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Thermal g-modes and unexpected convection in superfluid neutron stars

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We suggest that specific thermal low-frequency g-modes can exist in superfluid neutron stars. They can be excited in the matter with a nonzero gradient of entropy per electron. We determine the Brunt-Väisälä frequency for these modes and demonstrate that they can be unstable with respect to convection. The criterion for the instability onset (analogue of the well-known Schwarzschild criterion) is derived. It is very sensitive to the equation of state and a model of nucleon superfluidity. In particular, convection may occur for both positive and negative temperature gradients. Our results may have interesting implications for neutron star cooling and seismology.

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I. INTRODUCTION

This note is devoted to *gravity* oscillation modes (g-modes) and the related phenomenon of convection in neutron stars (NSs). The restoring force for g-modes is buoyancy that originates from the dependence of the pressure on at least two quantities (e.g., density and temperature or density and chemical composition). g-modes and convection are actively studied in laboratory experiments (e.g., [1–3]) and are widespread in nature. For instance, g-modes are observed in Earth's atmosphere and ocean, in white dwarfs [4], in slowly pulsating B-stars [5], and in other objects [6], while convection is typical for most of the stars (including the Sun). In application to NSs, g-modes were studied, e.g., in Refs. [7–12]. In all these works the NS matter was assumed to be nonsuperfluid (normal). However, according to microscopic calculations [13,14], baryons in the internal layers of NSs become superfluid (SF) at temperatures $T \lesssim 10^8 \div 10^{10}$ K which has a drastic impact on stellar dynamics and evolution [14,15]. Recent real-time observations [16] of the cooling NS in Cas A supernova remnant indicate that this NS has an SF core [17,18]. A number of attempts [19–21] have been made to theoretically predict g-modes in cold SF NSs, but they have failed. This led to a general belief that g-modes do not exist in SF interiors of NSs. In this paper we show that proper account of finite temperature effects leads to the presence of peculiar g-modes that can be *unstable* with respect to convection. Possible applications of these results are outlined. Below the Planck constant, the speed of light, and the Boltzmann constant equal unity, $\hbar = c = k_{\rm B} = 1$.

II. CONVECTION IN SF NSS AND THERMAL G-MODES

For simplicity, consider npe NS cores, composed of neutrons (n), protons (p), and electrons (e). To start with, assume that all particles are nonsuperfluid. Any thermodynamic quantity in npe-matter (e.g., the heat function $w = \varepsilon + P$, where ε is the energy density and P is the pressure) can be presented as a function of 3 variables, say, P, $x_e \equiv n_e/n_b$, and $x_S \equiv S/n_b$. Here n_i is the number density for particles $i = n$, p, and e; n_b is the baryon number density; S is the entropy density. What is the local criterion for the absence of convection in npe-matter? Assume that a spherically symmetric star is in hydrostatic equilibrium (but not necessarily in thermal or beta-equilibrium), that is $\nabla P = -w \nabla \phi$, where $\phi(r)$ is the gravitational potential and r is the radial coordinate. Here and below $\nabla \equiv d/dr$ because all quantities of interest depend on r only. Consider two close points 1 and 2 with $r = r_1$ and r_2. Let A_1 and A_2 be the values of some thermodynamic quantity A at points 1 and 2, respectively, and $\Delta A \equiv A_2 - A_1$. Displace adiabatically a small fluid element upward from point 1 to point 2. At point 2 P adjusts itself to the surrounding pressure $P_2 = P_1 + \Delta P$, while x_e and x_S remain unchanged and equal to x_{e1} and x_{S1} (we assume that beta-processes are slow). The matter is stable against convection if the inertial mass density of the lifted element (w for the relativistic matter [22]) is larger than the equilibrium density at point 2. Thus, stability requires $w(P_2, x_{e2}, x_{S2}) < w(P_2, x_{e1}, x_{S1})$. Expanding w in Taylor series near point 1, we obtain

$$\partial_{x_e} w(P, x_e, x_S) \nabla x_e + \partial_{x_s} w(P, x_e, x_S) \nabla x_S < 0, \qquad (1)$$

where $\partial_A \equiv \partial/\partial A$. When w is a function of P and x_S only, Eq. (1) reproduces the Schwarzschild criterion for the absence of convection (see, e.g., [23,24]). In a strongly degenerate matter the second term in Eq. (1) can be neglected. Similarly, to calculate the first term it is sufficient to set $T = 0$ and $x_S = 0$. Then, Eq. (1) reduces to $\partial_{x_e} w(P, x_e) \nabla x_e < 0$ [11]. This Ledoux-type criterion is always satisfied in beta-equilibrated NSs, i.e., they are stable against convection. Oscillations of such a matter near equilibrium correspond to temperature-independent *composition* g-modes, first studied in Ref. [10].

Assume now that neutrons (and possibly protons) are SF. What will be the analogue of criterion (1)? Nucleon SF leads to the appearance of two independent velocity fields: SF neutron velocity $\boldsymbol{V}_{sn}$ and velocity of normal liquid component $\boldsymbol{V}_{\rm q}$, composed of neutron thermal (Bogoliubov) excitations and charge-neutral conglomerate of protons and electrons [25,26]. The presence of extra velocity field $\boldsymbol{V}_{sn}$ results in an additional (besides equation $\nabla P = -w\nabla\phi$) condition of hydrostatic equilibrium in SF matter [26,27]: $\nabla(\mu_n {\rm e}^{\phi}) = 0$, where μ_n is the relativistic neutron chemical potential. As a result, when we displace the fluid element, "attached" to the *normal* liquid component, from point 1 to point 2, both P and μ_n adjust themselves to their equilibrium values P_2 and μ_{n2} at point 2. The pressure adjusts by contraction/expansion of the fluid element, while μ_n adjusts by the variation in the number of "SF neutrons" in this element. Note that, since SF neutrons can freely escape from the fluid element attached to the normal particles, the total number of neutrons in the element is not conserved, and neither are the quantities $x_e = n_e/n_b$ and $x_S = S/n_b$. In this situation the conserved quantity is $x_{eS} = S/n_e$, because both the entropy and electrons flow with the same velocity $\boldsymbol{V}_{\rm q}$ (e.g., [26–28]). Bearing this in mind, it is convenient to consider w as a function of P, μ_n, and x_{eS}. Then the condition for stability against convection reads $w(P_2, \mu_{n2}, x_{eS2}) < w(P_2, \mu_{n2}, x_{eS1})$ or

$$\partial_{x_{eS}} w(P, \mu_n, x_{eS})\nabla x_{eS} < 0. \tag{2}$$

A similar condition was derived in a different way in Ref. [29], where internal gravity waves were analyzed in a mixture of SF He-4 and a normal fluid (see also [30,31]). Note that the left-hand side of Eq. (2) vanishes at $T = 0$. Then the system is marginally stable, since there is no restoring force acting on a displaced fluid element. Thus, it is not surprising that the authors of Refs. [19–21], who assumed $T = 0$, did not find g-modes in SF NSs. In contrast, consistent treatment of the temperature effects should reveal g-modes.

To check it we performed a local analysis of SF hydrodynamic equations (see, e.g., [26,27]), describing oscillations of a NS in the weak-field limit ($\phi \ll 1$) at $T \neq 0$. We analyzed short-wave perturbations, proportional to $\exp(i\omega t)\exp[i\int^r dr' k(r')]{\rm Y}_{lm}$, where the wave number k of a perturbation weakly depends on r ($k \gg |d\ln k/dr|$, WKB approximation), ω is the frequency, and ${\rm Y}_{lm}$ is a spherical harmonic. Solving oscillation equations in the Cowling approximation (in which ϕ is not perturbed [32]), we found the standard [10] short-wave dispersion relation for the SF thermal g-modes,

$$\omega^2 = \mathcal{N}^2\frac{l(l+1)}{l(l+1)+k^2r^2}, \tag{3}$$

where

$$\mathcal{N}^2 = -\frac{g}{\mu_n n_b}\frac{(1+y)}{y}\partial_{x_{eS}} w(P, \mu_n, x_{eS})\nabla x_{eS} \tag{4}$$

is the corresponding Brunt-Väisälä frequency squared; $g = \nabla\phi$; $y = n_b Y_{pp}/[\mu_n(Y_{nn}Y_{pp} - Y_{np}^2)] - 1 > 0$, Y_{ik} being the relativistic entrainment matrix (analogue of the superfluid density for mixtures; see, e.g., [33,34]). The stability condition for these g-modes, $\mathcal{N}^2 > 0$, coincides with Eq. (2).

Introducing the thermal expansion coefficient for SF relativistic npe-matter $\beta \equiv -\partial_T w(P, \mu_n, T)/w$, one obtains from Eq. (4), following similar derivation as in the case of ordinary convection (e.g., Sec. 4 of Ref. [23]),

$$\mathcal{N}^2 = \alpha\beta w[\beta w/C + \nabla T^\infty/(gT^\infty)]. \tag{5}$$

Here $\alpha = g^2 T(1+y)/(y\mu_n n_b) > 0$, $C = n_e T\partial_T x_{eS}(P, \mu_n, T) \approx T\partial_T S(P, \mu_n, T) > 0$, and $T^\infty \equiv T{\rm e}^{\phi}$ is the red-shifted temperature. For an NS in thermal equilibrium, T^∞ is constant throughout the core, $\nabla T^\infty = 0$. In that case $\mathcal{N}^2$ in Eq. (5) is positive and reduces to

$$\mathcal{N}^2 = \alpha(\beta w)^2/C > 0. \tag{6}$$

If NS is not in thermal equilibrium, $\mathcal{N}^2$ [and hence ω^2, see Eq. (3)] can be negative for certain ∇T^∞. These gradients follow from Eq. (5) [or Eq. (2)] and are defined by the inequality $\beta\nabla T^\infty < -gT^\infty\beta^2 w/C$, which is the analogue of the usual Schwarzschild criterion for convection [24]. This inequality, as well as Eq. (2), is valid not only in the weak-field limit $\phi \ll 1$, but also in the full general relativity. When it is satisfied, convective instability occurs (see, however, comment [35]). Thus, the critical gradient for the instability onset is given by (cf. Eq. 4.4 of Ref. [23])

$$\nabla T^\infty_{\rm crit} = -gT^\infty\beta w/C. \tag{7}$$

Clearly, the thermal expansion coefficient β determines the sign of $\nabla T^\infty_{\rm crit}$. If $\beta > 0$ one gets the instability while heating the matter from below; if $\beta < 0$ the instability occurs when it is heated from above. Note that both signs of ∇T^∞ can be realized in cooling NSs [37]. Most of the substances (but not all) have $\beta > 0$ (e.g., $\beta < 0$ for water near 0 °C). For npe-matter of NSs the situation is different: Depending on equation of state (EOS) and/or model of SF β (at fixed n_b and T) can be either $>$ or <0 (see below). Notice, however, that in any case mature NSs, for which $\nabla T^\infty = 0$, will be always stable with respect to convection.

III. RESULTS

First, consider the limit $T \ll T_{cn}, T_{cp}$ (T_{ci} is the critical temperature for particles $i = n, p$), in which the nucleon thermal excitations are exponentially suppressed and only electrons determine the dependence of $w, P, C, \ldots$ on T. In this limit $\beta \propto T$, $C \propto T$ and hence both $\mathcal{N}$ and $\nabla T^\infty_{\rm crit}$ are $\propto T$. Figure 1(a) presents $\mathcal{N}$ [given by Eq. (6)] versus n_b for npe-matter in thermodynamic equilibrium ($\nabla T^\infty = 0$) for 5 EOSs. The figure is plotted assuming $T^\infty \approx T = 10^7$ K (the weak-field approximation, $\phi \ll 1$). One sees that for any EOS $\mathcal{N}$ vanishes at a certain n_b that

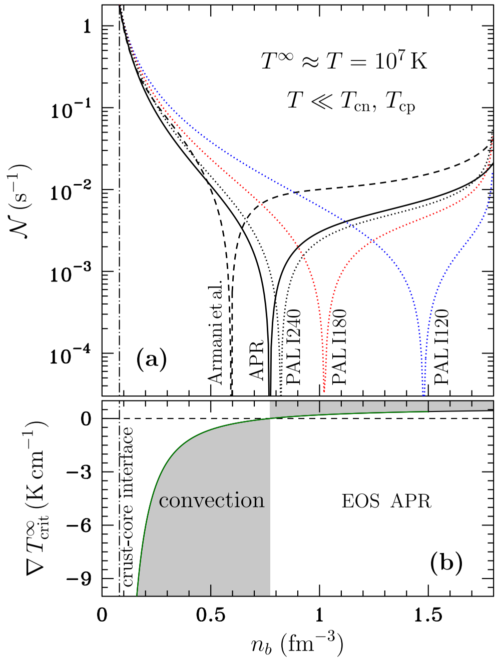

FIG. 1 (color online). (a): $\mathcal{N}$ given by Eq. (6) (isothermal NS matter, $\nabla T^\infty = 0$) versus n_b for EOSs of Armani *et al.* [48], Akmal-Pandharipande-Ravenhall (APR) [38], and Prakash-Ainsworth-Lattimer (PAL) [49]. We adopt the model I of PAL family with three values of the compression modulus, 120, 180, and 240 MeV. (b): $\nabla T^\infty_{\rm crit}$ given by Eq. (7) versus n_b for APR EOS. The convectively unstable regions are filled with gray. The vertical dot-dashed line corresponds to the crust-core interface. Both panels are plotted assuming $T^\infty \approx T = 10^7$ K (the weak-field approximation); they can be rescaled to any $T \ll T_{cn}$, T_{cp} since both $\mathcal{N}$ and $\nabla T^\infty_{\rm crit} \propto T$ in this limit. Here and in Fig. 2 $g = 10^{14}\ \mathrm{cm\,s^{-2}}$.

corresponds to $\beta = 0$. Figure 1(b) shows $\nabla T^\infty_{\rm crit}$ [given by Eq. (7)] versus n_b for the APR EOS [38]. The regions of parameters, where convection occurs are filled with gray. As expected, $\nabla T^\infty_{\rm crit}$ changes sign simultaneously with β. At small n_b one has $\beta > 0$ ($\nabla T^\infty_{\rm crit} < 0$), like in the vast majority of other substances. (It is interesting to note that β is always >0 for npe-matter treated as an ideal gas of noninteracting particles.) At large n_b the situation is opposite, $\beta < 0$. The reason for this can be understood if we rewrite β in the form [39]: $\beta = -[\mu_n \partial_T n_b(P, \mu_n, T) + C]/w$ and notice that the matter becomes stiffer (P grows faster and faster) with increasing n_b. As a consequence, n_b varies only weakly with T at fixed P and μ_n (the stiffer EOS the smaller the variation), that is at large densities $\mu_n|\partial_T n_b(P, \mu_n, T)| < C$ and hence $\beta < 0$.

When T is not too low ($0.1T_{ci} \lesssim T \lesssim T_{ci}$, $i = n$ and/or p), the contribution of nucleon thermal excitations to β and C (and hence to $\mathcal{N}$ and $\nabla T^\infty_{\rm crit}$) cannot be neglected. The results then strongly differ from those obtained in the limit $T \ll T_{cn}$, T_{cp} and are presented in Fig. 2. For illustration, we adopt the APR EOS and take $T^\infty \approx T = 1.5 \times 10^8$ K. Some realistic profiles of singlet proton

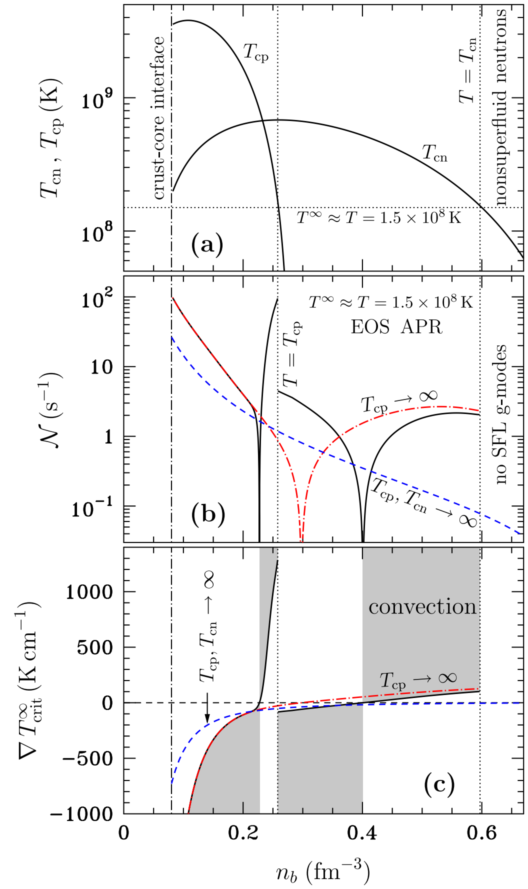

FIG. 2 (color online). Panel (a): T_{cn} and T_{cp} versus n_b. Panels (b) and (c): $\mathcal{N}$ and $\nabla T^\infty_{\rm crit}$ given by, respectively, Eqs. (6) and (7), versus n_b for EOS APR and $T^\infty \approx T = 1.5 \times 10^8$ K [see the horizontal dotted line in panel (a)]. Solid lines in panels (b) and (c) are obtained for $T_{cn}(n_b)$ and $T_{cp}(n_b)$ from panel (a); dot-dashed lines: $T_{cn}(n_b)$ is from panel (a), $T_{cp} \to \infty$; dashed lines: T_{cn}, $T_{cp} \to \infty$. The right vertical dotted line indicates the boundary between the SF and normal neutron matter (in the latter SF thermal g-modes are absent). The left vertical dotted line shows a similar boundary for protons. The convectively unstable regions are filled with gray. Other notations are the same as in Fig. 1.

$T_{cp}(n_b)$ and triplet neutron $T_{cn}(n_b)$ critical temperatures, employed in our numerical calculations, are shown in Fig. 2(a). Figures 2(b) and 2(c) demonstrate the functions $\mathcal{N}(n_b)$ and $\nabla T^\infty_{\rm crit}(n_b)$, given by Eqs. (6) and (7), respectively. The key role in their behavior is played by the nucleon thermal excitations, whose number is very sensitive to T_{ci}, which is in turn a very strong function of n_b [especially, on the slopes of $T_{ci}(n_b)$, see Fig. 2(a)]. To illustrate this point we present different limiting cases in Figs. 2(b) and 2(c). Solid lines in Figs. 2(b) and 2(c) are obtained for $T_{cn}(n_b)$ and $T_{cp}(n_b)$ from Fig. 2(a). Dot-dashed lines are plotted for $T_{cn}(n_b)$ from Fig. 2(a), but for $T_{cp} \to \infty$. Finally, dashed lines in Figs. 2(b) and 2(c) correspond to the limit T_{cn}, $T_{cp} \to \infty$ of Fig. 1 (note, however, that Fig. 1 is plotted for different T). One can see that accounting for the nucleon thermal excitations strongly affects $\mathcal{N}(n_b)$ and $\nabla T^{\infty}_{\rm crit}(n_b)$. Let us note that when $T \ll T_{cp}$ the solid and dot-dashed curves coincide, because in that case all protons are paired. At the phase transition (when $T = T_{cp}$, left vertical dotted line in Fig. 2), solid lines for $\mathcal{N}$ and $\nabla T^{\infty}_{\rm crit}$ are discontinuous, since discontinuous are β and C.

Summarizing, from the analysis of Figs. 1 and 2 it follows that the thermal g-modes and convection in the internal layers of SF NSs are *extremely sensitive* to the EOS and the model of nucleon SF. An account for the singlet neutron SF at $n_b \lesssim 0.08\ \mathrm{fm}^{-3}$ may additionally affect $\mathcal{N}$ and $\nabla T^{\infty}_{\rm crit}$ near the crust-core interface.

IV. DISCUSSION AND CONCLUSION

Our results indicate that NSs can have convective internal layers. This could affect the thermal evolution of young NSs (such as in Cas A), for which ∇T^{∞} is not completely smoothed out by the thermal conductivity, as well as the thermal relaxation of quasipersistent X-ray transients [40,41]. Notice that in this paper we only consider thermal g-modes and convection in SF NS cores (but not in the crust). If the NS crust is *elastic* then it is most likely that core SF (thermal) g-modes do not penetrate the crust, while the crustal SF g-modes are "mixed" with the shear modes [9] (for which the restoring force is elasticity), and pushed to frequencies $\omega \gg \mathcal{N}$ (but see comment [42]). Then convection is absent. However, if the inner crust (especially, mantle [15]) is *plastic* [43], the existence of crustal SF g-modes and convection cannot be excluded, which can have even more interesting implications for NS cooling. The detailed analysis of (not yet well understood) effects of the SF thermal g-modes and related convection on the dynamics of NSs is postponed for future work.

How can the predicted thermal g-modes in SF NSs be excited? Among the potential scenarios is the excitation of stable g-modes by unstable ones (i.e., by convective motions). Another possibility was considered in Refs. [11,44] in application to composition g-modes of normal NSs. It consists in resonant excitations of SF thermal g-modes by tidal interaction in coalescing binary systems. Finally, SF thermal g-modes in rotating NSs could be excited due to gravitational driven (CFS) instability [12], though this scenario does not seem very realistic because $\mathcal{N}$ is low, which results in a large gravitational radiation time scale.

In this paper we considered nonrotating NSs (rotation frequency $\Omega = 0$). However, it is well known for *nonsuperfluid* NSs that, at $\Omega \gtrsim \mathcal{N}$, rotation affects g-modes (which transform into inertial waves [45]) and modifies the criterion for convection [46,47] (stabilizes the star). It is quite likely that the same is also true for SF NSs. Bearing in mind that $\mathcal{N}$ is small (e.g., for our NS model $\mathcal{N} \sim 1 \div 10^2\ \mathrm{s}^{-1}$ for $T = 1.5 \times 10^8$ K), this would mean that the effects discussed in this paper would play a role only in not too rapidly rotating NSs. Clearly, the problem of rotation deserves future attention.

To conclude, we have predicted specific thermal g-modes in SF NSs. We have shown that these modes can propagate in npe-matter with nonzero gradient of entropy per electron. We have calculated their Brunt-Väisälä frequency $\mathcal{N}$, which strongly depends on T and vanishes at $T = 0$. The predicted g-modes appear to be unstable for certain temperature gradients (that correspond to $\mathcal{N}^2 < 0$). We have derived the criterion for convective instability (analogue of the Schwarzschild criterion) in SF NS cores. We have shown that convection in the NS core may occur for both positive and negative temperature gradients and is extremely sensitive to the model of EOS and nucleon SF.

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