [\partial_\theta \mathbf{a}(r, \theta) - \varepsilon \partial_r \mathbf{g}(r, \theta)]^2] dr d\theta. \quad (73)$$

The minimization problem for this functional can be simplified by treating the last term in (73) as small and replacing the functions $\mathbf{a}(r, \theta)$ and $\mathbf{g}(r, \theta)$ with their lower-order perturbative approximations: $\mathbf{a}(r, \theta) = \mathbf{a}(r)$ and $\mathbf{g}(r, \theta) = -(\kappa/\varepsilon) \cdot W_2/(2W_1)$; see Eqs. (51) and (64).¹⁶ Although this procedure is not entirely rigorous (the last term may become large for sufficiently large n), it significantly simplifies the problem of finding the optimal n . Substituting $\mathbf{g}(r, \theta)$, W_1 , W_2 , and W_3 into (73) using formulas (64), (60)–(62), we obtain¹⁷

¹⁵Note that among higher-order terms, there are none that could compete with the terms included in expression (73).

¹⁶One can check that substituting $\mathbf{g}(r, \theta) = -(\kappa/\varepsilon) \cdot W_2/(2W_1)$ into the functional (73) yields an integrand that vanishes outside the magnetized regions, so that only magnetized regions contribute. In these regions, $\mathbf{a}(r, \theta) \approx \mathbf{a}(r)$ holds to leading order; see Eq. (51), and the sentence following Eq. (69).

¹⁷It is important to emphasize that the procedure described here is not a mathematically rigorous minimization of the functional (73) (since the last term in this functional is treated perturbatively). However, if we prove that instability exists for this choice of $\mathbf{a}(r, \theta)$ and $\mathbf{g}(r, \theta)$, then by virtue of the spectral variational principle from Sec. III A, it must also exist for the true $\mathbf{a}(r, \theta)$ and $\mathbf{g}(r, \theta)$, and moreover, the growth rate of this instability must be greater than or equal to what can be found from the approximate analysis.

$$W[\xi] \simeq \frac{\kappa^2}{2} \int_0^R \{ -[a'(r)]^2 |F_1| + [a(r)]^2 F_2 - a(r)a'(r)F_3 + \kappa[a''(r)]^2 |F_4| \} dr, \quad (74)$$

where

$$F_1(r) = \int_0^\pi \frac{4\pi r^2 \mathcal{B}^4 \cos^2 \theta p_0 \gamma_{\text{fr},0} \gamma_{\text{eq},0}}{\sin \theta [p'_0(r)]^2 (\gamma_{\text{fr},0} - \gamma_{\text{eq},0})} d\theta, \quad (75)$$

$$F_2(r) = \int_0^\pi \frac{4\pi r \mathcal{B} \partial_\theta \Phi_1 \rho_0 (-\sin \theta \partial_\theta \mathcal{B} + r \cos \theta \partial_r \mathcal{B})}{p'_0(r)} d\theta, \quad (76)$$

$$F_3(r) = - \int_0^\pi \frac{4\pi r^2 \mathcal{B}^2 \cos \theta \partial_\theta \Phi_1 \rho_0}{p'_0(r)} d\theta, \quad (77)$$

$$F_4(r) = \int_0^\pi \frac{8\pi r^2 \mathcal{B}^6 \cos^2 \theta p_0^2 \gamma_{\text{fr},0}^2 \gamma_{\text{eq},0}^2}{\sin \theta (\gamma_{\text{fr},0} - \gamma_{\text{eq},0})^2 [p'_0(r)]^4} d\theta, \quad (78)$$

and R is the radius of the unperturbed star in the absence of the magnetic field. Note that the functions $F_1(r)$ and $F_4(r)$ never become negative. When deriving the expression for $F_4(r)$ in Eq. (73), we retained only the largest term depending on $[a''(r)]^2$ and neglected all other terms. Due to the conditions on the magnetic field (see Sec. III B), all functions $F_1(r), \dots, F_4(r)$ are bounded and behave well inside the star.

If the function $a(r)$ oscillates strongly, then in the functional (74) the dominant terms are the negative first term and the positive last term, proportional to $\kappa^2 n^2$ and $\kappa^3 n^4$, respectively. In this case, the second and third terms in (74) can be neglected. If, as a trial oscillating function $a(r)$, one chooses

$$a(r) = \cos\left(\frac{nr}{R}\right), \quad (79)$$

then one obtains from Eq. (74)

$$W[\xi] \approx \frac{\kappa^2}{4} \left[-\left(\frac{n}{R}\right)^2 \int_0^R |F_1| dr + \kappa \left(\frac{n}{R}\right)^4 \int_0^R |F_4| dr \right], \quad (80)$$

where we used that $\int_0^R \cos^2(\frac{nr}{R}) |F_i| dr \approx \int_0^R \sin^2(\frac{nr}{R}) |F_i| dr \approx \frac{1}{2} \int_0^R |F_i| dr$ (for $i = 1, 4$). Minimizing (80) with respect to n , we find that the optimal value of n within this approximate treatment is

$$n = \frac{R}{\sqrt{2\kappa}} \sqrt{\frac{\int_0^R |F_1| dr}{\int_0^R |F_4| dr}} \sim \frac{\sqrt{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}}}{\kappa^{1/2}} \sim \frac{\mathcal{N}}{\omega_A}, \quad (81)$$

in complete agreement with our expectations.¹⁸ This choice of n corresponds to a negative value of $W[\xi]$:

¹⁸Note that the optimal value of n is smaller than the maximum value for which the instability can persist [$W[\xi] < 0$, see Eq. (80)] by only a factor of $\sqrt{2}$.

$$W[\xi] \approx -\frac{\kappa}{16} \frac{[\int_0^R |F_1| dr]^2}{\int_0^R |F_4| dr} \sim -\frac{\pi\kappa}{8} \mathcal{B}^2 R^3 = -\frac{B^2 R^3}{64}. \quad (82)$$

That is, we indeed have instability with such a choice of the trial function $a(r)$. Obviously, the exact minimization of the functional (74) (with fixed normalization $I[\xi]$) may yield a more optimal (but not universal) function $a(r)$, which depends on the specific form of the toroidal magnetic field (see Sec. VI).

Thus, we have found that in ideal MHD the most unstable mode corresponds to $n \sim \mathcal{N}/\omega_A$. For such values of n , one has $\varepsilon n \sim 1$ [see Eq. (72)]. As a result, the last term in Eq. (73), which scales as $\mathcal{O}(\kappa \varepsilon^2 n^2) \sim \mathcal{O}(\kappa)$, is no longer a small correction: it becomes of the same order as the first term, which depends on $|W_1|$ and likewise scales as $\mathcal{O}(\kappa)$ [since $\varepsilon^2 \sim \kappa n \varepsilon \sim \kappa$]. Moreover, both of these terms become of the same order as the leading term, $\kappa \alpha_{01}$, in the expansion (44). Therefore, the perturbative treatment developed in the preceding sections is, strictly speaking, no longer justified in this regime. This means, in particular, that shellular rotation [Eq. (55)] may only be a rough approximation to the optimal displacement associated with the global instability considered here.

The situation changes, however, once dissipative effects are included in the MHD equations. As discussed in Secs. VIII B and VIII C, viscosity and magnetic diffusivity can substantially slow down the development of small-scale displacements with $n \sim \mathcal{N}/\omega_A$, which would otherwise be optimal in ideal MHD. As a consequence, larger-scale modes with smaller n become more favorable. Such modes can again be described within the perturbative framework, and for them shellular rotation provides a very good approximation. As will be shown below, this is precisely what is observed in the numerical experiments presented in Sec. IX.

F. Summary of Sec. IV

Let us summarize the main results of Sec. IV (see also Table I, which lists the key parameters used in constructing the unstable Lagrangian displacement). We found that a toroidal field is destabilized by a Lagrangian displacement that can, within the range of validity of the perturbation theory and to leading order in each spherical component, be written as

$$\xi(r, \theta, \varphi) = \xi_{\text{rot}}(r, \theta, \varphi) + \xi_r(r, \theta, \varphi) e_r, \quad (83)$$

where the vector ξ_{rot} gives the dominant θ and φ components of the displacement and corresponds to a shellular differential rotation about some axis perpendicular to the symmetry axis z , while $\xi_r e_r$ gives its dominant radial component. The θ and φ components also contain corrections of relative order $\mathcal{O}(\varepsilon n)$, which are not written explicitly here [see, in particular, Eq. (42)], whereas the

TABLE I. Key parameters controlling the unstable displacement found in Sec. IV. The expansion parameter κ is introduced in Eq. (26). The quantities n and the characteristic wave vector of the perturbation, k , are defined after Eq. (65). The small parameter ε is defined in Eq. (40) [see also Eq. (67)]. The second column gives the definitions of these quantities. The third and fourth columns provide order-of-magnitude estimates of their optimal values, for which the instability is particularly efficient (these values are close to the maximum values for which the instability still exists; see footnote 18). In the third column, these values are expressed in terms of κ and $\gamma_{\text{fr},0} - \gamma_{\text{eq},0}$; in the fourth column, the optimal values are expressed in terms of the typical Alfvén frequency ω_A and the Brunt-Väisälä frequency $\mathcal{N}$.

Parameter	Definition	Optimal/maximum value	
$\kappa \ll 1$	$\frac{B^2}{8\pi p}$	...	...
$n \gtrsim 1$	$\frac{\partial_r a(r,\theta)}{a(r,\theta)} R \approx \frac{a'(r)}{a(r)} R$	$\frac{\sqrt{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}}}{\kappa^{1/2}}$	$\frac{\mathcal{N}}{\omega_A}$
$k \gtrsim 1/R$	$\frac{n}{R}$	$\frac{\sqrt{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}}}{\kappa^{1/2} R}$	$\frac{\mathcal{N}}{\omega_A R}$
$\varepsilon \ll 1$	$\left \frac{\xi_r}{\xi_\theta} \right \sim \frac{n\kappa}{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}} \sim n \frac{\omega_A^2}{\mathcal{N}^2}$	$\frac{\kappa^{1/2}}{\sqrt{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}}}$	$\frac{\omega_A}{\mathcal{N}}$

radial component itself is smaller than the tangential one by a factor of order ε :

$$\left| \frac{\xi_r}{\xi_{\text{rot}}} \right| \sim \left| \frac{\xi_r}{\xi_\theta} \right| \sim \mathcal{O}(\varepsilon); \quad (84)$$

see Eqs. (40) and (41). Note that the expression (83) is valid only in the region of the star occupied by the magnetic field [see the discussion after Eq. (55)]. Without loss of generality, we take the rotation axis to be the x axis. Then $\xi_{\text{rot}}(r, \theta, \varphi)$ is given by Eq. (54):

$$\xi_{\text{rot}}(r, \theta, \varphi) = a(r) \hat{x} \times \mathbf{r}. \quad (85)$$

If the characteristic radial variation scale of the function $a(r)$, which quantifies the degree of differential rotation, is not too small (i.e., if n , k , and ε are below their optimal values listed in Table I), the radial component ξ_r is obtained by an exact minimization of the energy functional $W[\xi]$ and is given by Eqs. (40) and (64). Such a displacement does destabilize the toroidal field, but the instability develops more slowly than in the optimal case.

To minimize the growth time, one must consider larger (optimal) values of n , k , and ε , as listed in the third (or fourth) column of Table I.¹⁹ In this limiting case, the perturbative expansion breaks down, so that the corrections to shellular differential rotation cease to be higher order in κ and ε . Furthermore, Eq. (64) for ξ_r should be regarded as only approximate, while the function $a(r)$ can be determined by minimizing the approximate functional (74). Note that, as announced at the end of Sec. IV E, this

¹⁹For even larger values of these parameters, the instability is completely suppressed by the higher-order term $\kappa \varepsilon^2 \alpha_{21}$ in $W[\xi]$, discussed in Sec. IV D.

limiting case need not be realized in practice once dissipative terms are included in the MHD equations (see Secs. VIII B and VIII C).

Finally, it is worth emphasizing, that since both ξ_{rot} and ξ_r depend on the function $a(r)$, which may vary rapidly with r , the radial derivatives of ξ_{rot} and ξ_r should be estimated, in order of magnitude, as

$$\partial_r \xi_{\text{rot}} \sim n \xi_{\text{rot}} / R \quad \text{and} \quad \partial_r \xi_r \sim n \xi_r / R, \quad (86)$$

which (for large n) can differ substantially from the naive estimates ξ_{rot}/R and ξ_r/R .

V. CHARACTERISTIC INSTABILITY GROWTH TIME

Using the trial function (79) for $a(r)$ allows us not only to calculate $W[\xi]$, but also to obtain an estimate for the upper bound on the characteristic instability growth time τ_{min} . For this purpose, we will use the spectral variational principle from Sec. III A. We need to minimize the ratio $W[\xi]/I[\xi]$. In this ratio, the functional $W[\xi]$ is known and given by formula (80), and for $I[\xi]$ it is necessary to use Eq. (23), in which the Lagrangian displacement ξ should be substituted. Proceeding in the same spirit as in Sec. IV E, we adopt as ξ the shellular differential rotation ξ_{rot} [see Eq. (54)], which, as shown in Sec. IV B, provides a good approximation [to within terms of order $\sim \mathcal{O}(\varepsilon + \kappa)$] to the Lagrangian displacement within the regime of validity of the perturbation theory. As a result, we obtain

$$\begin{aligned} I[\xi] &\simeq \frac{1}{2} \int \rho \xi_{\text{rot}} \cdot \xi_{\text{rot}} dV \approx \frac{4\pi}{3} \int_0^R r^4 \rho_0 a(r)^2 dr \\ &= \frac{4\pi}{3} \int_0^R r^4 \rho_0 \cos^2 \left(\frac{nr}{R} \right) dr \approx \frac{2\pi}{3} \int_0^R r^4 \rho_0 dr. \end{aligned} \quad (87)$$

By order of magnitude, this integral can be estimated as $I[\xi] \sim (2\pi/15) \rho_0 R^5$. It is evident that $I[\xi]$ does not depend on n ; therefore the minimization of $W[\xi]/I[\xi]$ reduces to the minimization of $W[\xi]$, which was already performed in Sec. IV E. As a result, using (25), (82), and (87), we obtain

$$\begin{aligned} \tau_{\text{min}} &\leq \sqrt{\frac{I[\xi]}{W[\xi]}} \\ &= \sqrt{\frac{32\pi}{3\kappa} \frac{(\int_0^R r^4 \rho_0 dr)^{1/2} (\int_0^R |F_4| dr)^{1/2}}{\int_0^R |F_1| dr}} \sim \tau_A, \end{aligned} \quad (88)$$

where $\tau_A = 1/\omega_A \sim (4\pi\rho_0)^{1/2} R/B$ is the typical Alfvén timescale. We see that the typical instability growth time for our trial function $a(r) = \cos(nr/R)$ is $\sim \tau_A$. Of course, the actual time depends on the specific field model and may be smaller for a more optimal function $a(r)$. In the next section, using the example of a simple magnetic field model, we will show that the optimal function $a(r)$ can, in

fact, differ significantly from the trial function of the form (79).

VI. NUMERICAL EXAMPLE

We illustrate the general results of the previous sections with an example of a specific model of a toroidal magnetic field of the form

$$\mathbf{B}(r, \theta) = \mathcal{L} \rho r \sin \theta \mathbf{e}_\varphi, \quad (89)$$

and study the instability associated with it. In formula (89), $\mathcal{L}$ sets the magnetic field strength. We chose this field model specifically because it allows a relatively simple calculation of the hydrostatic structure of a magnetized star [58,59]. Note that the stability of this field has already been studied in the work [7] based on the approach developed by Tayler [2].

A. Hydrostatic structure

Before studying the instability, it is necessary to construct a hydrostatic equilibrium model of a star with the field (89). We will assume that the stellar matter obeys a polytropic equation of state, $P = K \rho^{1+1/\mathbf{n}}$, with a polytropic index $\mathbf{n} = 3$ as an approximation for a radiative (stably stratified) star. Accordingly, in this case, $\gamma_{\text{eq},0} = 1 + 1/\mathbf{n} = 4/3$.

We need to solve the equations of hydrostatic equilibrium (8) and the Poisson equation (7) simultaneously. In the case of a weak field, this is done by standard perturbation theory methods [58,59]; the solution [see Eqs. (92)–(94) below] is expressed in terms of the functions $\tilde{\theta}(\zeta)$, $\psi_0(\zeta)$, and $\psi_2(\zeta)$, which are shown in Fig. 1. The variable ζ is related to the radial coordinate by the equation

$$r = \alpha \zeta, \quad \alpha = \left(\frac{P_c(\mathbf{n} + 1)}{4\pi G \rho_c^2} \right)^{1/2}, \quad (90)$$

where P_c, ρ_c are the central pressure and density of the star, respectively; note that $K = P_c / \rho_c^{\gamma_{\text{eq},0}}$. The weakness of the field corresponds to the smallness of the dimensionless parameter $h^2 \equiv \mathcal{L}^2 / (16\pi^2 G) \ll 1$, which is equivalent to the condition $\kappa \ll 1$. Further in Sec. VI, without loss of generality, we will relate the parameters κ and h^2 through the equality

$$\kappa \equiv h^2. \quad (91)$$

Knowing the functions $\tilde{\theta}(\zeta)$, $\psi_0(\zeta)$, and $\psi_2(\zeta)$, one can calculate the function $\Theta(\zeta, \mu)$,

$$\Theta(\zeta, \mu) = \tilde{\theta}(\zeta) + h^2 [\psi_0(\zeta) + \psi_2(\zeta) P_2(\mu)], \quad (92)$$

where $\mu = \cos \theta$ and $P_2(\mu)$ is the Legendre polynomial of order 2. Having $\Theta(\zeta, \mu)$, one can, in turn, determine all

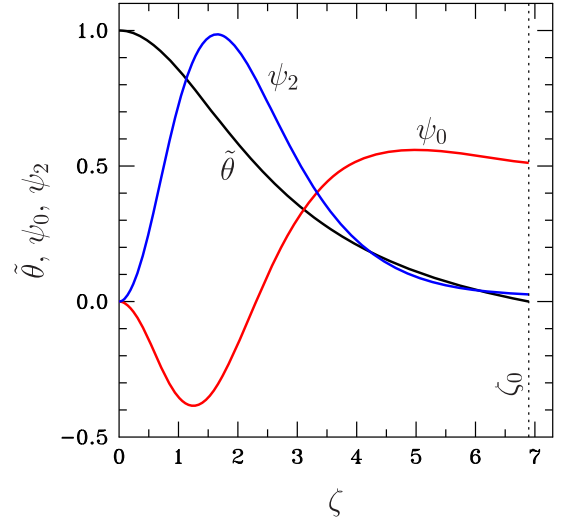


FIG. 1. Functions $\tilde{\theta}(\zeta)$, $\psi_0(\zeta)$, and $\psi_2(\zeta)$ versus dimensionless radial coordinate ζ .

other quantities, in particular ρ and Φ :

$$\rho = \rho_c \Theta(\zeta, \mu)^{\mathbf{n}}, \quad (93)$$

$$\Phi = -D \left\{ \Theta(\zeta, \mu) + c_0 + h^2 \left[c_1 + \frac{2}{3} \zeta^2 \tilde{\theta}(\zeta)^{\mathbf{n}} (1 - P_2(\mu)) \right] \right\}, \quad (94)$$

where $D = (\mathbf{n} + 1) P_c / \rho_c$. For $\mathbf{n} = 3$ one has $c_0 \approx 0.29263$, $c_1 \approx -0.31205$. To obtain from Eqs. (92)–(94) the corresponding expressions for a star without a magnetic field, it is sufficient to set $h = 0$. The functions of the variables ζ and μ are easily converted into functions of r and θ using the relations $\zeta = r/\alpha$ and $\mu = \cos \theta$.

The shape of the star, distorted by the magnetic field, follows from the condition $\Theta(\zeta, \mu) = 0$ and has the form $\zeta(\mu) = \zeta_0 + h^2 f(\mu)$, where $\zeta_0 \approx 6.897$ is the dimensionless radius of a spherically symmetric star without a magnetic field, and

$$f(\mu) = - \frac{[\psi_0(\zeta_0) + \psi_2(\zeta_0) P_2(\mu)]}{\frac{d\tilde{\theta}(\zeta)}{d\zeta} \big|_{\zeta=\zeta_0}}. \quad (95)$$

Using this equation, one can write

$$\frac{\zeta(1) - \zeta(0)}{\zeta(0)} = - \frac{3h^2}{2\zeta_0} \frac{\psi_2(\zeta_0)}{\frac{d\tilde{\theta}(\zeta)}{d\zeta} \big|_{\zeta=\zeta_0}}. \quad (96)$$

Keeping in mind that $\psi_2(\zeta_0) > 0$ and $d\tilde{\theta}/d\zeta < 0$, we see that the polar radius is always longer than the equatorial one, i.e., unlike the case of a rotating star, a magnetized star with toroidal field is elongated along the poles.

Below, as an illustration, we consider a model of a star with $\rho_c = 25 \text{ g cm}^{-3}$ and $P_c = 5 \times 10^{16} \text{ dyn cm}^{-2}$. These parameters correspond to a star with mass $M \approx 2.38 M_\odot$ and radius $R \approx 1.93 R_\odot$. The instability is studied for two magnetic field models with $h^2 = h_{500}^2 \approx 4.4035 \times 10^{-3}$ (model I) and $h^2 = h_{5000}^2 \approx 4.4035 \times 10^{-4}$ (model II) [the models differ only by normalization]. Model I corresponds to the ratio of gravitational to magnetic energy $|E_{\text{grav}}/E_{\text{mag}}| \approx 500$, while for model II this ratio is 5000. Figure 2 shows a color plot of the magnetic field used, normalized to the maximum field strength in the star.

For further analysis, it is convenient to introduce the star-averaged Alfvén time according to the formula

$$\tau_A = \frac{R(4\pi\bar{\rho})^{1/2}}{\bar{B}} \approx \frac{6745 \text{ s}}{h}, \quad (97)$$

where $\bar{\rho} = M/V$ is the average stellar density ($V = 4\pi R^3/3$) and $\bar{B} = (\int |B| dV)/V$ is the average absolute magnetic field value.

B. Instability

Now everything is prepared to determine the optimal function $a(r)$ that maximally destabilizes the star, and to estimate the characteristic instability growth time. To do this, we need to minimize the ratio $\Lambda = W[\xi]/I[\xi]$ [see Sec. III A], where $W[\xi]$ is given by Eq. (74), and $I[\xi]$ by Eq. (87) (in which, however, the function $a(r)$ should be treated as unknown). It is convenient to change the

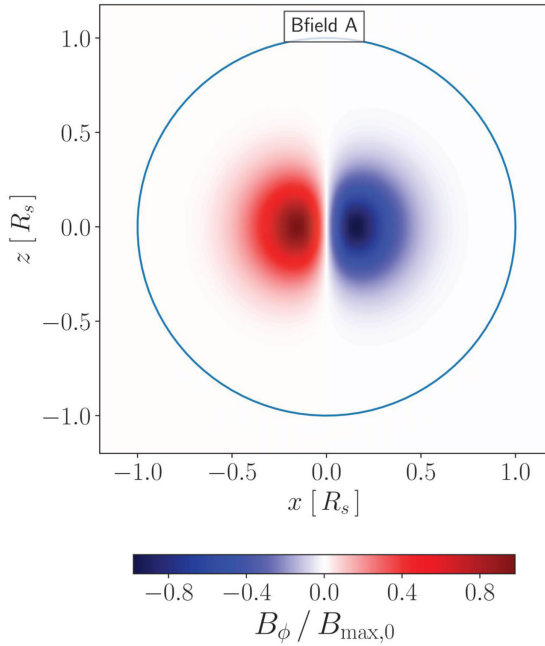


FIG. 2. Meridional cut of the star with the toroidal magnetic field used. The colors represent the toroidal-field magnitude (more precisely, its y projection) normalized to its amplitude.

integration over the radial coordinate r in the functionals $W[\xi]$ and $I[\xi]$ to integration over the dimensionless coordinate $\zeta = r/\alpha$. In this case, they can be rewritten as

$$W[\xi] \simeq \frac{1}{2} \int_0^{\zeta_0} \{ -[a'(\zeta)]^2 |\tilde{F}_1(\zeta)| + [a(\zeta)]^2 \tilde{F}_2(\zeta) - a(\zeta) a'(\zeta) \tilde{F}_3 + [a''(\zeta)]^2 |\tilde{F}_4| \} d\zeta, \quad (98)$$

$$I[\xi] \approx \int_0^{\zeta_0} \frac{4\pi\alpha^5}{3} \zeta^4 \rho_0 a(\zeta)^2 d\zeta, \quad (99)$$

where we introduced new functions defined as $\tilde{F}_1 \equiv \kappa^2 F_1/\alpha$, $\tilde{F}_2 \equiv \kappa^2 \alpha F_2$, $\tilde{F}_3 \equiv \kappa^2 F_3$, and $\tilde{F}_4 \equiv \kappa^3 F_4/\alpha^3$. To compute them numerically using formulas (75)–(78), one needs to take into account that $\kappa = h^2$, $8\pi\kappa B^2 = B^2$, and that Φ_1 , according to the definition (27) and Eqs. (92) and (94), is equal to (without the h^2 factor!)

$$\Phi_1 = -D \left[\psi_0(\zeta) + \psi_2(\zeta) P_2(\mu) + c_1 + \frac{2}{3} \zeta^2 \tilde{\theta}(\zeta)^n (1 - P_2(\mu)) \right]. \quad (100)$$

In addition, it is necessary to specify the “frozen” adiabatic index γ_{fr} . In all numerical calculations below, we set $\gamma_{\text{fr}} = 5/3$.

Figure 3 shows the functions $\tilde{F}_i(\zeta)$ ($i = 1, \dots, 4$) for magnetic field model I. All functions are normalized so that the maximum value of $\tilde{F}_1(\zeta)$ is equal to 1. As can be seen, the function $\tilde{F}_4$ is much smaller than the other functions. This is not surprising, since $\tilde{F}_4 \propto B^6/p^3 \sim h^6$, whereas the other functions are proportional to $B^4/p^2 \sim h^4$. All functions vanish near $\zeta = 0$ and $\zeta = \zeta_0$, and one can show that

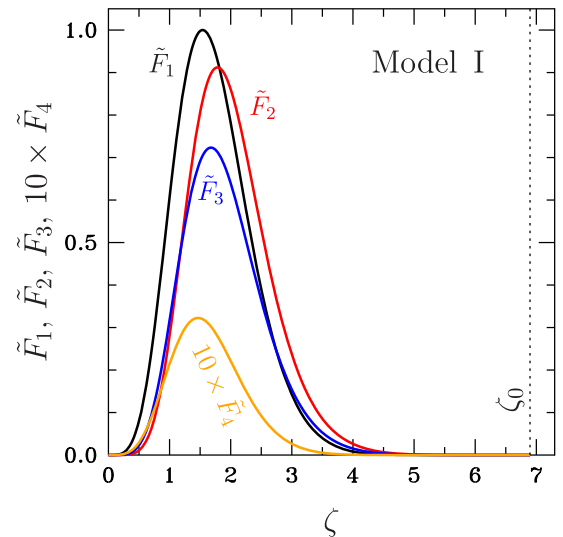
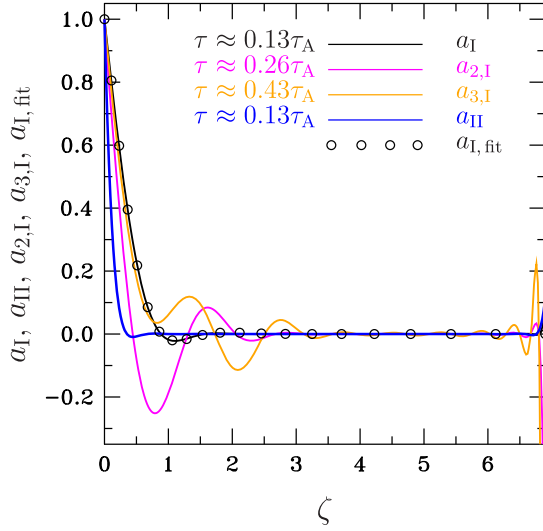


FIG. 3. Functions $\tilde{F}_i(\zeta)$ ($i = 1, \dots, 4$) for model I of the magnetic field.


 FIG. 4. Functions a_I , $a_{2,I}$, $a_{3,I}$, a_{II} , and $a_{I,fit}$ versus ζ .

both F_1 and F_4 go as ζ^4 when $\zeta \rightarrow 0$. This behavior of the functions $\tilde{F}_i(\zeta)$ suggests the simplest approach to perform the numerical minimization of $\Lambda = W[\xi]/I[\xi]$. Indeed, one could in principle vary this ratio with respect to $a(\zeta)$ and obtain a fourth-order differential equation for $a(\zeta)$.²⁰ However, such an equation would be highly singular, since its coefficients vanish both at the center and at the boundary of the star. Solving such an equation numerically is difficult. An alternative approach is to expand $a(\zeta)$ in some complete set of eigenfunctions and minimize $W[\xi]/I[\xi]$ by adjusting the coefficients in such an expansion. This is exactly what we did, expanding $a(\zeta)$ in Chebyshev polynomials $T_l(x(\zeta))$, where $x(\zeta) = -1 + 2\zeta/\zeta_0$, and retaining only the first 20 polynomials (the solution is practically insensitive to further increases in the number of polynomials).

The results of this standard minimization are presented in Fig. 4. The figure shows the optimal functions $a(\zeta)$ (corresponding to the largest in absolute value and negative eigenvalues Λ) for models I ($E_{\text{grav}}/E_{\text{mag}} = 500$) and II ($E_{\text{grav}}/E_{\text{mag}} = 5000$). These functions are denoted, respectively, by $a_I(\zeta)$ and $a_{II}(\zeta)$. In addition, for model I we also show the second and third most unstable eigenfunctions $a_{2,I}(\zeta)$ and $a_{3,I}(\zeta)$, which are orthogonal to the function $a_I(\zeta)$ (see footnote 20). For all the functions shown in the figure, the characteristic instability growth time $\tau = |\Lambda|^{-1/2}$ is also indicated.

²⁰Note that the differential operator corresponding to this equation is self-adjoint. Therefore, any two eigenfunctions $a_i(\zeta)$ and $a_j(\zeta)$, which are solutions to this equation and correspond to distinct eigenvalues Λ_i and Λ_j , are orthogonal with respect to the weight $\zeta^4 \rho_0(\zeta)$:

$$\int_0^{\zeta_0} \zeta^4 \rho_0(\zeta) a_i(\zeta) a_j(\zeta) d\zeta = 0.$$

Let us note the apparent (and unphysical) growth of these functions near the stellar surface. This occurs because in the surface layers the density, pressure, and magnetic field all tend to zero, so the contributions of these regions to $W[\xi]$ and $I[\xi]$ are negligible for essentially any choice of $a(\zeta)$; therefore, the minimization of $W[\xi]/I[\xi]$ does not constrain $a(\zeta)$ in those layers. As a result, our growth-time estimates are insensitive to the near-surface behavior.

The circles in Fig. 4 show the result of fitting the most unstable mode $a_I(\zeta)$ for model I:

$$a_{I,fit}(\zeta) = -22.73575e^{-1.034352\zeta^2} + 23.73575e^{-0.075038\zeta - 1.004375\zeta^2}. \quad (101)$$

A similar fitting formula also exists for the function $a_{II}(\zeta)$. For both of these functions, the characteristic instability growth time, measured in units of τ_A , is approximately

$$\tau = |\Lambda|^{-1/2} \approx 0.13\tau_A. \quad (102)$$

This fact can be explained by noting that the result of the minimization is only weakly dependent on $\tilde{F}_2$ and $\tilde{F}_3$. Assuming $\tilde{F}_2 = \tilde{F}_3 = 0$ in Eq. (98) and taking into account the asymptotic behavior $\tilde{F}_1 \propto \zeta^4$ and $\tilde{F}_4 \propto \zeta^4$ near 0, one can show that Λ scales with h as $\Lambda \sim h^2$, i.e., the time τ measured in units of τ_A should remain unchanged, which is consistent with the numerical results. Moreover, with $\tilde{F}_2 = \tilde{F}_3 = 0$, the optimal function $a(\zeta)$ becomes self-similar for different values of h , that is, $a(\zeta/h)$ is independent of h at small h . This explains why the optimal solution localizes toward the center as $h \rightarrow 0$ [$a_{II}(\zeta)$ is more localized near the center than $a_I(\zeta)$].

The localization of the unstable mode as $h \rightarrow 0$ is a direct consequence of neglecting dissipative effects in our treatment based on ideal magnetohydrodynamics. In reality, accounting for dissipative effects (e.g., introducing finite viscosity) can significantly, by a factor of approximately τ/τ_{diss} (see, e.g., [16]), slow down the growth of such a mode due to the large gradients involved (estimating the factor we have assumed that the dissipative timescale τ_{diss} is much shorter than τ). As a result, a different, less localized instability may have a better chance to develop: one that generates smaller gradients and is therefore less affected by dissipative processes (see also Sec. IX).

Such instabilities, of course, do exist. For instance, if we take the less localized solution $a_I(\zeta)$ obtained for model I and use it as the trial function $a(\zeta)$ in model II, the resulting displacement will still be unstable. Moreover, the corresponding characteristic growth time will be $\tau \approx 0.3\tau_A$, that is, it will increase by less than a factor of 3 [cf. Eq. (102)].

Another interesting example is provided by the “universally unstable” trial function $a(\zeta) = \cos(n\zeta/\zeta_0)$ from Sec. IV E [see Eq. (79)]. The instability timescale τ for this function is plotted in Fig. 5 as a function of n for

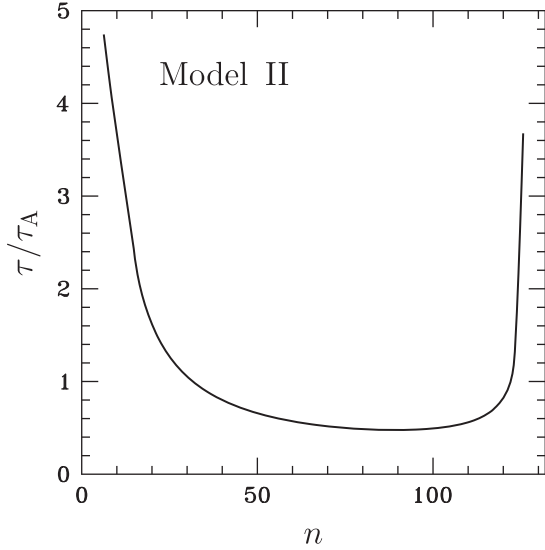


FIG. 5. Instability timescale for trial function $a(\zeta) = \cos(n\zeta/\zeta_0)$ versus number n for model II of the magnetic field.

model II. In agreement with Eq. (81), the minimum of $\tau(n)$ is reached at $n \approx 89.2$ and equals $\tau \approx 0.477\tau_A$.

C. A few subtle points

In this section, we discuss several subtle points related to the results obtained above.

(1) Short instability timescale.

As follows from Eq. (102), the characteristic instability growth time we obtained does not exceed $0.13\tau_A$, i.e., it is much shorter than the average Alfvén time τ_A [see Eq. (97)]. This fact can be easily explained by noting that the instability occurs in a small region near the center of the star. Accordingly, the magnetic field there is stronger than the average, and the characteristic length scale is much smaller than R . Therefore, the *actual* Alfvén time characterizing this region must be much shorter than the mean stellar Alfvén time τ_A defined by Eq. (97). For example, estimating the Alfvén time for the inner region $r \lesssim R/3$, where the magnetic field is mainly localized, using Eq. (97), we obtain $\tau_A(R/3) \approx 0.115\tau_A$, which is nearly 1 order of magnitude smaller than the mean stellar Alfvén time.

This result is consistent with the findings of Ref. [7], which studied the local Tayler instability for the field model (89) and found the minimum instability time [see their equation (66)]

$$\tau_{GV} \simeq \left(8.5 \times 10^{-5} \frac{E_{\text{mag}}}{|E_{\text{grav}}|} \frac{MR_{\odot}^3}{R^3 M_{\odot}} \right)^{-1/2}. \quad (103)$$

Substituting numbers, we find $\tau_{GV} \approx 0.05\tau_A$, an even shorter time for the development of local instability.

(2) Verification of the self-consistency of our approach and the “optimality” of the Lagrangian displacement we have found.

In deriving the functional (74) from the original functional (19), a kind of perturbation theory in the parameters κ and ε was developed. This allowed us to significantly simplify the problem by discarding all small terms in Eq. (19). It is of considerable interest to verify the self-consistency of our calculation by substituting the obtained Lagrangian displacement into the original *exact* expressions (19)–(21) and (23) for $W[\xi]$ and $I[\xi]$, and calculating $\Lambda = W[\xi]/I[\xi]$ and $\tau = |\Lambda|^{-1/2}$ using these exact formulas.

Let us perform such a verification using the example of magnetic field model I. We have already defined the function $a(r)$ [see the fitting formula (101), where $\zeta = r/\alpha$]. Now we set $\mathbf{a}(r, \theta) = a(r)$ and use Eq. (64) to find $\varepsilon \mathbf{g}(r, \theta)$. Knowing $\mathbf{a}(r, \theta)$ and $\varepsilon \mathbf{g}(r, \theta)$, we determine $\tilde{\xi}_r^{(m)}$, $\tilde{\xi}_\theta^{(m)}$, and $\tilde{\xi}_\varphi^{(m)}$ (with $m = 1$) from Eqs. (40)–(42). Thus, we obtain the full Lagrangian displacement $\xi(r, \theta, \varphi)$ given by Eq. (39). Substituting it into the integrals (20), (21), and (23), we calculate the characteristic instability growth time corresponding to our displacement: $\tau \approx 0.12\tau_A$. The resulting time turns out to be close to, and even slightly less than, the value $\tau \approx 0.13\tau_A$ from the approximate calculation [see Eq. (102)]. This suggests that our approximations capture the essential physics and that our approach is self-consistent.

(3) Angular momentum and its conservation.

Looking at Fig. 4, one may get the impression that the star predominantly rotates in a certain direction, since $a(\zeta)$ in the figure is mostly positive. This raises the question: how is the conservation of angular momentum maintained in our star? Indeed, the correct Lagrangian displacement, which minimizes the ratio $\Lambda = W[\xi]/I[\xi]$ (see Sec. III A) and satisfies the equations of magnetohydrodynamics, should not change the angular momentum of the system, which is zero in our case.

Let us check how well angular momentum is conserved for our solution using the example of model I. Neglecting the small contribution from the electromagnetic field, the angular momentum of the system is proportional to the integral

$$\begin{aligned} L &\propto \int \rho \mathbf{r} \times \xi dV \approx \int \rho_0(r) \mathbf{r} \times \xi_{\text{rot}}(r, \theta, \varphi) dV \\ &= \int \rho_0(r) \mathbf{r} \times [a(r) \hat{\mathbf{x}} \times \mathbf{r}] dV \\ &= \frac{8\pi}{3} \hat{\mathbf{x}} \int_0^R r^4 \rho_0(r) a(r) dr \\ &= \frac{8\pi \alpha^5 \rho_c}{3} \hat{\mathbf{x}} \int_0^{\zeta_0} \zeta^4 \frac{\rho_0(\zeta)}{\rho_c} a(\zeta) d\zeta, \end{aligned} \quad (104)$$

where we used the expression (54) and changed the integration variable from r to $\zeta = r/\alpha$. If we substitute the fitting formula (101) for $a(r)$ into Eq. (104), we obtain a nonzero but small value for the integral on the right-hand side of (104): $\int_0^{\zeta_0} \zeta^4 [\rho_0(\zeta)/\rho_c] a(\zeta) d\zeta \approx 0.0076$. To make this integral vanish exactly, a small solid-body rotation must be added to the system, i.e., $a(\zeta)$ should be replaced: $a(\zeta) \rightarrow a(\zeta) + \delta a$, where δa is some constant. It turns out that in our case this constant is very small, $\delta a \approx -0.0007$ [to be compared to the typical value of $a(\zeta) \sim 1$], and it has almost no effect on the potential energy of the system.²¹ Moreover, such a solid-body rotation contribution practically does not change the characteristic growth time of the instability. This result shows that the obtained Lagrangian displacement satisfies the law of angular momentum conservation to a good accuracy.

Let us recall that, as already discussed, the procedure for obtaining the function $a(r)$ in Sec. VI B is not absolutely rigorous. The fact that the angular momentum associated with the resulting Lagrangian displacement remains practically zero (and can be made to vanish exactly without noticeably affecting the instability timescale) provides additional support for the claim that the displacement we found is close to optimal.

VII. STABILIZING EFFECT OF THE POLOIDAL MAGNETIC FIELD

Let us estimate the strength of the poloidal magnetic field needed to suppress the instability of the toroidal field. To do that, we allow for the presence of the weak poloidal field B_{pol} of the form

$$\mathbf{B}_{\text{pol}} = \nabla \Psi(r, \theta) \times \nabla \varphi. \quad (105)$$

The idea is to determine the strength of the poloidal field at which the unstable Lagrangian displacement found in the previous sections becomes stabilized (i.e., ceases to be energetically favorable). As can be easily seen, the contribution of the poloidal field to $W[\xi]$ appears already at first order in the perturbation theory in κ and ε (see Sec. IV B), when the Lagrangian displacement is taken to be pure

²¹In general, adding solid-body rotation to the system should not change its potential energy. The fact that replacing $a(\zeta) \rightarrow a(\zeta) + \delta a$ does lead to a change in the potential energy, as approximately determined by the functional (74) [or (98)], is a direct consequence of the approximations we have made. In particular, following Ref. [54], we have neglected from the very beginning the last term in expression (20), which depends on the gravitational field perturbation, $\delta\Phi$ (see Sec. III B). It can be shown that if this term is retained, then any solid-body rotation of the system indeed does not change its potential energy, since it is equivalent to a simple rotation of the coordinate system.

shellular differential rotation $\xi_{\text{rot}}(r, \theta, \varphi) = a(r)\hat{x} \times \mathbf{r}$ [see Eq. (54)]. In this case, the main contribution to $W[\xi]$ comes from the magnetic part of the functional W^{mag} , given by expression (21). Substituting $B(r, \theta)\mathbf{e}_\varphi + \mathbf{B}_{\text{pol}}(r, \theta)$ and $\xi(r, \theta, \varphi) = \xi_{\text{rot}}(r, \theta, \varphi)$ into (21), we obtain a series of terms proportional to the functions $[a'(r)]^2$, $a'(r)a(r)$, and $a(r)^2$, and containing *no* mixed terms depending simultaneously on both the poloidal and toroidal fields. Taking into account that $Ra'(r) \gg a(r)$, the dominant contribution from the poloidal field will come from the term $\propto [a'(r)]^2$:

$$\frac{1}{2} \int_0^R |F_5| [a'(r)]^2 dr, \quad \text{where} \\ F_5(r) = \int_0^\pi \frac{1}{4} \left(\frac{\cos^2 \theta + 1}{\sin \theta} \right) (\partial_\theta \Psi)^2 d\theta. \quad (106)$$

This non-negative term will always act “against” the instability. Thus, taking into account the poloidal field, the problem of determining the optimal function $a(r)$ reduces to minimizing the ratio $\tilde{W}[\xi]/I[\xi]$, where $\tilde{W}[\xi]$ is the potential energy functional including the contribution (106) from the poloidal field [cf. Eq. (74)]:

$$\tilde{W}[\xi] \simeq \frac{1}{2} \int_0^R \{ [a'(r)]^2 (|F_5| - \kappa^2 |F_1|) + \kappa^2 [a(r)]^2 F_2 - \kappa^2 a(r) a'(r) F_3 + \kappa^3 [a''(r)]^2 |F_4| \} dr. \quad (107)$$

Even without performing an exact minimization of the ratio $\tilde{W}[\xi]/I[\xi]$, it is clear that the poloidal field will completely suppress the instability once the poloidal contribution $(1/2) \int_0^R [a'(r)]^2 |F_5| dr$ exceeds the main destabilizing term $-(1/2) \int_0^R \kappa^2 [a'(r)]^2 |F_1| dr$ in the functional (107), that is (substituting F_1 and F_5)

$$\int \frac{1}{4} \left(\frac{\cos^2 \theta + 1}{\sin \theta} \right) (\partial_\theta \Psi)^2 [a'(r)]^2 dr d\theta \\ \gtrsim \int \frac{r^2 B^4 \cos^2 \theta p_0 \gamma_{\text{fr},0} \gamma_{\text{eq},0}}{16\pi \sin \theta [p'_0(r)]^2 (\gamma_{\text{fr},0} - \gamma_{\text{eq},0})} [a'(r)]^2 dr d\theta. \quad (108)$$

If the regions of localization of the toroidal and poloidal magnetic fields coincide, then the integrals can be roughly estimated by order of magnitude, yielding the following lower bound for the stabilizing poloidal magnetic field B_{pol} :

$$B_{\text{pol}}^2 \gtrsim \frac{B^2 B^2}{4\pi p_0 \gamma_{\text{fr},0} - \gamma_{\text{eq},0}} \sim 2B^2 \frac{\omega_A^2}{\mathcal{N}^2}. \quad (109)$$

This estimate can be rewritten by introducing the gravitational energy of the star, E_{grav} , the energy of the poloidal field, $E_{\text{pol}} = \int B_{\text{pol}}^2 / (8\pi) dV$, and the energy of the toroidal field, $E_{\text{tor}} = \int B^2 / (8\pi) dV$. Estimating $B^2 / (8\pi p_0) \sim E_{\text{tor}} / E_{\text{grav}}$, one can rewrite (109) as

$$\frac{E_{\text{pol}}}{E_{\text{tor}}} \gtrsim \beta \frac{E_{\text{tor}}}{E_{\text{grav}}} \frac{\gamma_{\text{fr},0} \gamma_{\text{eq},0}}{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}}, \quad (110)$$

where β is some model-dependent coefficient. Let us estimate β for model I of the star with a toroidal magnetic field given by Eq. (89) and the optimal function $a(r)$ specified by Eq. (101) (recall that for model I, $E_{\text{grav}}/E_{\text{tor}} = 500$). To this end, we choose the poloidal magnetic field $\mathbf{B}_{\text{pol}}$ from Ref. [60], for which $\Psi(r, \theta) = B_{\text{max}} R^2 f(r/R) \sin^2 \theta$, where $f(x) = x^2/2 - 3x^4/5 + 3x^6/14$, and B_{max} is the normalization constant.²² Now substituting these data into inequality (108), we obtain a lower bound estimate for the constant B_{max} . This estimate can then be rewritten in the form of inequality (110), in which β turns out to be approximately $\beta \approx 14.6$. According to this estimate, inequality (108) for the chosen parameters begins to hold when $E_{\text{pol}} \gtrsim 0.2E_{\text{tor}}$. It is worth emphasizing, however, that for a more realistic choice of the ratio $E_{\text{grav}}/E_{\text{tor}}$, the energy of the poloidal field required to suppress the instability will be significantly smaller. For example, choosing $E_{\text{grav}}/E_{\text{tor}} \sim 10^6$ [55] reduces the ratio to $E_{\text{pol}}/E_{\text{tor}} \sim 10^{-4}$.

One sees that a very small poloidal field is sufficient to suppress the instability. It remains to note that the scaling (110) is well-known and was suggested from heuristic arguments in [61] and later confirmed in [15,56].

VIII. TAYLER INSTABILITY VERSUS THE GLOBAL INSTABILITY IDENTIFIED IN THIS WORK

A. Ideal MHD

Here we compare Tayler's approach with ours. In Ref. [2], Tayler analyzed the stability of a toroidal axisymmetric magnetic field using the energy functional method. In contrast to our approach, Tayler did not expand the energy functional in series in ε and κ . Instead, he reorganized the terms to explicitly identify a non-negative component. Although the original paper used cylindrical coordinates, spherical coordinates are more suitable for global problems in stars. Thus, Tayler's results were rederived in spherical coordinates [7] by presenting the functional for azimuthal number $m \neq 0$ as

$$W[\xi] = \int \left[\frac{B^2}{4\pi r^2} \left(r \frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_\theta}{\partial \theta} - \xi_\theta \cot \theta \right)^2 + (a_m \xi_r^2 + b_m \xi_r \xi_\theta + c_m \xi_\theta^2) \right] dV, \quad (111)$$

where

²²We emphasize that this particular choice of the poloidal magnetic field is made solely for the purpose of obtaining a more quantitative estimate of β , and it is not in equilibrium with the toroidal field of the form given by Eq. (89).

$$a_m = -\partial_r \rho \partial_r \Phi - \frac{\rho^2}{\gamma_{\text{fr}} p} (\partial_r \Phi)^2 + \frac{m^2 B^2}{4\pi r^2 \sin^2 \theta} - \frac{B^2}{2\pi r^2} - \frac{1}{2\pi r} \mathbf{B} \cdot \partial_r \mathbf{B}, \quad (112)$$

$$b_m = -\frac{1}{r} \partial_r \Phi \partial_\theta \rho - \frac{1}{r} \partial_\theta \Phi \partial_r \rho - \frac{2\rho^2}{\gamma_{\text{fr}} p r} \partial_r \Phi \partial_\theta \Phi - \frac{B^2}{\pi r^2} \cot \theta - \frac{1}{2\pi r} \mathbf{B} \cdot \partial_r \mathbf{B} \cot \theta - \frac{1}{2\pi r^2} \mathbf{B} \cdot \partial_\theta \mathbf{B}, \quad (113)$$

$$c_m = -\frac{1}{r^2} \partial_\theta \rho \partial_\theta \Phi - \frac{\rho^2}{\gamma_{\text{fr}} p r^2} (\partial_\theta \Phi)^2 + \frac{m^2 B^2}{4\pi r^2 \sin^2 \theta} - \frac{B^2}{2\pi r^2} \cot^2 \theta - \frac{1}{2\pi r^2} \mathbf{B} \cdot \partial_\theta \mathbf{B} \cot \theta. \quad (114)$$

The first term in Eq. (111) is non-negative, while the second term is a quadratic form of the functions ξ_r and ξ_θ , with coefficients a_m , b_m , and c_m being certain functions of the thermodynamic quantities and the magnetic field [see Eqs. (112)–(114)]. In what follows, we discuss the Tayler instability using the energy functional $W[\xi]$ given by Eq. (111). Tayler [2] proved the following (see [7] for details). If at least one of the conditions,

$$a_m < 0 \quad \text{or} \quad c_m < 0 \quad \text{or} \quad b_m^2 > 4a_m c_m, \quad (115)$$

is fulfilled at some point of the star, then the field is unstable.²³ He demonstrated this by explicitly constructing a displacement that makes the functional negative. The displacement satisfies two properties. First, it perturbs the system only in the vicinity of a point where (115) is fulfilled. Second, it eliminates the stabilizing non-negative term [the first term in Eq. (111)] in the vicinity of that point. To make the problem tractable, he adopted the WKB approximation with respect to the r and θ coordinates, and substituted $\xi \propto \exp^{i(l\theta + kr + \phi)}$ into $W[\xi]$, assuming that $l \gg 1$ and $k \gg 1/R$. Interestingly, the condition $l \gg 1$ is not necessary for the onset of instability in many toroidal field configurations (see, e.g., [7] and Sec. IX below for examples). Tayler (and we here) adopted it to make the argument universal, i.e., applicable to any model of the toroidal magnetic field. In the azimuthal direction the perturbation was global with azimuthal number $m=1$ since, as Tayler showed, $m=1$ is the most unstable mode. Indeed, m enters the energy functional $W[\xi]$ only as a factor m^2 in the non-negative terms of Eqs. (112) and (114). In light of the above considerations, the displacement should satisfy the following condition to neutralize the non-negative term:

²³Subsequent work [11,15] has shown that such a point exists for any model of the stellar toroidal magnetic field.

$$rk\xi_r + l\xi_\theta \approx 0, \quad (116)$$

while the energy functional reduces to

$$\begin{aligned} W[\xi] &= \int (a_m \xi_r^2 + b_m \xi_r \xi_\theta + c_m \xi_\theta^2) dV \\ &\approx \int \left(a_m \frac{l^2}{r^2 k^2} - b_m \frac{l}{rk} + c_m \right) \xi_\theta^2 dV, \end{aligned} \quad (117)$$

where a_m , b_m , and c_m are evaluated at $m = 1$. Notably, the coefficients b_m and c_m contain only terms that depend on the magnetic field and vanish as the field vanishes. In the leading order in the magnetic field both coefficients scale quadratically with the field value: $b_m, c_m \sim \mathcal{O}(B^2)$. In contrast, the coefficient a_m also includes magnetic field-independent contributions arising from the stellar matter stable stratification. These contributions are always positive and tend to stabilize the system, leading to $a_m > 0$ for reasonable field values. It follows from this analysis that to make the quadratic form in Eq. (117) negative one must choose small values of $l/(kr) \sim \mathcal{O}(B)$. In that case the b_m -dependent term in (117) can be neglected and the functional can be rewritten as

$$W[\xi] \approx \int \left(a_m \frac{l^2}{r^2 k^2} + c_m \right) \xi_\theta^2 dV. \quad (118)$$

Clearly, the necessary and sufficient condition for this form to become negative is

$$c_m < 0. \quad (119)$$

Instability occurs when kr exceeds the threshold value,

$$kr \gtrsim l \left(\frac{a_m}{|c_m|} \right)^{1/2}. \quad (120)$$

Expressing a_m and $|c_m|$ as [see Eqs. (112) and (114)]

$$a_m \approx -\partial_r \rho_0 \partial_r \Phi_0 - \frac{\rho_0^2}{\gamma_{\text{fr},0} p_0} (\partial_r \Phi_0)^2 = \rho_0 \mathcal{N}^2, \quad (121)$$

$$|c_m| \sim \frac{B^2}{r^2}, \quad (122)$$

where $\mathcal{N}$ is the Brunt-Väisälä frequency (35), one can represent (120) as (see also, e.g., [16,28])

$$kr \gtrsim \frac{\mathcal{N}}{\omega_A} l, \quad (123)$$

where $\omega_A = 1/\tau_A$ is the typical Alfvén frequency. Obviously, the smaller the magnetic field, the larger kr/l must be to fulfill the condition (123). Near the threshold [at $k \sim \mathcal{N}l/(\omega_A r)$], the growth rate of the instability is low but

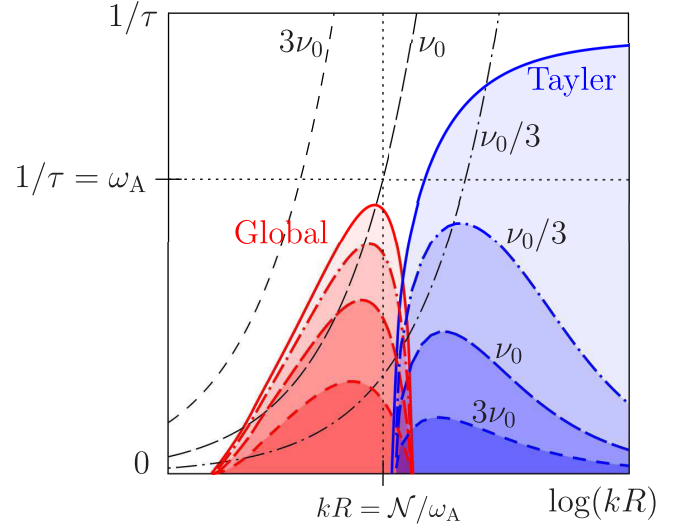


FIG. 6. A schematic plot of the growth rates for the global (red) and Tayler (blue) instabilities as functions of the radial perturbation wave number k . Dots mark the threshold value of k [see Eqs. (125) and (126)] and the Alfvénic growth rate ω_A . Solid lines show the ideal-MHD limit; colored dash-dotted and dashed lines illustrate the effect of three representative shear-viscosity values: $\nu_0/3$ (dash-dotted), ν_0 (long-dashed), and $3\nu_0$ (short-dashed); see text for details. The corresponding dissipative rates $1/\tau_v$ are shown by black curves, which scale as k^2 .

increases with increasing k approaching an asymptote at $k \rightarrow \infty$ (see the blue solid curve in Fig. 6). It should be emphasized that Tayler did not claim that the proposed displacement was either optimal or unique.

In contrast to Tayler, we assume that the functional $W[\xi]$ can be expanded in the small parameters ε and $\varkappa$. We then eliminate the non-negative leading-order term [not the same as the first term in Eq. (111)], and consequently find that the unstable displacement is the shellular differential rotation in the leading order of the perturbation theory. Analysis of the next-order terms yields corrections to this differential rotation. The shellular differential rotation is the only possible unstable displacement, but only for displacements ξ that do not mix orders of the expansion (i.e., do not make some next-order terms comparable to the leading-order terms).

On the other hand, Tayler's displacement mixes orders, as seen from Eq. (116): small ξ_r multiplied by k becomes of order unity. This allows Tayler's displacement to be unstable as well, even though it is not a shellular differential rotation. While we find unstable displacements of the differential-rotation type that respect the ordering, the fastest-growing among them mixes orders [see Eqs. (67) and (81), which imply $n\varepsilon \sim 1$].

In this sense our initial requirement that the leading-order term of $W[\xi]$ vanish is not compulsory for such “mixing” displacements. Similarly, Tayler's assumption that the first term in Eq. (111) must vanish is not

compulsory. Here, Tayler's method and ours share a common strategy: both rely on an ansatz (distinct in each case) to derive unstable displacements, which themselves differ fundamentally.

One of the important features of the unstable displacements is the existence of limiting values of the typical radial wave numbers k in both cases. However, these limits differ *qualitatively*. In the case of the global instability discussed in this paper, the typical value of k is limited from above. According to the results of Sec. IV E [see, in particular, Eq. (81)], the displacement is unstable, i.e., it is not stabilized by the effect of the third-order term, as long as $n \sim kR \lesssim \sqrt{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}}/\chi^{1/2} \sim \mathcal{N}/\omega_A$, or²⁴

$$k \lesssim \frac{\mathcal{N}}{\omega_A R}, \quad (124)$$

but k practically is not limited from below [it should only be $\gtrsim 1/R$; see the comment after Eq. (69)]. On the contrary, in the Tayler instability k has the lower bound, $kr \gtrsim \mathcal{N}l/\omega_A$, or equivalently, for $l \sim 1$ and $r \sim R$, $k \gtrsim \mathcal{N}/(\omega_A R)$ (at lower k the displacement is stable), but at the same time it is not limited from above. Taken together, the relevant k ranges read as

$$\frac{\mathcal{N}}{\omega_A R} \lesssim k \lesssim \infty \quad \text{Tayler instability}, \quad (125)$$

$$\frac{1}{R} \lesssim k \lesssim \frac{\mathcal{N}}{\omega_A R} \quad \text{global instability}. \quad (126)$$

It is clear that the threshold value of $\mathcal{N}/(\omega_A R)$ in these inequalities is determined only very approximately and may, in reality, differ by a factor of a few in either direction.

The properties of the global and Tayler instabilities in ideal MHD are summarized in the schematic Fig. 6 (see the solid lines there). The dashed curves account for dissipation and are discussed below in Sec. VIII B. In Fig. 6, for both types of instability, the qualitative behavior of the ideal-MHD growth rate $1/\tau_0(k)$ (defined by $\xi \propto e^{t/\tau_0}$) is shown as a function of $\log(kR)$, where k is the radial wave number. In accordance with the discussion above, the (red) solid curve for the global instability has a bell-shaped form with a maximum $1/\tau_0(k) \sim \omega_A$ near $kR \sim \mathcal{N}/\omega_A$. The Tayler instability (blue solid curve), in turn, comes into play only at $kR \gtrsim \mathcal{N}/\omega_A$ and becomes most efficient at large kR , where $1/\tau_0(k)$ approaches its asymptotic maximal value, also of order ω_A . In the figure this maximum slightly exceeds the maximum of $1/\tau_0(k)$ for the global instability, which is not required in general but is consistent with our numerical experiments performed for a set of toroidal field models.

²⁴This is why this instability was not found, for example, in Refs. [4,5,13], where the limit $k \rightarrow \infty$ was considered.

B. Dissipative effects

Note that the weaker the magnetic field, the larger the value of $\mathcal{N}/\omega_A$. Typically, $\mathcal{N}/\omega_A \gtrsim 100$ for various stellar types and magnetic field strengths. For instance, $\mathcal{N}/\omega_A \sim 100$ for magnetars, while for rotation-powered pulsars $\mathcal{N}/\omega_A \sim 10^5$. Such high values of $\mathcal{N}/\omega_A$, which imply large k [see Eqs. (125), (126), and Fig. 6], raise the question of whether dissipative processes such as, e.g., shear viscosity, magnetic diffusivity, thermal conductivity, and ambipolar diffusion, can significantly affect the development of these instabilities. A comprehensive analysis of this problem is well beyond the scope of this paper and merits a separate study (see, e.g., Refs. [8,16], where such an analysis was performed within the short-wavelength approximation in studies of the Tayler instability). Here, for illustration, we examine how shear viscosity, one of the effective dissipation mechanisms in neutron stars, could affect the instabilities under consideration. We expect magnetic diffusivity (which is more relevant for ordinary nondegenerate stars) to influence the instability in a qualitatively similar way [8,27,28].

The characteristic dissipative timescale due to shear viscosity depends on the perturbation wave number and can be estimated as

$$\tau_\nu \approx 1/(\nu k^2), \quad (127)$$

where ν is the kinematic shear-viscosity coefficient.²⁵ Shear viscosity slows the growth of the instability (either global or Tayler) but cannot suppress it completely. The corresponding instability timescale $\tau(k)$ can be written as (see Appendix C for details)

$$\tau(k) = \frac{\tau_0(k)^2}{2\tau_\nu(k)} + \tau_0(k) \sqrt{1 + \frac{\tau_0(k)^2}{4\tau_\nu(k)^2}}, \quad (128)$$

where $\tau_0(k)$ is the instability timescale in the ideal MHD (see solid lines in Fig. 6). As it should be, $\tau(k) = \tau_0(k)$ for $\tau_0(k) \ll \tau_\nu(k)$ and $\tau(k) = \tau_0^2(k)/\tau_\nu(k) \gg \tau_0(k)$ for $\tau_0(k) \gg \tau_\nu(k)$ (e.g., [16]). As follows from Eq. (128), both instabilities are slowed down by shear viscosity, but the Tayler instability is more sensitive to dissipation than the global one due to its shorter spatial scales. Since the (minimal) nondissipative timescales $\tau_0 \sim 1/\omega_A$ are similar for both instabilities (see Sec. VI), the more efficient suppression of the Tayler instability by shear viscosity makes the global instability comparatively faster and allows it to develop. Thus, if

$$\tau_\nu \left(k = \frac{\mathcal{N}}{\omega_A R} \right) \lesssim \tau_0 \left(k = \frac{\mathcal{N}}{\omega_A R} \right) \sim \omega_A^{-1}, \quad (129)$$

²⁵In Appendix B we present a more accurate expression for the dissipative timescale τ_ν .

we expect the global instability to dominate, while in the opposite limit, the Tayler instability may prevail. Note that a large dissipation, in a sense, guarantees the dominance of the global instability. However, depending on the specific field model, the global instability might still prevail even in the dissipationless limit or in the presence of only small dissipation.

For illustration, Fig. 6 qualitatively shows how viscosity modifies the ideal-MHD dependence $1/\tau_0(k)$ (solid lines) for the global (red lines) and Tayler (blue lines) instabilities. The red and blue curves $1/\tau(k)$ are plotted for three viscosity values, $\nu_0/3$ (dash-dotted), ν_0 (long-dashed), and $3\nu_0$ (short-dashed), where the reference viscosity ν_0 is chosen such that $\tau_\nu(k = \mathcal{N}/(\omega_A R)) = 1/\omega_A$. For each viscosity, $\tau(k)$ is computed from Eq. (128). Additionally, Fig. 6 shows the function $1/\tau_\nu(k)$, calculated for the same three viscosity values (see the black dash-dotted, long-dashed, and short-dashed curves, which scale as k^2). The figure indicates that as viscosity increases, the global instability becomes the increasingly dominant destabilizing mechanism. In summary, for systems with a relatively weak magnetic field [i.e., large $k = \mathcal{N}/(\omega_A R)$] and high viscosity (or high magnetic diffusivity), the global magnetic-field instability is expected to develop in the star *earlier* than the Tayler instability.

Returning to Eq. (129), it can be rewritten in the form

$$\nu \gtrsim \frac{\omega_A^2 R^2}{\mathcal{N}^2 \tau_A} \approx \frac{E_{\text{mag}}}{(\gamma_{\text{fr},0} - \gamma_{\text{eq},0}) E_{\text{grav}}} \frac{R^2}{\tau_A}. \quad (130)$$

Using values typical of neutron stars, we find that the global instability is expected to dominate over the Tayler instability when

$$\begin{aligned} \nu \gtrsim 10^9 B_{15}^3 \left(\frac{\rho}{7 \times 10^{14} \text{ g cm}^{-3}} \right)^{-3/2} \left(\frac{\mathcal{N}}{10^3 \text{ s}^{-1}} \right)^{-2} \\ \times \left(\frac{R}{12 \text{ km}} \right)^{-1} \text{ cm}^2 \text{ s}^{-1}, \end{aligned} \quad (131)$$

where $B_{15} \equiv B/(10^{15} \text{ G})$. For nondegenerate stars, the magnetic diffusivity η is a more relevant dissipative parameter, since it is usually larger than ν . At the same time, as already mentioned above, we expect that the constraints obtained for ν in this section should be equally applicable to the coefficient η . With this in mind, for Sun-like stars the inequality (130) can be written as

$$\begin{aligned} \eta \gtrsim 10^3 B_3^3 \left(\frac{\rho}{0.2 \text{ g cm}^{-3}} \right)^{-3/2} \left(\frac{\mathcal{N}}{2 \times 10^{-3} \text{ s}^{-1}} \right)^{-2} \\ \times \left(\frac{R}{0.7 R_\odot} \right)^{-1} \text{ cm}^2 \text{ s}^{-1}, \end{aligned} \quad (132)$$

where $B_3 \equiv B/(10^3 \text{ G})$ and $R_\odot$ is the solar radius. One finds that, for $B \sim 1 \text{ kG}$, the threshold value of η given by

Eq. (132) is comparable to typical values of the Spitzer magnetic diffusivity η in the radiative layers of solar-type stars (e.g., [62]).

Returning to the estimate (131) for neutron stars, it is interesting to assess how effective the viscosity ν can be in practice. Under typical neutron-star conditions, the kinematic shear viscosity is dominated by particle-particle collisions (see, e.g., [63]). As a rough estimate it can be approximated as

$$\nu \approx 3 \times 10^4 \left(\frac{T}{10^8 \text{ K}} \right)^{-5/3} \text{ cm}^2 \text{ s}^{-1}, \quad (133)$$

where T stands for the stellar temperature, and the dependence of ν on the matter density and composition is omitted. Substituting now (133) into the inequality (131), we find that particle shear viscosity effectively damps the Tayler instability for magnetic fields

$$B_{15} \lesssim 0.03 \left(\frac{T}{10^8 \text{ K}} \right)^{-5/9}. \quad (134)$$

By contrast, during the early stages of neutron-star evolution, when temperatures are still high enough that stellar matter is opaque to neutrinos, neutrino viscosity, associated with momentum transport by neutrinos, may play a significant role. Assuming nondegenerate neutrinos and degenerate other particle species, Ref. [64] obtained the following expression for the neutrino viscosity (see also [65] and the recent review [66]):

$$\nu \approx 5 \times 10^7 \left(\frac{T}{3 \times 10^{10} \text{ K}} \right) \left(\frac{\rho}{3 \times 10^{14} \text{ g cm}^{-3}} \right)^{-4/3} \text{ cm}^2 \text{ s}^{-1}, \quad (135)$$

where we omit a factor of order unity in this formula that depends on the proton fraction. Comparing (135) with the threshold in the inequality (131), we find that neutrino viscosity effectively damps the Tayler instability for magnetic fields

$$B_{15} \lesssim 0.4 \left(\frac{T}{3 \times 10^{10} \text{ K}} \right)^{1/3}. \quad (136)$$

Summarizing, in nascent neutron stars, the global instability should dominate for magnetic-field strengths given by Eq. (136) and possibly even higher, while at later evolutionary stages the corresponding threshold value is given by (134). In addition to the viscosities discussed above, turbulent (eddy) viscosity driven by momentum transport by eddies may also play a role in nascent neutron stars (see, e.g., [66]). Its magnitude is uncertain, so we do not consider it here, but it may be efficient.

C. Optimal wave numbers in the presence of dissipation

At the end of Sec. IV E, we mentioned that including dissipation (viscosity or magnetic diffusivity) in the MHD equations can substantially slow down the development of small-scale perturbations with $n \sim kR \sim \mathcal{N}/\omega_A$, which would be optimal in ideal MHD. Under these conditions, the most favorable displacements are those with smaller n (that is, with smaller radial wave numbers k), lying within the range of validity of the perturbation theory developed in Sec. IV.

In essence, this effect already follows from the discussion in the previous section. In particular, it is clearly visible in Fig. 6, where the maximum of $1/\tau(k)$ for the global instability shifts to the left, i.e., toward smaller k , as the viscosity ν increases. Below we illustrate this behavior with a simple analytic estimate.

Let us introduce the growth rates $\omega_0(k) \equiv 1/\tau_0(k)$ and $\omega(k) \equiv 1/\tau(k)$. We denote the maximum value of $\omega_0(k)$ by $\omega_{0,\text{opt}}$, and the corresponding value of k by $k_{0,\text{opt}}$. According to our previous results, we have $\omega_{0,\text{opt}} \sim \omega_A$ and $k_{0,\text{opt}} \sim \mathcal{N}/(\omega_A R)$. Similarly, we denote the maximum of $\omega(k)$ by ω_{opt} , and the corresponding value of k by k_{opt} .

To estimate the effect of viscosity, we first need an approximate form of $\omega_0(k)$. Consider an unstable mode ξ_k characterized by the radial wave number k . For such a mode, the ideal-MHD growth rate is given by the Rayleigh quotient, $\omega_0^2(k) = -W[\xi_k]/I[\xi_k]$ (see Sec. III A). Using the approximate functional $W[\xi]$ from Sec. IV E, the functional $I[\xi]$ from Sec. V, and the estimate $a'(r) \sim ka(r)$ (Sec. IV C), we obtain the following simple k dependence: $\omega_0^2(k) = \alpha_1 k^2 - \alpha_2 k^4$ [cf. Eq. (80), where an analogous dependence on $n \sim kR$ arises for the same reason]. This form reflects the competition between the destabilizing contribution, which grows with k , and the stabilizing contribution associated with excessively rapid radial variation, which arises from the third-order term discussed in Sec. IV E and scales as k^4 . Here α_1 and α_2 are some positive constants. It is convenient to re-express them in terms of $\omega_{0,\text{opt}}$ and $k_{0,\text{opt}}$, which gives

$$\omega_0^2(k) = \omega_{0,\text{opt}}^2 \left(2 - \frac{k^2}{k_{0,\text{opt}}^2} \right) \frac{k^2}{k_{0,\text{opt}}^2}. \quad (137)$$

This expression reaches its maximum value $\omega_{0,\text{opt}}^2$ at $k = k_{0,\text{opt}}$.

Let us now include viscosity. Rewriting Eq. (128) in terms of $\omega(k)$, $\omega_0(k)$, and $\omega_\nu(k) \equiv 1/\tau_\nu(k) = \nu k^2$, we obtain $\omega(k) = -\omega_\nu(k)/2 + \sqrt{\omega_\nu(k)^2/4 + \omega_0^2(k)}$. Maximizing this expression with respect to k , we find that its maximum is

$$\omega_{\text{opt}} = \frac{2\omega_{0,\text{opt}}^2}{2\omega_{0,\text{opt}} + \nu k_{0,\text{opt}}^2}, \quad (138)$$

and is attained at $k = k_{\text{opt}}$, where

$$k_{\text{opt}} = k_{0,\text{opt}} \sqrt{\frac{2\omega_{0,\text{opt}}}{2\omega_{0,\text{opt}} + \nu k_{0,\text{opt}}^2}}. \quad (139)$$

Thus, as the viscosity increases, both the optimal growth rate ω_{opt} and the optimal wave number k_{opt} decrease relative to their nondissipative counterparts, $\omega_{0,\text{opt}} \sim \omega_A$ and $k_{0,\text{opt}} \sim \mathcal{N}/(\omega_A R)$. In particular, in the large-viscosity regime, $\nu \gg 2\omega_{0,\text{opt}}/k_{0,\text{opt}}^2$, which coincides with the condition for the global instability to dominate over the Tayler instability [see Eq. (130)], we have

$$\omega_{\text{opt}} \approx \frac{2\omega_{0,\text{opt}}^2}{\nu k_{0,\text{opt}}^2} \ll \omega_{0,\text{opt}}, \quad k_{\text{opt}} \approx \sqrt{\frac{2\omega_{0,\text{opt}}}{\nu}} \ll k_{0,\text{opt}}. \quad (140)$$

Note that the inequality $k_{\text{opt}} \ll k_{0,\text{opt}}$ implies that the stabilizing third-order term $\propto k^4$ in the expression (137) for $\omega_0(k)$ can be neglected. In this regime, the ratio of the viscous term to the Lorentz force perturbation δF_L in the Euler equation for the optimal mode with $k = k_{\text{opt}}$ can be estimated as

$$\frac{\text{Viscous term}}{\text{Lorentz force perturbation}} \sim \frac{\nu \omega_{\text{opt}} k_{\text{opt}}^2}{\omega_A^2} \sim 2 \frac{2\omega_{0,\text{opt}}}{\nu k_{0,\text{opt}}^2} \ll 1. \quad (141)$$

In energy terms, the smallness of this ratio means that the magnetic energy functional $W^{\text{mag}}[\xi]$ is much larger than the corresponding viscous dissipation term.²⁶ Combined with the inequality $k_{\text{opt}} \ll k_{0,\text{opt}}$ [see Eq. (140)], this shows that, for the optimal ($k = k_{\text{opt}}$) mode in the presence of viscosity, perturbation theory in the parameters χ and ε can be applied. According to estimate (141), the viscous term does not contribute at first order in this perturbative expansion. Thus, at this order, the energy functional $W[\xi]$ [see Eq. (44)] is determined solely by its magnetic part $W^{\text{mag}}[\xi]$, yielding the same term $\chi \alpha_{01}$ as in the first-order perturbation theory developed in Sec. IV B. We therefore conclude that, although viscosity determines the optimal radial scale of the fastest-growing mode, it does not modify the leading-order form of its Lagrangian displacement. Accordingly, in the high-viscosity regime, this displacement should still be well described by the first-order perturbation theory of Sec. IV, that is, it should remain close to *shellular differential rotation*. This conclusion is further supported by the numerical simulations presented in the next section.

IX. SIMULATIONS

To illustrate our theoretical results presented in the previous sections, we carried out a series of numerical

²⁶Indeed, the perturbed Lorentz force δF_L is related to the magnetic energy functional $W^{\text{mag}}[\xi]$ by Eq. (D11) in Appendix D.

TABLE II. Parameters for the simulations presented in this paper. All simulations are performed in a computational box with side length $L_{\text{box}} = 3R$. The adiabatic index is $\gamma_{\text{fr}} = 5/3$, and the equilibrium index is $\gamma_{\text{eq}} = 4/3$. See text for details.

Simulation	Resolution	$E_{\text{grav}}/E_{\text{mag}}$	$\mathcal{N}/\omega_A$	$\nu/[R^2(G\rho_c)^{1/2}]$	$\tau_A/(G\rho_c)^{-1/2}$	$\tau_s/(G\rho_c)^{-1/2}$
I	288^3	358	11	10^{-5}	27	3.9
II	288^3	358	11	10^{-4}	27	3.9
III	288^3	358	11	2×10^{-4}	27	3.9
IV	128^3	806	16	10^{-5}	41	3.9
V	128^3	358	11	10^{-5}	27	3.9
VI	128^3	159	7.3	10^{-5}	18	3.9
VII	128^3	89	5.4	10^{-5}	13	3.9

simulations using the PENCIL Code²⁷ [51], a high-order finite-difference code for compressible MHD flows. The code uses sixth-order centered spatial derivatives and a third-order Runge-Kutta time-stepping scheme to solve the MHD equations given in Appendix A of [56]. The simulations are performed in a cubic box of side $L_{\text{box}} = 3R$, centered on the star, with a Cartesian grid with equally spaced points and periodic boundary conditions. We verified that our results are insensitive to the box size; increasing the box size at fixed point density does not change them.

We describe stellar matter as an ideal monatomic gas, with the density and specific entropy being the independent variables. The adiabatic index then equals $\gamma_{\text{fr}} = 5/3$. For the initial nonmagnetized background configuration we choose a polytropic relation between the gas pressure and density, $p \propto \rho^{\gamma_{\text{eq}}}$, with $\gamma_{\text{eq}} = 4/3$. We define the stellar surface as the sphere at which the matter density ρ equals $2 \times 10^{-3}\rho_c$, where ρ_c is the central density. The stellar radius R then equals the radius of this sphere. We assume that the star is surrounded by a low-density ideal-gas atmosphere with a uniform temperature smoothly matching with its value at the stellar surface. The magnetic diffusivity is set to 0 inside the star and has a finite value outside, with a smooth transition between these two regions.

Starting from a nonmagnetized equilibrium stellar model, we introduce a magnetic field, which perturbs the star from equilibrium. To obtain the initial magnetic equilibrium, we evolve the system for several sound-crossing timescales with the magnetic field held fixed, so that the hydrodynamical forces relax against the Lorentz force. The perturbations of the gravitational potential by the magnetic field are neglected in all our simulations to reduce the numerical computation time (Cowling approximation). That is, the gravitational potential is calculated for the nonmagnetized spherical background stellar configuration and fixed along the simulations. For more details on the initial setup, we refer the reader to Refs. [56,67].

The evolution and stability of purely toroidal, purely poloidal, and mixed magnetic-field configurations were investigated in detail in [56]. Here we use the magnetic-field model “Field I” from [56], with the poloidal component set to zero (note that the model “Field I” was originally proposed in [15]). The amplitude of the toroidal component is adjusted so that the ratio of gravitational to magnetic energy, $E_{\text{grav}}/E_{\text{mag}}$,²⁸ equals 806, 358, 159, or 89 (see the third column of Table II and the corresponding values of $\mathcal{N}/\omega_A$ in the fourth column). These values are clearly far too low to be representative of real stars. However, adopting realistic magnetic field values is technically challenging for MHD simulations. A color map of the chosen field model is shown in the left panel of Fig. 7. The right panel of the figure shows the value of the c_m coefficient [see Eq. (114)]. According to [2,7] [see also Sec. VIII A], in the regions of the star where $c_m < 0$, the unstable (predominantly tangential to spherical surfaces) displacements associated with the Tayler instability are expected to develop.

In what follows, we discuss seven representative simulations (see the Supplemental Material [68] and the snapshots in Fig. 9). They differ in the numerical grid resolution, in the magnetic field amplitude, and in the values of the shear-viscosity coefficient ν . Namely, the first three simulations (simulations I, II, and III) have high spatial resolution (288^3) and differ only in the value of the shear viscosity, whereas the last four (simulations IV, V, VI, and VII) have lower resolution (128^3) and differ in the magnetic-field amplitude. The corresponding parameters are listed in Table II (see the columns 2–5). We define the Alfvén timescale τ_A , given in column six, as $\tau_A = R\sqrt{4\pi\bar{\rho}}/B_{\text{rms}}$, where $\bar{\rho}$ is the average density (stellar mass over stellar volume) and B_{rms} is the root-mean square magnetic field in the star.²⁹ In column seven, we give τ_s , the sound crossing time across the distance R . It is calculated as $\tau_s = R/c_{\text{s,rms}}$, where $c_{\text{s,rms}}$ is the root-mean square value of

²⁸For the definition of E_{grav} and E_{mag} we use equations (19) and (22) of [67], replacing the integration over the box by the integration over the stellar volume.

²⁹Note that this definition differs from that given by Eq. (97).

²⁷<https://pencil-code.nordita.org>

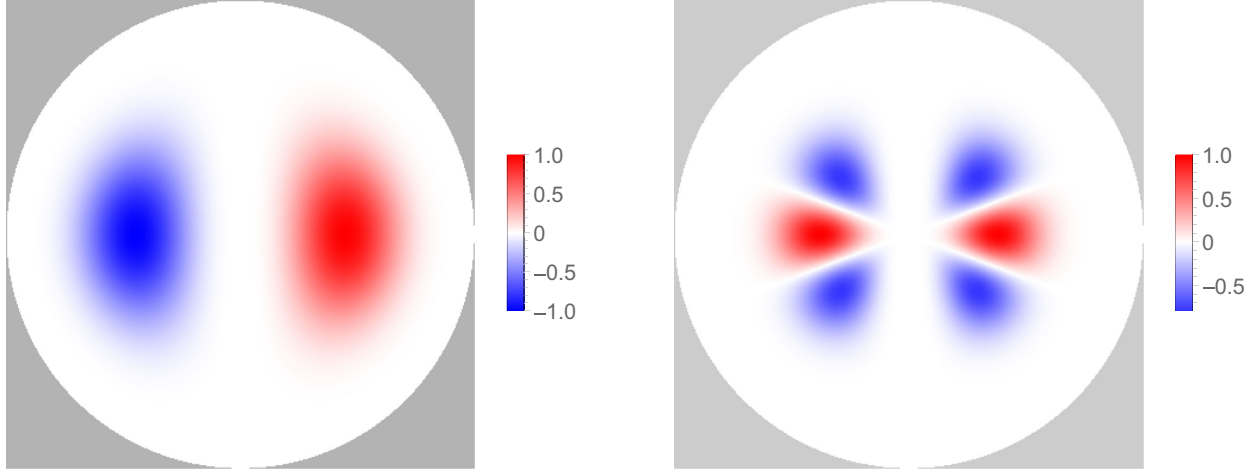


FIG. 7. Meridional cuts of the star showing color maps of the initial magnetic field used in our simulations (left) and the corresponding coefficient c_m (right). Both quantities are normalized to their respective maxima. The Tayler instability is expected to develop in the blue regions of the right panel.

the sound speed $c_s = \sqrt{\gamma_{\text{fr}} P / \rho}$, averaged over the stellar volume.

In addition to dissipation due to shear viscosity ν , MHD motions also experience numerical dissipation (see, e.g., [69] and references therein). By analyzing the evolution of the magnetic, kinetic, gravitational, and thermal energies during the simulations, we found that numerical resistivity can be neglected, whereas numerical viscosity may play a role. To gain qualitative insight into the contribution of numerical viscosity, we ran sound-wave tests in nonmagnetized homogeneous matter in a box, with the shear-viscosity term removed from the equations. Varying the grid resolution and the sound wave number k , we found that the decrement $1/\tau_{\text{num}}$ of the sound-wave velocity amplitude, $u = u_0 \exp(-t/\tau_{\text{num}})$ (where u_0 is a small initial perturbation), for the adopted numerical scheme is given by

$$\frac{1}{\tau_{\text{num}}(k)} \approx 9.6 \times 10^{-6} \left(\frac{N_{\text{gp}}}{96} \right)^{-3} \left(\frac{kR}{\pi} \right)^4 \frac{1}{\tau_s}. \quad (142)$$

This is consistent with the third-order scheme of the Pencil code [51]. Here, N_{gp} is the number of grid points *per length* R . In our magnetic simulations, N_{gp} is either $288/3 = 96$ (for the simulations I, II, and III) or $128/3$ (for the simulations IV, V, VI, and VII). Finally, kR/π is the number of sound wave zeros per length R . In the remainder of this section, instead of the perturbation wave number k , we will use the typical length scale d . For harmonic perturbations it is defined as $d = \pi/k$, and equals half the perturbation wavelength. As follows from Eq. (142), smaller length scales d and coarser grid resolutions result in faster numerical dissipation. Note, however, that simulations VI and VII are likely subject to different numerical dissipation, since a different numerical scheme (an upwind

scheme) was employed in these simulations for numerical stability when computing spatial derivatives.

Figure 8 shows d as a function of time for the simulations I–VII. We estimate the typical value of d for each simulation using the following formula, averaging the length scale over the meridional cuts shown in Fig. 9:

$$d \equiv \pi \frac{\int_{0.1}^{0.8} \int_0^\pi (|v_\theta| + |v_\phi|) r d\theta dr}{\int_{0.1}^{0.8} \int_0^\pi r (|\frac{\partial(v_\theta/r)}{\partial r}| + |\frac{\partial(v_\phi/r)}{\partial r}|) r d\theta dr}. \quad (143)$$

The radial integration is restricted to $0.1 \leq r \leq 0.8$, excluding the central region, where the instability does not develop and the coordinate singularity may affect the estimate, as well as the outer layers, where residual postrelaxation noise is still present.

Figure 8 shows that the value of d is rather noisy and therefore provides only a rough estimate of the length scale of the most unstable perturbation. The least viscous simulation I exhibits the instability on the smallest scale. Just as increasing the shear viscosity leads to an increase in the characteristic scale (see the curves for simulations II and III), increasing the numerical viscosity (i.e., lowering N_{gp} ; simulations IV–VII) also leads to a larger characteristic scale than in simulation I.

Now, let us analyze the simulations. Simulation I (see the top left panel of Fig. 9) shows a clear onset of the Tayler instability in the stellar regions with $c_m < 0$, implying that, in the low-viscosity regime (including both numerical and shear viscosity), the Tayler instability grows faster than the global one.

As discussed in Sec. VIII, the typical stellar values of $\mathcal{N}/\omega_A$ are large, usually exceeding ~ 100 . Consequently, the Tayler instability in real stars develops on very small spatial scales in the radial direction, $d \sim \pi/k \lesssim \pi R \omega_A / \mathcal{N} \lesssim 0.03R$. Under these conditions, there exists

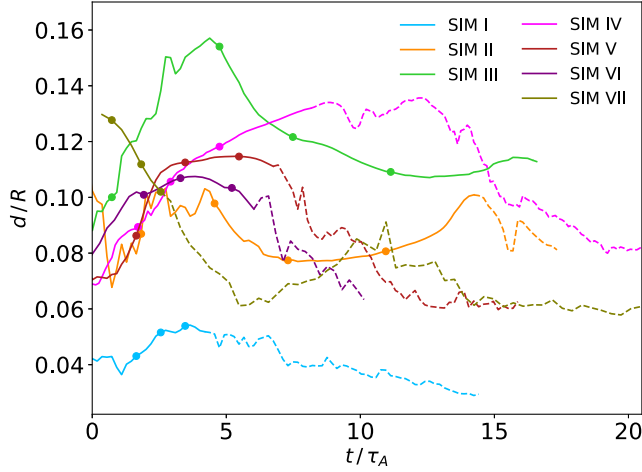


FIG. 8. Typical length scale d as a function of time for simulations I–VII. The solid lines correspond to the linear stages of the instability, while the dashed lines indicate the subsequent nonlinear stages. The circles correspond to the snapshots shown in Figs. 11 and 12.

a value of shear viscosity that efficiently suppresses perturbations at $d \lesssim 0.03R$, while still allowing the global instability to grow at scales $0.03R \ll d \ll R$. Our estimates in Sec. VIII indicate that such values of the shear viscosity can be reached in stars. To illustrate this situation, we ran simulations II and III with the same parameters as in simulation I, but with an increased shear viscosity ν . Unfortunately, for technical reasons, our simulations require unrealistically strong magnetic fields, corresponding to $\mathcal{N}/\omega_A \sim 16, 11, 7.3$, or 5.4 ; see Table II. This allows the Tayler instability to develop on larger length scales, while narrowing the range of length scales available to the global instability [see inequalities (125) and (126)]. Under these conditions, viscosities sufficient to damp the Tayler instability also suppress the global instability across nearly the entire range of accessible scales. Consequently, the snapshots of simulations II and III (middle and lower left panels of Fig. 9) represent a mixture of the Tayler and global instabilities, with the latter contributing more prominently in the more viscous simulation III. Indeed, the global instability manifests itself through the development of increasingly pronounced shellular rotation about the x axis, which is perpendicular to the plane of the figure and passes through the center of the star. The presence of the global instability is also corroborated by the behavior of the characteristic parameter $\mathcal{R}_x$, discussed below in this section.

To artificially reproduce the suppression of the Tayler instability and isolate the global instability in its pure form, we carried out simulation V, which had the same $E_{\text{grav}}/E_{\text{mag}} = 358$ as simulations I–III, but a lower grid resolution (128^3). A snapshot from this simulation is presented in the right-hand panel of the second row of Fig. 9. The plane of the figure is aligned with the plane of

the unstable displacement (shellular differential rotation). One clearly sees colored concentric circles in the right subpanel, corresponding to the spherical shells participating in the shellular rotation, a typical characteristic feature of the global instability (see Sec. VIII C). These circles are much more pronounced than similar structures in simulations II and III. According to Eq. (142), a lower grid resolution results in a larger numerical viscosity. Therefore, for the adopted parameters, numerical viscosity suppresses the entire k range relevant to the Tayler instability, while decreasing sharply toward lower k values relevant to the global instability. This mimics the situation described above in this section, in which the Tayler instability is completely suppressed by shear viscosity, whereas the global instability is not.

The sensitivity of the instability simulations to the magnetic field strength is demonstrated by simulations IV–VII, carried out for $\mathcal{N}/\omega_A$ equal to, respectively, 16, 11, 7.3, and 5.4, with all other parameters kept the same (see Table II). The results are compared with those of the fiducial simulation V, for which $\mathcal{N}/\omega_A = 11$. Increasing the magnetic field relative to its value in simulation V (simulations VI and VII) strengthens the Lorentz force and thus reduces the role of viscosity. Moreover, as the magnetic field increases, the threshold wave number, $\mathcal{N}/(\omega_A R)$, decreases [see Eqs. (125) and (126)]. Consequently, the range of k relevant to the Tayler instability shifts to lower values, where dissipation is weaker. This makes the Tayler instability more likely to dominate over the global one, as seen in the simulations: simulation VI exhibits the global instability with some admixture of Tayler instability, rather than the purely global instability found in simulation V, while simulation VII, with an even stronger magnetic field, shows a purely Tayler instability. By contrast, reducing the magnetic field (simulation IV) makes the onset of the Tayler instability even less favorable, and in this case we observe a purely global instability.

Figure 10 shows the time evolution of the kinetic energy (in units of the initial magnetic energy) for the simulations discussed above (solid lines). The dotted lines indicate exponential fits to the linear stages. Simulations I, IV, V, VI, and VII exhibit a clear linear growth with a constant rate, corresponding, respectively, to the development of either Tayler or global instabilities. By contrast, in simulations II and III the slope during the linear stage changes with time: the instability initially grows with a steep slope, which later becomes flatter. We interpret this behavior as follows. The initial steep slope corresponds to the more rapidly growing Tayler instability, which saturates at some point (probably because of the generated poloidal magnetic field that suppresses it). After that, the initially slower global instability, which may be further slowed down by the poloidal field, takes over, leading to the subsequent flattening of the slope. The corresponding instability

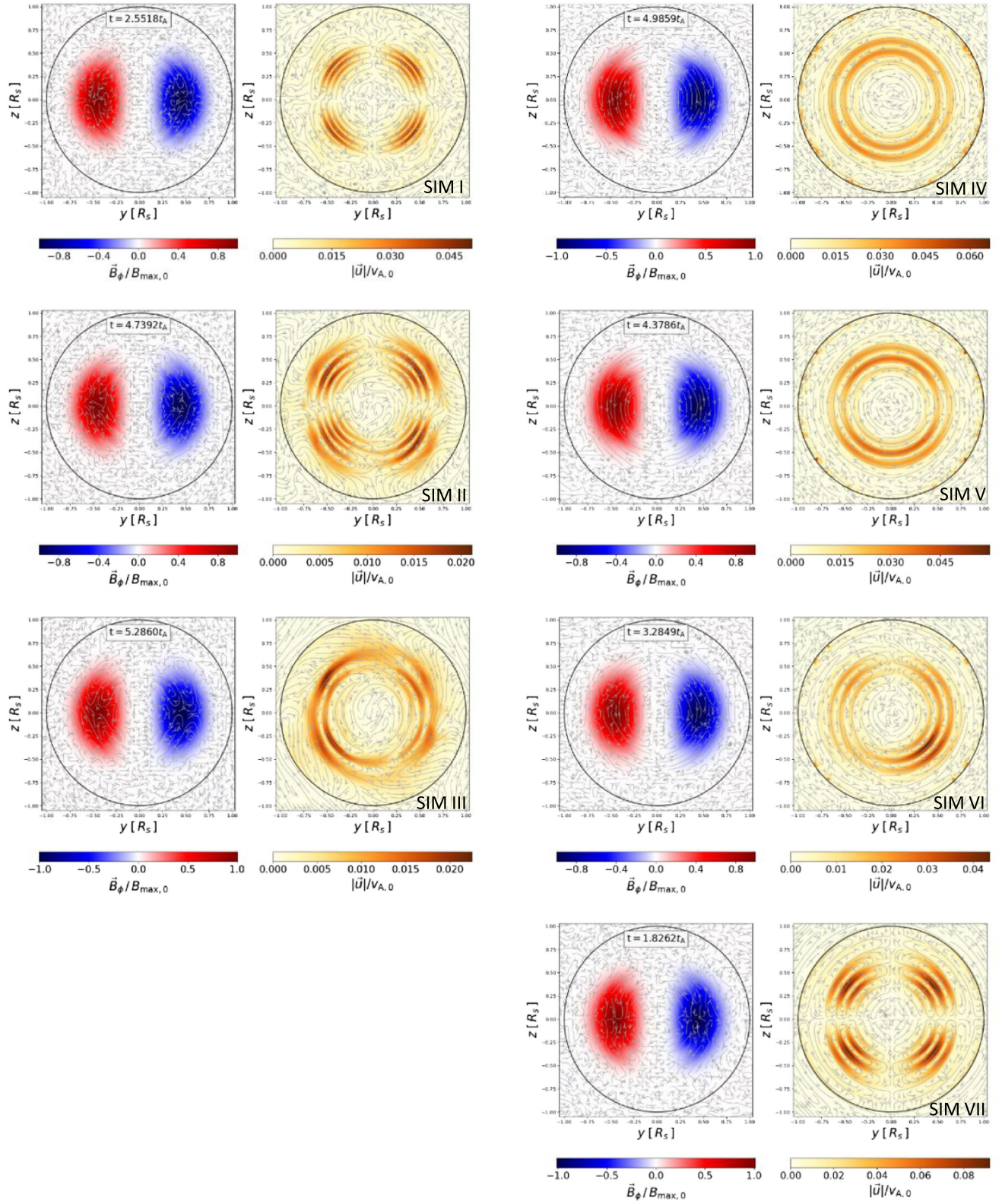


FIG. 9. Snapshots of the seven simulations. Simulations I, II, and III are displayed in the left column, and simulations IV, V, VI, and VII in the right column, ordered from top to bottom. All panels show meridional cuts of the star. The x axis, perpendicular to the plane of the figure, is selected to maximize in-plane displacements along the θ direction. The left subpanel of each panel shows the magnetic field. The colors represent the toroidal-field magnitude (more precisely, its x projection) normalized to its initial amplitude; arrows indicate the small, noisy poloidal component. The times for each snapshot are listed in the left subpanels in units of the initial Alfvén time. The right subpanel of each panel shows the velocity field. Colors indicate its magnitude in units of the Alfvén speed; arrows indicate its direction.

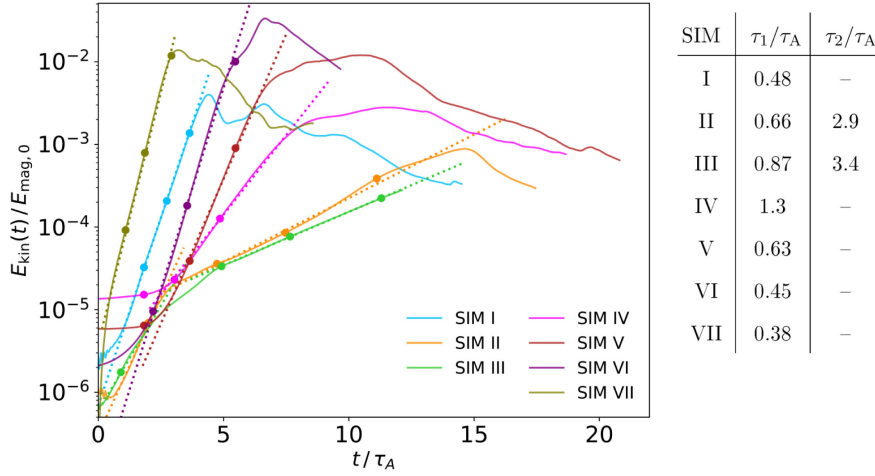


FIG. 10. Evolution of the kinetic energy, normalized to the initial magnetic energy, as a function of time, normalized to the Alfvén time. The seven solid lines correspond to the seven simulations. The dotted lines indicate exponential fits to the linear growth stages of the instability. For simulations II and III, the two growth stages are fitted separately. The table lists the corresponding growth times inferred from these fits. The circles correspond to the snapshots shown in Figs. 11 and 12.

timescales inferred from the fits are listed in the table next to the figure. They are of the order of the Alfvén time. These timescales, however, are affected by both numerical and shear viscosity. When the relevant viscous timescales are known, Eq. (128) can be used to infer the corresponding ideal-MHD instability timescales. We apply this procedure to the initial stages of simulations I–V, but not to the later stages of simulations II and III, since those stages may already be affected by the generated poloidal field, nor to simulations VI and VII, since τ_{num} is unknown in these simulations.

Figure 8 allows us to (very roughly) estimate the typical length scale for a given simulation; the resulting values for simulations I–V are listed in column three of Table III. Using this length scale, we evaluate the numerical and shear viscous timescales from Eqs. (142) and (127), respectively. Their ratio, $\tau_\nu/\tau_{\text{num}}$, is shown in column four, while the combined viscous timescale, $\tau_{\text{visc}} = (1/\tau_\nu + 1/\tau_{\text{num}})^{-1}$, is given in column five. The corresponding ideal-MHD timescale, τ_0 , inferred from Eq. (128) with τ_ν replaced by τ_{visc} , is listed in column six.

It is clear that these estimates should be treated with considerable caution. One of the main reasons is that the

perturbations excited in the simulations span a range of length scales, whereas our procedure replaces this range by a single representative value. This simplification may affect the inferred timescales. Still, column four indicates that shear viscosity is unimportant for simulations I, IV, and V, but contributes for simulations II and III, as expected. Moreover, in all cases, it appears that $\tau_0(k = \pi/d) \sim \tau_{\text{visc}}(k = \pi/d)$, suggesting that the perturbation length scale d is likely set by viscosity: perturbations with smaller length scales are viscously damped, whereas those with larger length scales grow more slowly even without dissipation (see Sec. VIII B). In the case of simulation I, we expect that increasing the grid resolution will increase τ_{num} and shift the Tayler instability toward smaller length scales d (larger k).

To characterize the nature of the instabilities more quantitatively, we introduce the parameter $\mathfrak{R}_x$, which measures how close the unstable perturbations are to shellular rotation.

To evaluate $\mathfrak{R}_x$, we divide the star radially into spherical shells of thickness $\sqrt{(\gamma_{\text{fr}} - \gamma_{\text{eq}})E_{\text{grav}}/E_{\text{mag}}}R/5 \approx (\omega_A/\mathcal{N})R/5$. Here $(\omega_A/\mathcal{N})R$ is an approximate characteristic radial scale of the instability (more precisely, it provides a lower estimate for the global instability and an upper estimate for the Tayler instability). The shells should be thinner than this scale. At the same time, choosing a thickness below the grid resolution is not meaningful. We therefore adopt $(\omega_A/\mathcal{N})R/5$ as a convenient shell thickness. For each shell, we compute the following vector:

$$\mathfrak{R} \equiv \int_{\text{shell}} \frac{\mathbf{r} \times \mathbf{u} dV}{r_{\text{shell}} u_{\text{av}} V_{\text{shell}}}, \quad (144)$$

where $\mathbf{r}$ is the radius vector, r_{shell} is the characteristic radial coordinate of the shell, V_{shell} is the shell volume, $\mathbf{u}$ is the fluid velocity, and u_{av} is the shell-averaged absolute value of the

TABLE III. Inferred viscous and ideal-MHD timescales for simulations I–V. See text for details.

SIM	τ_1/τ_A	$d = \pi/k$	$\tau_\nu/\tau_{\text{num}}$	$\tau_{\text{visc}}/\tau_A$	$\tau_0(k = \pi/d)/\tau_A$
I	0.48	0.05	10	0.17	0.19
II	0.66	0.09	0.31	0.46	0.34
III	0.87	0.13	0.074	0.59	0.43
IV	1.3	0.12	20	0.35	0.44
V	0.63	0.1	28	0.25	0.26

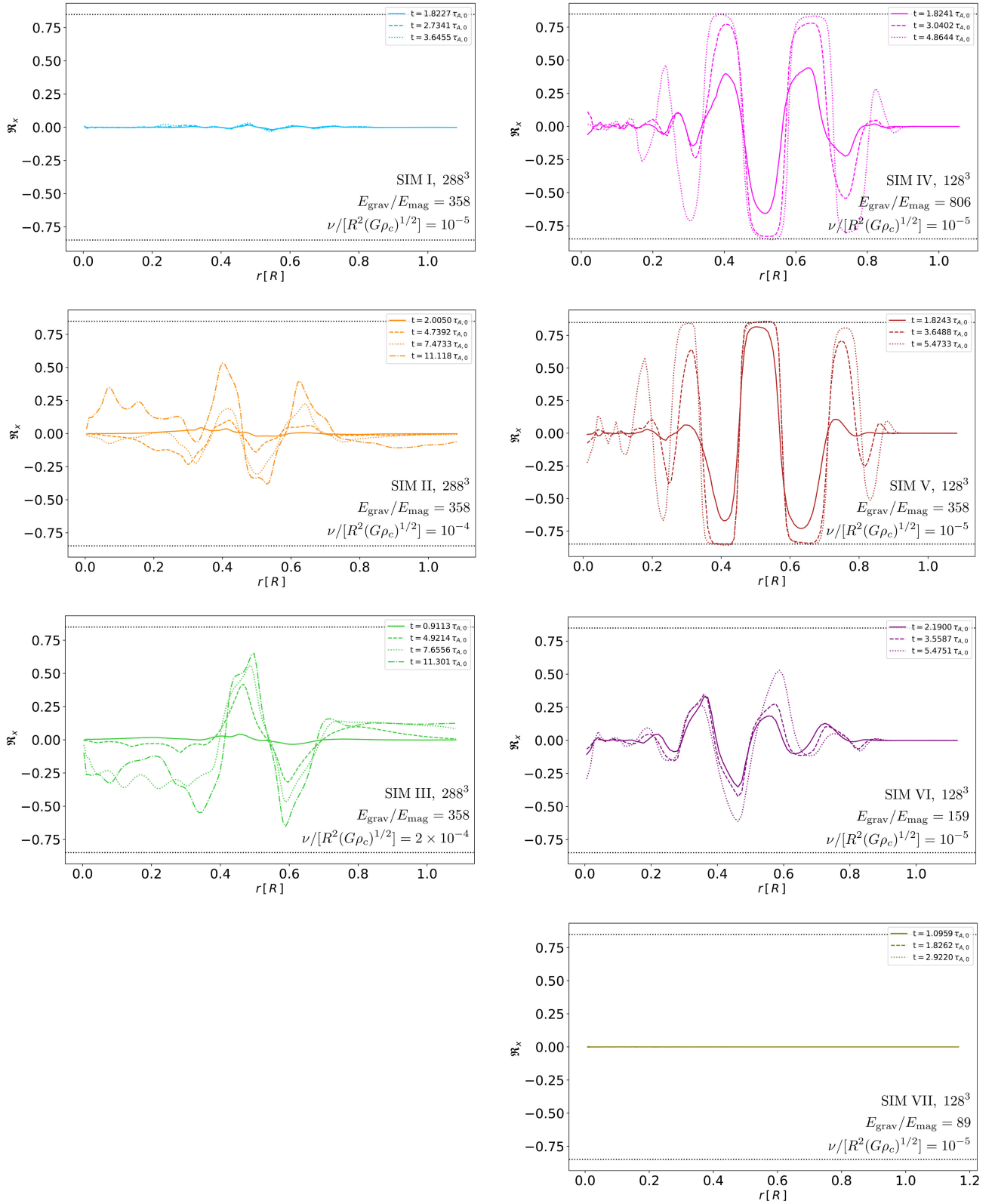


FIG. 11. $\mathfrak{R}_x$ as a function of radial coordinate for the simulations discussed in the paper. In each panel $\mathfrak{R}_x(r)$ is plotted for several snapshots (see labels in the figure and the circles in Figs. 8 and 10).

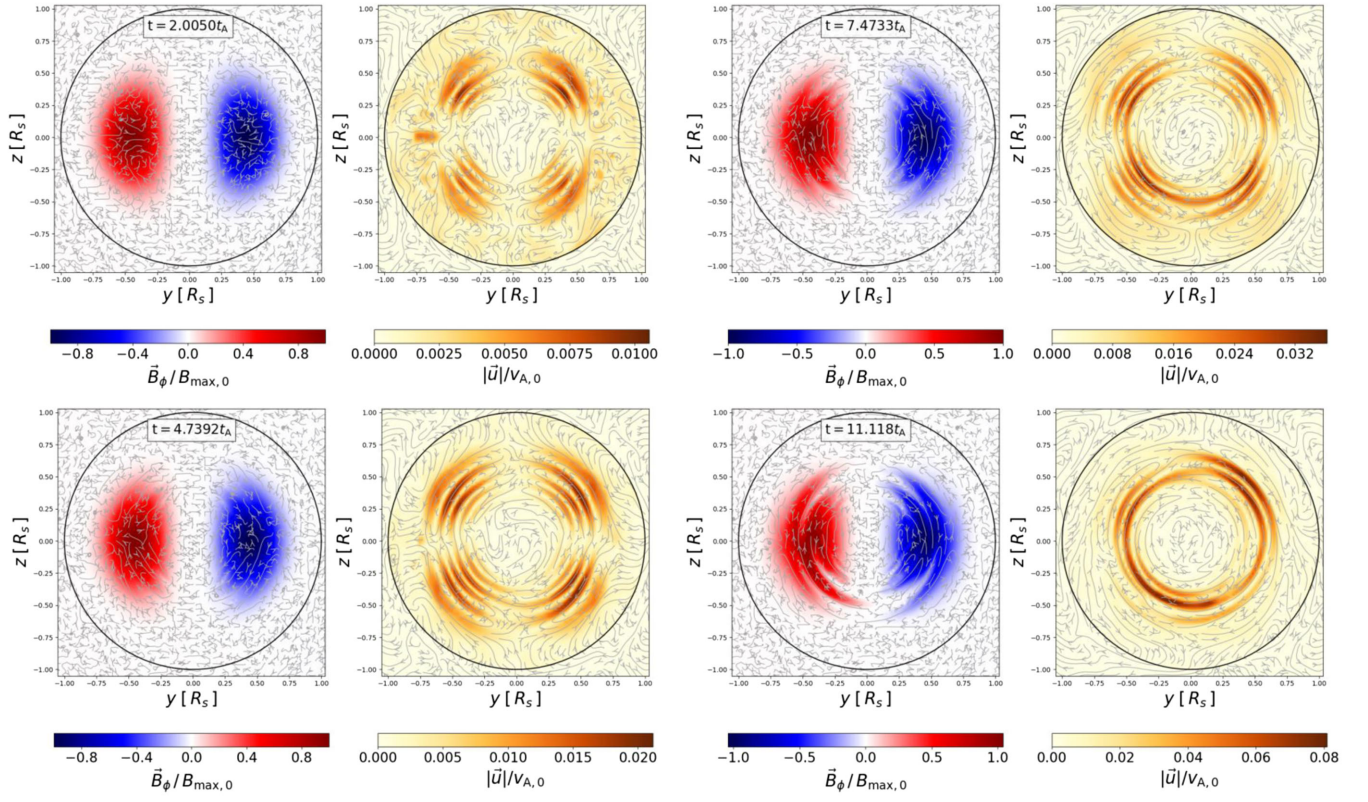


FIG. 12. Snapshots of simulation II. Notations are the same as in Fig. 9.

velocity, $u_{av} \equiv 1/V_{\text{shell}} \int_{\text{shell}} |\mathbf{u}| dV$. In the case of an ideal shellular rotation, this vector should be aligned in the same direction in all shells and should have magnitude $8/(3\pi)$ (in the limit of an infinitesimally thin shell). We then define the parameter $\mathfrak{R}_x$ as the projection of $\mathfrak{R}$ onto the x axis. Since we choose the x and y axes so as to localize the motions mainly in the $y - z$ plane, we expect $\mathfrak{R}_x$ to approach $8/(3\pi)$ in the case of global instability. For the Tayler instability, by contrast, $\mathfrak{R}_x$ is expected to be smaller.

Figure 11 shows $\mathfrak{R}_x$ as a function of radial coordinate for simulations I–VII. Each panel corresponds to one particular simulation and contains several curves corresponding to different moments in time. These moments are also marked by circles in Figs. 8 and 10. Horizontal dotted lines indicate $\mathfrak{R}_x = \pm 8/(3\pi)$. One can see that in the case of the Tayler instability $\mathfrak{R}_x$ remains close to zero [see panels, which show simulations I and VII, corresponding to the pure Tayler instability]. By contrast, in the case of shellular rotation [see panels, corresponding to simulations IV and V], $\mathfrak{R}_x$ reaches the value $\pm 8/(3\pi)$. In this case, the antinodes of the function $\mathfrak{R}_x$ correspond to the regions where the velocities are maximal, and in these regions $\mathfrak{R}_x$ approaches $\pm 8/(3\pi)$. Meanwhile, the nodes of $\mathfrak{R}_x$ correspond to the nodes of the shellular rotation. In these regions, shellular rotation does not contribute to $\mathfrak{R}_x$, and the value of $\mathfrak{R}_x$ is determined by higher-order velocity corrections and numerical noise.

For simulations II and III, $\mathfrak{R}_x$ illustrates the transition from Tayler-type perturbations to the global instability discussed above. In both simulations, the instability initially develops through Tayler-like displacements, and the parameter $\mathfrak{R}_x$ remains small. The Tayler instability then saturates (probably due to the generation of a poloidal field component), after which the global instability, although initially slower, takes over, so that $\mathfrak{R}_x$ tends toward $\pm 8/(3\pi)$. This interpretation is also supported by Fig. 10 (see the discussion above), where the initial steep slope associated with the Tayler instability is followed by a flatter one associated with the global instability.

To further illustrate the transition from one type of instability to the other, Fig. 12 shows snapshots from simulation II corresponding to the time moments indicated in the left middle panel of Fig. 11. One can see that the Tayler instability gradually gives way to the global instability, in agreement with the discussion above.

X. SUMMARY AND CONCLUSION

In this paper we analyzed the stability of toroidal magnetic fields in stably stratified nonrotating stars, using both analytical methods based on the energy principle of Bernstein *et al.* [50] and numerical simulations with the PENCIL Code [51]. Although most of the paper discusses stably stratified stellar matter, some of our results also

apply to the barotropic case; see the comment after Eq. (64) in Sec. IV C.

Analytically, we showed that, in addition to the well-known Tayler instability [2], there exists a distinct class of unstable displacements that destabilize a toroidal field of essentially arbitrary (physically acceptable) form. Although both instabilities grow on the Alfvén timescale, they are qualitatively different. The Tayler instability is generally local in the r and θ directions, i.e., it develops in specific regions of the star. In contrast to the Tayler instability, the instability we identify is global and, under certain conditions discussed in Sec. IV (see also Secs. VIII C and IX), may reveal itself by generating large-scale shellular differential rotation about an arbitrary axis perpendicular to the symmetry axis of the toroidal field. The corresponding Lagrangian displacement takes the form [see Eq. (55)] $\xi_{\text{rot}}(r, \theta) = a(r)\hat{\mathbf{O}} \times \mathbf{r}$, where $\hat{\mathbf{O}}$ is a unit vector that sets the rotation-axis direction. In Sec. IV E we propose an approximate variational method to compute the function $a(r)$ that describes the differential rotation.

Another characteristic difference between the two instabilities is the scale of inhomogeneity in the radial coordinate r that arises as the instability develops. The Tayler instability is characterized by small radial scales with characteristic wave numbers $\mathcal{N}/(\omega_A R) \lesssim k \lesssim \infty$ [see Eq. (125)]. In turn, the instability we find corresponds to smaller k , $1/R \lesssim k \lesssim \mathcal{N}/(\omega_A R)$ [see Eq. (126)]. Consequently, the Tayler instability is more readily damped by dissipative effects (Sec. VIII). Depending on the dissipation level, either instability can dominate. For example, according to our estimates, in young neutron stars where dissipation is governed by neutrino shear viscosity, the global instability is more likely to prevail.

Our general conclusions about the global instability were illustrated by (in)stability calculations for a star with magnetic field $\mathbf{B} = \mathcal{L}pr \sin \theta \mathbf{e}_\phi$ within the Bernstein *et al.* energy principle (Sec. VI). For this field we numerically determined the function $a(r)$ and obtained the growth time of the instability, following the analytical approach developed in Sec. IV.

In addition, we independently verified the existence of the global instability in a numerical experiment. To this end, we performed three-dimensional numerical simulations using the PENCIL Code of the nonlinear, dissipative MHD equations that describe an initially equilibrated magnetized star with a toroidal field (Sec. IX). We showed that, as anticipated, depending on the dissipation strength, the simulations showed either Tayler-dominated or global-dominated behavior, with some runs exhibiting a mixture of the two. For the chosen magnetic-field model, we found that the Tayler instability prevails for weak dissipation. However, for strong dissipation, we observed suppression of Tayler-type perturbations and the

dominance of the new global instability, with the emergence and growth of the analytically predicted large-scale modes corresponding to shellular differential rotation.

We also estimated the strength of the poloidal magnetic field required to stabilize the system by opposing the growth of the unstable shellular differential rotation (Sec. VII). We showed that it constitutes only a small fraction of the toroidal field, and that the corresponding criterion is consistent with estimates reported in the literature [15,56,61], although our criterion is more quantitative.

Frequently, numerical modeling of stellar magnetic-field evolution employs approximations that substantially simplify the governing MHD equations. We verify (see Appendix D) that the global instability persists under four widely used approximations: the equilibrium Cowling approximation, the anelastic approximation, the Lantz-Braginsky-Roberts (LBR) formulation [70–72], and the Boussinesq approximation.

Taken together, the results of this work indicate that, in addition to the classical Tayler instability, stars may be subject to a new current-driven $m = 1$ global instability that can become important and possibly dominant in some astrophysically relevant regimes. This may call for a revision of theoretical models of stellar magnetism, in particular the Tayler-Spruit dynamo scenario, in which the presence of an additional class of unstable modes may significantly affect angular momentum transport and magnetic-field evolution in stellar interiors. A natural next step is therefore to understand how these conclusions are modified by rotation.

It is clear that rotation should substantially affect the instability considered here once the Coriolis force in the equations of motion becomes comparable to the Lorentz force. A simple estimate shows (see, e.g., Refs. [12,27]) that, in the absence of dissipative effects, this occurs when $\Omega \gtrsim \omega_A$. Therefore, for objects that do not satisfy this inequality (for example, magnetars), the results obtained in the present work are likely to remain qualitatively applicable.

By contrast, for sufficiently rapidly rotating objects, $\Omega \gtrsim \omega_A$, which include most stars, a systematic treatment of rotation is required. In this case, the analytical study of the global instability becomes considerably more complicated, although it is by no means hopeless. By contrast, a numerical investigation of the problem including rotation presents no fundamental difficulties. We have already made progress in both directions. According to our preliminary results, rotation does not suppress the global instability, but it does significantly slow down its development: at large Ω , the characteristic growth time of the instability increases in proportion to Ω . In this respect, the situation resembles the effect of rotation on the Tayler instability [12]. A detailed study of the global instability in the presence of rotation will be presented elsewhere.

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DATA AVAILABILITY

The data that support the findings of this article are openly available (see Supplemental Material, Ref. [68]).

APPENDIX A: ACCURATE MINIMIZATION OF THE RATIO $W[\xi]/I[\xi]$ WITH RESPECT TO $\nabla \cdot \xi$

In Sec. III D we presented Eq. (37), which was used later in the text and which was obtained by minimizing the functional $W[\xi]$ with respect to the φ component of the Lagrangian displacement ξ_φ or, equivalently, with respect to the function $\nabla \cdot \xi$. As discussed in footnote 5, if we are interested in the growth time of the instability and in the optimal Lagrangian displacement, it would be more appropriate to minimize the ratio $W[\xi]/I[\xi]$. The result of such a minimization with respect to $\nabla \cdot \xi$ is discussed in this Appendix. As in the main text of the paper, we assume $m \neq 0$ and without any loss of generality present $\xi(r, \theta, \varphi) = [\xi_r(r, \theta, \varphi), \xi_\theta(r, \theta, \varphi), \xi_\varphi(r, \theta, \varphi)]$ in the form (39). We also introduce a new vector $\bar{\xi}(r, \theta, \varphi) \equiv [\xi_r(r, \theta, \varphi), \xi_\theta(r, \theta, \varphi), 0]$. Then minimization of $W[\xi]/I[\xi]$ with respect to $\nabla \cdot \xi$ at fixed ξ_r and ξ_θ yields

$$\begin{aligned} \nabla \cdot \xi = & -\frac{\rho m^2}{-m^2 p \gamma_{\text{fr}} + \rho \Lambda r^2 \sin^2 \theta} (\xi \cdot \nabla \Phi) \\ & + \frac{\rho \Lambda r^2 \sin^2 \theta}{-m^2 p \gamma_{\text{fr}} + \rho \Lambda r^2 \sin^2 \theta} \nabla \cdot \bar{\xi}. \end{aligned} \quad (\text{A1})$$

We know (see Sec. V) that for the global instability under consideration and for ideal MHD

$$|\Lambda| \sim \omega_A^2 \propto B^2. \quad (\text{A2})$$

Accordingly, $-m^2 p \gamma_{\text{fr}} + \rho \Lambda r^2 \sin^2 \theta \approx -m^2 p \gamma_{\text{fr}}$ and Eq. (A1) simplifies:

$$\nabla \cdot \xi = \frac{\rho}{p \gamma_{\text{fr}}} (\xi \cdot \nabla \Phi) - \frac{\rho \Lambda r^2 \sin^2 \theta}{m^2 p \gamma_{\text{fr}}} \nabla \cdot \bar{\xi}. \quad (\text{A3})$$

This formula differs from expression (37) by the presence of the second term on the right-hand side. Let us estimate how large the contribution of this additional term is. To this end, note that for the optimal Lagrangian displacement found in the paper, corresponding to the global instability, the following estimates hold: $\nabla \cdot \bar{\xi} \sim a(r)$, $\xi_r \sim \epsilon r a(r) \sim (\chi^{1/2} / \sqrt{\gamma_{\text{fr}, 0} - \gamma_{\text{eq}, 0}}) r a(r)$ [see Eqs. (40), (41), (67), and (81)]. Using these estimates, it is easy to see that the ratio of the second to the first term on the right-hand side of (A3) is approximately

$$\frac{\text{2nd term in Eq. (A3)}}{\text{1st term in Eq. (A3)}} \sim \left(\frac{\omega_A}{\mathcal{N}} \right) \left(\frac{r \mathcal{N}^2}{g} \right) \ll 1, \quad (\text{A4})$$

and is the product of two small dimensionless factors, $\omega_A/\mathcal{N}$ and $r \mathcal{N}^2/g$. The latter factor is, in order of magnitude, equal to the ratio of the typical acceleration caused by the buoyancy force to the acceleration of gravity g .

The above estimate shows that performing a more accurate minimization of the ratio $W[\xi]/I[\xi]$, rather than of $W[\xi]$ alone, with respect to $\nabla \cdot \xi$, does not noticeably change Eq. (37) and hence does not affect the optimal Lagrangian displacement or the characteristic growth time of the global instability.

APPENDIX B: DISSIPATIVE TIMESCALE DUE TO SHEAR VISCOSITY

In this section we derive a more accurate expression for the dissipative timescale τ_ν . We define τ_ν as

$$\tau_\nu = \frac{2I[\xi]}{W_\nu[\xi]}, \quad (\text{B1})$$

where $I[\xi]$ is the kinetic energy functional (23) and $W_\nu[\xi]$ is the dissipative integral,

$$W_\nu[\xi] = \frac{1}{2} \int \rho \nu \sigma_{ik}(\xi) \sigma^{ik}(\xi) dV. \quad (\text{B2})$$

In Eq. (B2), σ_{ik} is the stress tensor (see, e.g., [73]). Since the kinetic energy functional $I[\xi]$ is expressed in terms of the Lagrangian displacement ξ (rather than the velocity), the argument of the stress tensor in (B2) should also be taken to be ξ .

For the global instability considered here, within the range of validity of the perturbation theory of Sec. IV, the Lagrangian displacement is given by Eq. (54) up to a small correction. To compute τ_ν to leading order, it is therefore sufficient to take $\xi = \xi_{\text{rot}}(r, \theta, \varphi)$, where ξ_{rot} is given by Eq. (54). Then the angular integrals in the expressions for $I[\xi]$ and $W_\nu[\xi]$ can be evaluated analytically, and we obtain³⁰

³⁰An expression for $I[\xi]$ in this form has already been used; see Eq. (87).

$$I[\xi] \approx \frac{4\pi}{3} \int_0^R r^4 \rho_0 a(r)^2 dr, \quad (\text{B3})$$

$$W_\nu[\xi] \approx \frac{8\pi}{3} \int_0^R r^4 \rho_0 \nu [a'(r)]^2 dr. \quad (\text{B4})$$

Knowing the function $a(r)$, the integrals in these expressions are readily evaluated, allowing us to obtain τ_ν from Eq. (B1).

APPENDIX C: ONE-MODE ESTIMATE FOR THE VISCOUS GROWTH TIMESCALE AND STATEMENT OF A GENERAL VARIATIONAL THEOREM

In this Appendix we first present a simple one-mode estimate leading to Eq. (128). This derivation is intended only as a physically transparent argument. A more general variational theorem is stated at the end of the Appendix.

Consider a trial displacement $\xi_k(\mathbf{r})$ characterized by a radial wave number k . In the one-mode approximation, we represent the perturbation as

$$\xi(\mathbf{r}, t) = A(t) \xi_k(\mathbf{r}), \quad (\text{C1})$$

where $A(t)$ is a scalar amplitude, and dots below denote derivatives with respect to time. Suppose that, in the absence of dissipation, this spatial pattern grows on the timescale $\tau_0(k)$. Then $A(t)$ satisfies

$$\ddot{A} - \frac{1}{\tau_0^2(k)} A = 0. \quad (\text{C2})$$

Now assume that shear viscosity does not substantially modify the spatial structure of the displacement and mainly affects its temporal evolution. Then the effect of viscosity can be modeled by adding a linear damping term with the characteristic timescale $\tau_\nu(k)$:

$$\ddot{A} + \frac{1}{\tau_\nu(k)} \dot{A} - \frac{1}{\tau_0^2(k)} A = 0. \quad (\text{C3})$$

Looking for a growing solution in the form

$$A(t) \propto e^{t/\tau(k)}, \quad (\text{C4})$$

we obtain

$$\frac{1}{\tau^2(k)} + \frac{1}{\tau_\nu(k)\tau(k)} - \frac{1}{\tau_0^2(k)} = 0. \quad (\text{C5})$$

Hence

$$\tau(k) = \frac{\tau_0^2(k)}{2\tau_\nu(k)} + \tau_0(k) \sqrt{1 + \frac{\tau_0^2(k)}{4\tau_\nu^2(k)}}. \quad (\text{C6})$$

This is Eq. (128).

The viscous timescale $\tau_\nu(k)$ entering Eqs. (C3)–(C6) should be evaluated for the same trial displacement ξ_k ; in our case it is estimated from the dissipation functional derived in Appendix B.

Equation (C6) reproduces the expected limiting cases:

$$\tau(k) \approx \tau_0(k), \quad \tau_0(k) \ll \tau_\nu(k), \quad (\text{C7})$$

and

$$\tau(k) \approx \frac{\tau_0^2(k)}{\tau_\nu(k)}, \quad \tau_0(k) \gg \tau_\nu(k). \quad (\text{C8})$$

Thus, viscosity slows down the instability, and it slows down short-scale modes more efficiently because their viscous timescale is shorter.

The derivation above is a one-mode estimate. A more general variational theorem for linear dissipative systems explains why the resulting estimate has the form of Eq. (128). We state the corresponding variational result below. Its proof, given by one of us (M. E. Gusakov), is beyond the scope of the present paper and will be presented elsewhere.

Consider the abstract linear evolution equation

$$\ddot{\xi}(t) + \hat{D} \dot{\xi}(t) + \hat{L} \xi(t) = 0. \quad (\text{C9})$$

Here $\xi(t)$ is a vector in a Hilbert space with scalar product $(\cdot, \cdot)$ and norm $\|\cdot\|$. We assume that the operator $\hat{L}$ is self-adjoint and bounded from below, while the operator $\hat{D}$ is self-adjoint and non-negative:

$$(\hat{D}z, z) \geq 0 \quad \text{for any admissible } z. \quad (\text{C10})$$

In the MHD problem considered in the main text, $\hat{L}$ is the force operator of the ideal system, while $\hat{D}$ represents viscous dissipation. In this language, the operator $\hat{L}$ determines the quadratic potential-energy form of the ideal system, $W[z] = (\hat{L}z, z)/2$, while $\hat{D}$ determines the viscous dissipation functional $(\hat{D}z, z)/2$. The kinetic-energy functional, in turn, is given by $I[z] = \|z\|^2/2$.

For a trial displacement z , one may define the corresponding ideal growth timescale $\tau_0[z]$ and viscous damping timescale $\tau_\nu[z]$ by

$$\frac{1}{\tau_0^2[z]} = -\frac{(\hat{L}z, z)}{\|z\|^2}, \quad \frac{1}{\tau_\nu[z]} = \frac{(\hat{D}z, z)}{\|z\|^2}. \quad (\text{C11})$$

Thus, $\tau_0[z]$ and $\tau_\nu[z]$ characterize, respectively, the ideal growth and viscous damping of the same spatial pattern z .

For any $\lambda \geq 0$ and any admissible trial function z , let us define the quadratic form

$$q_\lambda[z] \equiv (\hat{L}z, z) + \lambda(\hat{D}z, z) + \lambda^2\|z\|^2. \quad (\text{C12})$$

Suppose now that we have found a trial function η such that

$$(\hat{L}\eta, \eta) < 0, \quad (\text{C13})$$

i.e., the corresponding potential-energy variation is negative. Then the system is unstable already in the absence of dissipation. Since $q_\lambda[\eta]$ is a convex quadratic polynomial in λ , negative at $\lambda = 0$, and tends to $+\infty$ as $\lambda \rightarrow \infty$, it has one and only one positive root, $\lambda = \lambda_\eta$, given by

$$\lambda_\eta = \frac{-(\hat{D}\eta, \eta) + \sqrt{(\hat{D}\eta, \eta)^2 - 4(\hat{L}\eta, \eta)\|\eta\|^2}}{2\|\eta\|^2}. \quad (\text{C14})$$

The quantity λ_η provides a lower bound on the exact maximal growth rate of the dissipative system (C9). In other words, the true fastest-growing solution cannot grow slower than $e^{\lambda_\eta t}$.

Thus, solving the scalar equation

$$q_\lambda[\eta] = 0 \quad (\text{C15})$$

for a given trial function η yields a variational estimate of the growth rate from below. In the conservative limit, $\hat{D} = 0$, Eq. (C14) reduces to the standard variational formulation (24).

The one-mode derivation given at the beginning of this Appendix is recovered from this general variational estimate by restricting the trial functions to the form $\xi(\mathbf{r}, t) = A(t)\xi_k(\mathbf{r})$.

APPENDIX D: GLOBAL INSTABILITY UNDER VARIOUS APPROXIMATIONS

When investigating magnetohydrodynamic phenomena (both numerically and analytically), researchers often employ various simplifying approximations. However, the global instability discussed here may be sensitive to certain ones. In this section, we examine how these approximations affect the instability.

First, we emphasize that, as follows from the discussion in Secs. III C and IV C, the instability of a barotropic magnetized star should not necessarily exhibit the characteristic features described here; in particular, we do not expect to observe the suppression of the radial displacement in barotropic matter. Therefore, modeling a toroidal magnetic field instability under the barotropic approximation is not suitable. To observe the distinctive features of the

described instability one should adopt a nonbarotropic equation of state in the numerical simulations.

With this in mind, we begin by discussing the common approximation of neglecting magnetic field-induced perturbations to the background gravitational potential, which is often used to simplify calculations (in particular, we adopt it in the simulations of Sec. IX). Such an approximation can be called the “equilibrium Cowling approximation.” In this case, following the derivation of Sec. IV, we find the same expression for W , but with vanishing $\partial_\theta \Phi_1$ in W_2 and $W_3 = 0$. The argumentation of Sec. IV remains valid with the only modification: the expression (69) simplifies to

$$W[\xi] \simeq \frac{\kappa^2}{2} \int r^2 \sin \theta \left(-\frac{W_2^2}{4W_1} \right) dr d\theta, \quad (\text{D1})$$

implying that $W[\xi]$ is always nonpositive and is negative for a wide class of $\mathbf{a}(r, \theta)$ functions (namely, it is zero for solid-body rotation and negative for differential rotation). Notice that now we do not need $n \sim \partial_r \mathbf{a}(r, \theta) R / \mathbf{a}(r, \theta)$ to be $\gg 1$; however, it should not be $\ll 1$ either, to ensure that the contribution of the next-order terms in κ and ε is negligible. Nevertheless, since $W_2 \propto \partial_r \mathbf{a}(r, \theta)$, the stronger radial gradients in the function $\mathbf{a}(r, \theta)$ correspond to shorter instability timescales [see Eq. (25)]. Thus, the displacements with larger n should develop faster, and the n value is again bounded from above by the third-order terms. Following the minimization procedure described in Sec. IV C and retaining the leading contribution from the third-order terms, one arrives at an approximate energy functional of the form (74), but with $F_2 = F_3 = 0$. Minimizing this functional yields the desired function $a(r)$. Since, in the general case (i.e., beyond the equilibrium Cowling approximation), the contributions of the terms $\propto F_2$ and $\propto F_3$ are negligible in the limit $n \gg 1$ [see the discussion preceding Eq. (79)], the optimal function $a(r)$ obtained within the equilibrium Cowling approximation practically coincides with that derived beyond this approximation. In other words, the global instability analyzed in the equilibrium Cowling approximation should yield an $a(r)$ that is very close to the “true” one obtained in a self-consistent treatment.

Let us now restore magnetic field-induced perturbations to the background gravitational potential and consider one of the most frequently used approximations, the anelastic approximation (see, e.g., [74] and references therein), which filters out fast variability in the equations (sound waves), replacing the continuity equation (11) with $\nabla \cdot (\rho \xi) = 0$. Indeed, as discussed in Sec. III C, for slow perturbations, $\nabla \cdot (\rho \xi)$ is suppressed to ensure that the second term on the right-hand side of Eq. (31) becomes comparable to or smaller than the Lorentz force.

In addition to modifying the continuity equation, the Euler equation is often adjusted as well. We focus on the so

called LBR formulation [70–72], widely used in numerical simulations. This formulation conserves energy [74], justifying the use of the energy functional, the approach adopted in this paper. Note, however, that the hydrodynamic part of the LBR energy functional differs from (20). Before deriving it below, we examine the approximations used in the LBR formulation. To this end, let us rewrite Eq. (31) as

$$-\omega^2 \xi = -\nabla \left(\frac{\delta p}{\rho} \right) - \left(\frac{\delta p}{\rho^2} \nabla \rho + \frac{\delta p}{p \gamma_{\text{fr}}} \nabla \Phi \right) - \frac{\partial \rho}{\partial x} \delta x \frac{\nabla \Phi}{\rho} + \frac{\delta F_L}{\rho} \quad (\text{D2})$$

and estimate the terms on the right-hand side of this equation. When the magnetic field is small, one can approximately rewrite the second term as

$$\begin{aligned} \frac{\delta p}{\rho^2} \nabla \rho + \frac{\delta p}{p \gamma_{\text{fr}}} \nabla \Phi &\approx \frac{\delta p}{\rho^2} \nabla \rho - \frac{\delta p}{p \gamma_{\text{fr},0} \rho} \nabla p \\ &\approx \frac{\delta p}{\rho^2} \left(\frac{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}}{\gamma_{\text{fr},0}} \right) \nabla \rho. \end{aligned} \quad (\text{D3})$$

Matching two expressions for $\delta \rho$, $\delta \rho = \frac{\rho}{p \gamma_{\text{fr}}} \delta p + \frac{\partial \rho}{\partial x} \delta x$ and $\delta \rho = \frac{\rho}{p \gamma_{\text{fr}}} \delta p - [\nabla \rho \cdot \xi - \frac{\rho}{p \gamma_{\text{fr}}} (\nabla p \cdot \xi)]$ [see Eq. (17)], one can present the third term on the right-hand side of Eq. (D2) as

$$\begin{aligned} -\frac{\partial \rho}{\partial x} \delta x \frac{\nabla \Phi}{\rho} &= \left[\nabla \rho \cdot \xi - \frac{\rho}{p \gamma_{\text{fr}}} (\nabla p \cdot \xi) \right] \frac{\nabla \Phi}{\rho} \\ &\sim -\frac{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}}{\gamma_{\text{fr},0}} \frac{\nabla \rho}{\rho^2} (\nabla p \cdot \xi). \end{aligned} \quad (\text{D4})$$

The results of Sec. III C imply that for perturbations of a general form caused by the Lorentz force, $\nabla p \cdot \xi \sim \frac{\gamma_{\text{eq},0}}{\gamma_{\text{fr},0} - \gamma_{\text{eq},0}} \delta p$ [see Eq. (17) and the estimates in Sec. III C], and hence the third term is of the same order of magnitude as the first one. At the same time, the second term is smaller than the first term in Eq. (D2) by a factor of $\frac{\gamma_{\text{fr}} - \gamma_{\text{eq},0}}{\gamma_{\text{fr}}}$. Typically, in stellar matter, $\frac{\gamma_{\text{fr}} - \gamma_{\text{eq},0}}{\gamma_{\text{fr}}}$ is small, indicating that the second term can be neglected,³¹ and we arrive at the LBR anelastic approximation of the Euler equation,

$$-\omega^2 \xi = -\nabla \left(\frac{\delta p}{\rho} \right) - \frac{\partial \rho}{\partial x} \delta x \frac{\nabla \Phi}{\rho} + \frac{\delta F_L}{\rho}, \quad (\text{D5})$$

where the second term stays for the buoyancy force. In what follows we will use Eq. (D5) in its equivalent form:

³¹Note that for small-scale perturbations, the difference between the first and second terms becomes even larger (an additional factor, the ratio of the perturbation length scale to R , appears), while the third term remains of the same order of magnitude as the first one.

$$\begin{aligned} -\omega^2 \xi &= -\nabla \left(\frac{\delta p}{\rho} \right) + \left[\nabla \rho \cdot \xi - \frac{\rho}{p \gamma_{\text{fr}}} (\nabla p \cdot \xi) \right] \frac{\nabla \Phi}{\rho} \\ &\quad + \frac{\delta F_L}{\rho}. \end{aligned} \quad (\text{D6})$$

Let us now derive the perturbation of the potential energy of the system under the displacement ξ . Following [72], it can be calculated as the integral over the stellar volume

$$W[\xi] = -\frac{1}{2} \int \xi \cdot F(\xi) dV, \quad (\text{D7})$$

where $F(\xi)$ is a linear, self-adjoint operator defined by the equation

$$\rho \ddot{\xi} = F(\xi). \quad (\text{D8})$$

Then

$$\begin{aligned} W[\xi] &= \frac{1}{2} \int \left[\rho \xi \cdot \nabla \left(\frac{\delta p}{\rho} \right) - \left(\nabla \rho \cdot \xi - \frac{\rho}{p \gamma_{\text{fr}}} (\nabla p \cdot \xi) \right) \rho \xi \cdot \frac{\nabla \Phi}{\rho} \right. \\ &\quad \left. - \delta F_L \cdot \xi \right] dV. \end{aligned} \quad (\text{D9})$$

After integration by parts, the first term cancels out in the anelastic approximation for perturbations that obey the suitable boundary condition $(\nabla p \cdot \xi) \xi \cdot \mathbf{n} = 0$ (where $\mathbf{n}$ is the unit surface normal vector of the boundary), and we are left with

$$W[\xi] = -\frac{1}{2} \int \left[\xi \cdot \left(\nabla \rho - \rho \frac{F_L - \rho \nabla \Phi}{p \gamma_{\text{fr}}} \right) (\nabla \Phi \cdot \xi) + \delta F_L \cdot \xi \right] dV, \quad (\text{D10})$$

where the hydrostatic equilibrium condition has been used. The magnetic part of $W[\xi]$, $W^{\text{mag}}[\xi]$, remains unchanged, and we present its derivation below for completeness.

$$\begin{aligned} W^{\text{mag}}[\xi] &= -\frac{1}{2} \int \xi \cdot \delta F_L dV \\ &= \frac{1}{8\pi} \int \xi \cdot [\mathbf{B} \times (\nabla \times \delta \mathbf{B})] dV \\ &\quad + \frac{1}{8\pi} \int \xi \cdot [\delta \mathbf{B} \times (\nabla \times \mathbf{B})] dV \\ &\equiv \frac{1}{8\pi} \int \mathbf{I} dV + \frac{1}{8\pi} \int \mathbf{II} dV, \end{aligned} \quad (\text{D11})$$

where

$$\begin{aligned} \mathbf{I} &= (\nabla \times \delta \mathbf{B}) \cdot (\xi \times \mathbf{B}) \\ &= \delta \mathbf{B} \cdot (\nabla \times (\xi \times \mathbf{B})) - \nabla \cdot ((\xi \times \mathbf{B}) \times \delta \mathbf{B}) \\ &= \delta \mathbf{B} \cdot \delta \mathbf{B} - \nabla \cdot ((\xi \times \mathbf{B}) \times \delta \mathbf{B}), \end{aligned} \quad (\text{D12})$$

$$\Pi = (\nabla \times \mathbf{B}) \cdot (\boldsymbol{\xi} \times \delta \mathbf{B}) = \frac{4\pi}{c} \mathbf{J} \cdot (\boldsymbol{\xi} \times \delta \mathbf{B}). \quad (\text{D13})$$

Hence,

$$W^{\text{mag}}[\boldsymbol{\xi}] = \frac{1}{2} \int \left(\frac{1}{4\pi} \delta \mathbf{B} \cdot \delta \mathbf{B} + \frac{1}{c} \mathbf{J} \cdot (\boldsymbol{\xi} \times \delta \mathbf{B}) \right) dV + \text{surface term}. \quad (\text{D14})$$

Since only the magnetic part of $W[\boldsymbol{\xi}]$, namely $W^{\text{mag}}[\boldsymbol{\xi}]$, contributes at leading order in κ and ε , the conclusions of Sec. IV B remain unchanged: for a toroidal field, the most

unstable mode is the one with azimuthal number $m = 1$. To first order in the perturbation theory, the optimal Lagrangian displacement for this mode is given by shellular differential rotation of the star about an arbitrary axis in the $x - y$ plane, for example, the x axis [see Eq. (54)]. At the same time, this Lagrangian displacement is “neutral” in the sense that, in first-order perturbation theory, the perturbation-induced change in the energy functional vanishes, $W[\boldsymbol{\xi}_{\text{rot}}] \simeq \kappa \alpha_{01} = 0$.

In the next order in κ and ε , using Eq. (D10), we find that $W[\boldsymbol{\xi}]$ has the same structure as that discussed in Sec. IV C, with the functions W_1 , W_2 , and W_3 given by

$$W_1 = \frac{\pi(\gamma_{\text{fr},0} - \gamma_{\text{eq},0})[p'_0(r)]^2}{p_0\gamma_{\text{fr},0}\gamma_{\text{eq},0}}, \quad (\text{D15})$$

$$W_2 = \frac{2\pi}{p_0\gamma_{\text{fr},0}\gamma_{\text{eq},0}} [-2\mathcal{B}^2 \cot \theta p_0\gamma_{\text{fr},0}\gamma_{\text{eq},0} \partial_r \mathbf{a}(r, \theta) + p'_0(r)(\rho_0 \partial_\theta \phi_1 + \mathcal{B}^2 \cot \theta + \mathcal{B} \partial_\theta \mathcal{B})(\gamma_{\text{fr},0} - \gamma_{\text{eq},0}) \mathbf{a}(r, \theta)], \quad (\text{D16})$$

$$W_3 = \frac{\pi(\rho_0 \partial_\theta \phi_1 + 2\mathcal{B}^2 \cot \theta + 2\mathcal{B} \partial_\theta \mathcal{B}) \mathbf{a}(r, \theta)^2}{r p'_0(r) p_0 \gamma_{\text{fr},0} \gamma_{\text{eq},0}} [r p'_0(r) \partial_\theta \phi_1 (\gamma_{\text{fr},0} - \gamma_{\text{eq},0}) \rho_0 + 4\mathcal{B} p_0 \gamma_{\text{fr},0} \gamma_{\text{eq},0} (-\partial_\theta \mathcal{B} + r \cot \theta \partial_r \mathcal{B})]. \quad (\text{D17})$$

As we see, the function W_1 is not affected by the anelastic LBR approximation, while W_2 and W_3 undergo minor modifications (additional underlined terms appear). Thus, the reasoning developed in Sec. IV applies to the LBR formulation of MHD and leads to the same conclusions. In other words, the approximations of the anelastic LBR do not eliminate the global instability.

Another widely used approach is the Boussinesq approximation (used, e.g., in the MagIC code [75]), developed for MHD problems where the background

length scale strongly exceeds the perturbation scale. Perturbations are assumed to be slow in the Boussinesq approximation, similarly to the anelastic one. Under this approximation, the density in the first term of the Euler equation (D6) is factored out, while the continuity equation reduces to the incompressibility condition, $\nabla \cdot \boldsymbol{\xi} = 0$. The energy functional $W[\boldsymbol{\xi}]$ in the Boussinesq approximation then coincides with (D10), indicating that the resulting equations also support the global toroidal instability.

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- [1] J.-F. Donati and J.D. Landstreet, Magnetic fields of non-degenerate stars, *Annu. Rev. Astron. Astrophys.* **47**, 333 (2009).
 - [2] R.J. Tayler, The adiabatic stability of stars containing magnetic fields-I. Toroidal fields, *Mon. Not. R. Astron. Soc.* **161**, 365 (1973).
 - [3] Y.V. Vandakurov, Theory for the stability of a star with a toroidal magnetic field, *Sov. Astron.* **16**, 265 (1972), <https://ui.adsabs.harvard.edu/abs/1972SvA....16..265V/abstract>.
 - [4] L. Kitchatinov and G. Rüdiger, Stability of toroidal magnetic fields in rotating stellar radiation zones, *Astron. Astrophys.* **478**, 1 (2008).
 - [5] G. Rüdiger and L.L. Kitchatinov, The kink-type instability of toroidal stellar magnetic fields with thermal diffusion, *Geophys. Astrophys. Fluid Dyn.* **104**, 273 (2010).
 - [6] R. Arlt and G. Rüdiger, Amplification and stability of magnetic fields and dynamo effect in young A stars, *Mon. Not. R. Astron. Soc.* **412**, 107 (2011).
 - [7] M. Goossens and R. Veugelen, Hydromagnetic instabilities in polytropic stars containing toroidal magnetic fields, *Astron. Astrophys.* **70**, 277 (1978), <https://ui.adsabs.harvard.edu/abs/1978A%26A....70..277G/abstract>.

- [8] D. J. Acheson, On the instability of toroidal magnetic fields and differential rotation in stars, *Phil. Trans. R. Soc. A* **289**, 459 (1978).
- [9] M. Goossens, σ -stability analysis of toroidal magnetic fields in stars, *Geophys. Astrophys. Fluid Dyn.* **15**, 123 (1980).
- [10] M. Goossens and R. J. Tayler, On unstably stratified toroidal magnetic fields in stars, *Mon. Not. R. Astron. Soc.* **193**, 833 (1980).
- [11] M. Goossens, D. Biront, and R. J. Tayler, Additional results for unstable stratified toroidal magnetic fields in stars, *Astrophys. Space Sci.* **75**, 521 (1981).
- [12] E. Pitts and R. J. Tayler, The adiabatic stability of stars containing magnetic fields. IV—The influence of rotation, *Mon. Not. R. Astron. Soc.* **216**, 139 (1985).
- [13] L. L. Kitchatinov, Stability of toroidal magnetic fields in the radiation zone of a star, *Astronomy Reports* **52**, 247 (2008).
- [14] A. Bonanno and V. Urpin, Stability of the toroidal magnetic field in stellar radiation zones, *Astrophys. J.* **747**, 137 (2012).
- [15] T. Akgün, A. Reisenegger, A. Mastrano, and P. Marchant, Stability of magnetic fields in non-barotropic stars: an analytic treatment, *Mon. Not. R. Astron. Soc.* **433**, 2445 (2013).
- [16] V. A. Skoutnev and A. M. Beloborodov, Tayler instability revisited, *Astrophys. J.* **974**, 290 (2024).
- [17] V. A. Skoutnev and A. M. Beloborodov, Zones of Tayler instability in stars, *Astrophys. J.* **988**, 195 (2025).
- [18] J. Braithwaite, The stability of toroidal fields in stars, *Astron. Astrophys.* **453**, 687 (2006).
- [19] V. Duez, J. Braithwaite, and S. Mathis, On the stability of non-force-free magnetic equilibria in stars, *Astrophys. J. Lett.* **724**, L34 (2010).
- [20] K. Kiuchi, S. Yoshida, and M. Shibata, Nonaxisymmetric instabilities of neutron star with toroidal magnetic fields, *Astron. Astrophys.* **532**, A30 (2011).
- [21] S. K. Lander and D. I. Jones, Instabilities in neutron stars with toroidal magnetic fields, *Mon. Not. R. Astron. Soc.* **412**, 1394 (2011).
- [22] J. C. Ibáñez-Mejía and J. Braithwaite, Stability of toroidal magnetic fields in stellar interiors, *Astron. Astrophys.* **578**, A5 (2015).
- [23] G. Guerrero, F. Del Sordo, A. Bonanno, and P. K. Smolarkiewicz, Global simulations of Tayler instability in stellar interiors: The stabilizing effect of gravity, *Mon. Not. R. Astron. Soc.* **490**, 4281 (2019).
- [24] S. Ji, J. Fuller, and D. Lecoanet, Magnetohydrodynamic simulations of the Tayler instability in rotating stellar interiors, *Mon. Not. R. Astron. Soc.* **521**, 5372 (2023).
- [25] G. Monteiro, G. Guerrero, F. Del Sordo, A. Bonanno, and P. K. Smolarkiewicz, Global simulations of Tayler instability in stellar interiors: A long-time multistage evolution of the magnetic field, *Mon. Not. R. Astron. Soc.* **521**, 1415 (2023).
- [26] D. G. Meduri, R. Arlt, A. Bonanno, and G. Licciardello, Gravity's role in taming the Tayler instability in red giant cores, *Astron. Astrophys.* **702**, A44 (2025).
- [27] H. C. Spruit, Differential rotation and magnetic fields in stellar interiors, *Astron. Astrophys.* **349**, 189 (1999), <https://ui.adsabs.harvard.edu/abs/1999A%26A...349..189S/abstract>.
- [28] H. C. Spruit, Dynamo action by differential rotation in a stably stratified stellar interior, *Astron. Astrophys.* **381**, 923 (2002).
- [29] P. Barrère, J. Guilet, A. Reboul-Salze, R. Raynaud, and H. T. Janka, A new scenario for magnetar formation: Tayler-Spruit dynamo in a proto-neutron star spun up by fallback, *Astron. Astrophys.* **668**, A79 (2022).
- [30] P. Barrère, J. Guilet, R. Raynaud, and A. Reboul-Salze, Numerical simulations of the Tayler-Spruit dynamo in proto-magnetars, *Mon. Not. R. Astron. Soc.* **526**, L88 (2023).
- [31] P. Barrère, J. Guilet, R. Raynaud, and A. Reboul-Salze, Tayler-Spruit dynamo in stably stratified rotating fluids: Application to proto-magnetars, *Astron. Astrophys.* **695**, A183 (2025).
- [32] A. Igoshev, P. Barrère, R. Raynaud, J. Guilet, T. Wood, and R. Hollerbach, A connection between proto-neutron-star Tayler-Spruit dynamos and low-field magnetars, *Nat. Astron.* **9**, 541 (2025).
- [33] A. Maeder and G. Meynet, Stellar evolution with rotation and magnetic fields. II. General equations for the transport by Tayler-Spruit dynamo, *Astron. Astrophys.* **422**, 225 (2004).
- [34] A. Maeder and G. Meynet, Stellar evolution with rotation and magnetic fields. III. The interplay of circulation and dynamo, *Astron. Astrophys.* **440**, 1041 (2005).
- [35] L. Asatiani, P. Eggenberger, M. Marchand, F. D. Moyano, G. Meynet, and A. Choplin, Impact of the Tayler magnetic instability on the surface abundance of boron in massive stars, *Astron. Astrophys.* **699**, L8 (2025).
- [36] G. Buldgen, A. Noels, A. M. Amarsi, D. Nandal, C. Pezzotti, R. Scuflaire, M. Deal, and N. Grevesse, Constraints on the properties of macroscopic transport in the Sun from combined lithium and beryllium depletion, *Astron. Astrophys.* **694**, A285 (2025).
- [37] J. Fuller, A. L. Piro, and A. S. Jermyn, Slowing the spins of stellar cores, *Mon. Not. R. Astron. Soc.* **485**, 3661 (2019).
- [38] C. Aerts, S. Mathis, and T. M. Rogers, Angular momentum transport in stellar interiors, *Annu. Rev. Astron. Astrophys.* **57**, 35 (2019).
- [39] P. A. Denissenkov and M. Pinsonneault, A revised prescription for the Tayler-spruit dynamo: Magnetic angular momentum transport in stars, *Astrophys. J.* **655**, 1157 (2007).
- [40] J.-P. Zahn, A. S. Brun, and S. Mathis, On magnetic instabilities and dynamo action in stellar radiation zones, *Astron. Astrophys.* **474**, 145 (2007).
- [41] L. Petitdemange, F. Marcotte, and C. Gissinger, Spin-down by dynamo action in simulated radiative stellar layers, *Science* **379**, 300 (2023).
- [42] L. Petitdemange, F. Marcotte, C. Gissinger, and F. Daniel, Tayler-Spruit dynamo simulations for the modeling of radiative stellar layers, *Astron. Astrophys.* **681**, A75 (2024).
- [43] A. Igoshev, P. Barrère, R. Raynaud, J. Guilet, T. Wood, and R. Hollerbach, A connection between proto-neutron-star Tayler-Spruit dynamos and low-field magnetars, *Nat. Astron.* **9**, 541 (2025).
- [44] M. Cantiello, C. Mankovich, L. Bildsten, J. Christensen-Dalsgaard, and B. Paxton, Angular momentum transport within evolved low-mass stars, *Astrophys. J.* **788**, 93 (2014).

- [45] P. Eggenberger, J. W. den Hartogh, G. Buldgen, G. Meynet, S. J. A. J. Salmon, and S. Deheuvels, Asteroseismology of evolved stars to constrain the internal transport of angular momentum. II. Test of a revised prescription for transport by the Tayler instability, *Astron. Astrophys.* **631**, L6 (2019).
- [46] P. Eggenberger, G. Buldgen, and S. J. A. J. Salmon, Rotation rate of the solar core as a key constraint to magnetic angular momentum transport in stellar interiors, *Astron. Astrophys.* **626**, L1 (2019).
- [47] J. W. den Hartogh, P. Eggenberger, and S. Deheuvels, Asteroseismology of evolved stars to constrain the internal transport of angular momentum. III. Using the rotation rates of intermediate-mass stars to test the Fuller-formalism, *Astron. Astrophys.* **634**, L16 (2020).
- [48] J. Fuller and W. Lu, The spins of compact objects born from helium stars in binary systems, *Mon. Not. R. Astron. Soc.* **511**, 3951 (2022).
- [49] F. D. Moyano, P. Eggenberger, and S. J. A. J. Salmon, Angular momentum transport near convective-core boundaries of gamma doradus stars, *Astron. Astrophys.* **681**, L16 (2024).
- [50] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, An energy principle for hydromagnetic stability problems, *Proc. R. Soc. A* **244**, 17 (1958).
- [51] A. Brandenburg *et al.* (Pencil Code Collaboration), The PENCIL Code, a modular MPI code for partial differential equations and particles: Multipurpose and multiuser-maintained, *J. Open Source Software* **6**, 2807 (2021).
- [52] J. P. H. Goedbloed and S. Poedts, *Principles of Magneto-hydrodynamics* (Cambridge University Press, Cambridge, England, 2004).
- [53] T. Akgün and I. Wasserman, Toroidal magnetic fields in type II superconducting neutron stars, *Mon. Not. R. Astron. Soc.* **383**, 1551 (2008).
- [54] D. L. Moss and R. J. Tayler, The influence of a poloidal magnetic field on convection in spherical shells, *Mon. Not. R. Astron. Soc.* **145**, 217 (1969).
- [55] A. Reisenegger, Stable magnetic equilibria and their evolution in the upper main sequence, white dwarfs, and neutron stars, *Astron. Astrophys.* **499**, 557 (2009).
- [56] L. Becerra, A. Reisenegger, J. A. Valdivia, and M. Gusakov, Stability of axially symmetric magnetic fields in stars, *Mon. Not. R. Astron. Soc.* **517**, 560 (2022).
- [57] J. L. Friedman and B. F. Schutz, Secular instability of rotating Newtonian stars., *Astrophys. J.* **222**, 281 (1978).
- [58] N. K. Sinha, Structure of polytropes with purely toroidal magnetic fields, *Aust. J. Phys.* **21**, 283 (1968).
- [59] N. K. Sood and S. K. Trehan, Oscillations of a polytrope with a toroidal magnetic field, *Astrophys. Space Sci.* **16**, 451 (1972).
- [60] M. E. Gusakov, E. M. Kantor, and D. D. Ofengeim, Evolution of the magnetic field in neutron stars, *Phys. Rev. D* **96**, 103012 (2017).
- [61] J. Braithwaite, Axisymmetric magnetic fields in stars: Relative strengths of poloidal and toroidal components, *Mon. Not. R. Astron. Soc.* **397**, 763 (2009).
- [62] C. J. Hansen, S. D. Kawaler, and V. Trimble, *Stellar Interiors: Physical Principles, Structure, and Evolution* (Springer, New York, 2004).
- [63] A. Schmitt and P. Shternin, Reaction rates and transport in neutron stars, in *Astrophysics and Space Science Library*, Astrophysics and Space Science Library, edited by L. Rezzolla, P. Pizzochero, D. I. Jones, N. Rea, and I. Vidaña (2018), Vol. 457, p. 455.
- [64] C. Thompson and R. C. Duncan, Neutron star dynamos and the origins of pulsar magnetism, *Astrophys. J.* **408**, 194 (1993).
- [65] N. Iwamoto and C. J. Pethick, Effects of nucleon-nucleon interactions on scattering of neutrinos in neutron matter, *Phys. Rev. D* **25**, 313 (1982).
- [66] D. Radice and I. Hawke, Turbulence modelling in neutron star merger simulations, *Living Rev. Comput. Astrophys.* **10**, 1 (2024).
- [67] L. Becerra, A. Reisenegger, J. A. Valdivia, and M. E. Gusakov, Evolution of random initial magnetic fields in stably stratified and barotropic stars, *Mon. Not. R. Astron. Soc.* **511**, 732 (2022).
- [68] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/2h7n-g6nq> for mp4 movies illustrating the evolution of the seven representative simulations discussed in Sec. IX.
- [69] T. Rembiasz, M. Obergaulinger, P. Cerdá-Durán, M.-Á. Aloy, and E. Müller, On the measurements of numerical viscosity and resistivity in Eulerian MHD codes, *Astrophys. J. Suppl. Ser.* **230**, 18 (2017).
- [70] S. R. Lantz, Dynamical Behavior of Magnetic Fields in a Stratified, Convecting Fluid Layer, Ph.D. thesis, Cornell University, New York (1992).
- [71] S. I. Braginsky and P. H. Roberts, Equations governing convection in Earth's core and the geodynamo, *Geophys. Astrophys. Fluid Dyn.* **79**, 1 (1995).
- [72] J. Goldstein, R. H. D. Townsend, and E. G. Zweibel, The Tayler instability in the anelastic approximation, *Astrophys. J.* **881**, 66 (2019).
- [73] L. Landau and E. Lifshitz, *Fluid Mechanics* (Pergamon Press, Oxford, 1987).
- [74] B. P. Brown, G. M. Vasil, and E. G. Zweibel, Energy conservation and gravity waves in sound-proof treatments of stellar interiors. Part I. Anelastic approximations, *Astrophys. J.* **756**, 109 (2012).
- [75] <https://magic-sph.github.io/>.