Relativistic entrainment matrix of a superfluid nucleon-hyperon mixture.
II. Effect of finite temperatures

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We calculate the important quantity of superfluid hydrodynamics, the relativistic entrainment matrix for a nucleon-hyperon mixture at *arbitrary* temperature. In the nonrelativistic limit this matrix is also termed the Andreev-Bashkin or mass-density matrix. Our results can be useful for modeling the pulsations of massive neutron stars with superfluid nucleon-hyperon cores and for studies of the kinetic properties of superfluid baryon matter.

I. INTRODUCTION

It is generally accepted that observations of pulsating neutron stars may potentially provide unique information on properties of superdense matter. For a correct interpretation of the observations one has to develop a hydrodynamic theory describing global pulsations.

In this paper we mainly focus on the hydrodynamics of superfluid nucleon-hyperon matter of massive neutron-star cores. More specifically, we study the important set of quantities of such hydrodynamics, the relativistic entrainment matrix. We assume the following matter composition: electrons (e), muons (μ), neutrons (n), protons (p), Λ-, and Σ^--hyperons (Λ and Σ, respectively). According to majority of calculations, Λ and Σ^- are the hyperons, which appear first in the stellar matter with the increasing density (see, however, Ref. [1] and the discussion in Sec. I of Ref. [2]).

Let us explain the physical meaning of the relativistic entrainment matrix. It is well known, that any baryon species i ($i = n$, p, Λ, or Σ) becomes superfluid when temperature T falls below some critical value T_{ci}. The microscopic calculations [3–10] predict that protons and hyperons pair in the spin singlet ($^1S_0$) state, while neutrons in the core pair in the spin triplet ($^3P_2$) state. In the most general case, when all baryon species are superfluid, we have four condensates and a normal (nonsuperfluid) liquid component. The normal component includes electrons, muons, and baryon Bogoliubov excitations. We assume that, because of collisions, all these 'normal' particles have the same hydrodynamic velocity. Below in this paper we always work in the frame, *comoving* with the normal liquid component.

If there are no superfluid currents in the system, then Cooper pairs are formed by particles with strictly opposite momenta (e.g., $\boldsymbol{p}$ and $-\boldsymbol{p}$). The presence of superfluid currents corresponds to a situation when pairing occurs between particles with momenta ($\boldsymbol{p} + \boldsymbol{Q}_i$) and ($-\boldsymbol{p} + \boldsymbol{Q}_i$). The total momentum of a Cooper pair is then $2\boldsymbol{Q}_i$. In this case the particle current density $\boldsymbol{j}_i$ of a species i can be written as [2,11]

$$\boldsymbol{j}_i = c^2 \sum_k Y_{ik} \boldsymbol{Q}_k. \tag{1}$$

Here the summation is performed over all baryon species,

$k = n$, p, Λ, and Σ; c is the speed of light. Finally, Y_{ik} is the 4×4 relativistic entrainment matrix. It is symmetric, $Y_{ik} = Y_{ki}$, and generally depends on the four baryon number densities n_i and temperature T. It follows from Eq. (1) that the superfluid motion of one species contributes to particle current density of another species (and vice versa). This 'entrainment' effect was first suggested by Andreev and Bashkin [12] in the context of superfluid solutions of ^{3}He in ^{4}He.

In the nonrelativistic limit the matrix Y_{ik} (more precisely, its analog, the matrix ρ_{ik}) was analyzed for a neutron-proton mixture at zero temperature in Refs. [13–15] and at arbitrary temperature in Ref. [16]. In our recent paper [2] (hereafter GKH09) we calculate the relativistic entrainment matrix Y_{ik} for the nucleon-hyperon matter in the approximation of zero temperature. The aim of the present study is to extend the analysis of GKH09 to finite temperatures. For that, the strongly interacting nucleon-hyperon mixture is considered in the frame of *relativistic* Landau theory of Fermi liquids, generalized to allow for possible superfluidity of baryons.

In Eq. (1) it is assumed that the matrix elements Y_{ik} are scalars. Strictly speaking, this is the case only if the matter is isotropic in the absence of superfluid currents, i.e., if all baryon species pair in the spin singlet ($^1S_0$) state. Meanwhile, neutrons undergo triplet-state pairing. This leads to anisotropic neutron energy gap $\Delta_{\boldsymbol{p}}^{(n)}$ and to appearance of the preferred direction along the neutron quantization axis. As a consequence, the matrix elements Y_{ik} are tensors with respect to spatial rotations (rather than scalars, as in the isotropic case). This makes the problem of calculation of the matrix Y_{ik} much more complex and model dependent [16].

To avoid this difficulty we assume, following Refs. [16,17], that the unperturbed matter can be treated as a collection of microscopic domains with chaotically directed neutron quantization axis. After averaging over the large number of such domains, the elements Y_{ik} will become scalars. One can obtain then qualitatively correct results for the averaged matrix elements Y_{ik}, assuming $^1S_0$ pairing of neutrons, and introducing an *effective* isotropic energy gap $\Delta_{\text{eff}}^{(n)}(T)$ in the neutron dispersion relation

$$\Delta_{\text{eff}}^{(n)}(T) = \min\{\Delta^{(n)}(|\boldsymbol{p}| = p_{\mathrm{F}n})\}. \tag{2}$$

Here $\Delta_{\rm eff}^{(n)}(T)$ is defined as the minimum of the angle-dependent gap $\Delta_{\boldsymbol{p}}^{(n)}$ on the neutron Fermi surface, see Refs. [16,17] for more details. Below in this paper we follow this strategy.

The paper is organized as follows. In Sec. II we formulate a relativistic Hamiltonian describing superfluid nucleon-hyperon mixture. In Sec. III we diagonalize this Hamiltonian, determine baryon dispersion relations in the presence of superfluid currents, and calculate the relativistic entrainment matrix Y_{ik}. Section IV presents the summary.

Throughout the paper, unless otherwise stated, indices i and k refer to baryons, $i, k = n, p, \Lambda$, and Σ. We use the system of units in which the Planck constant $\hbar$, the speed of light c, and the Boltzmann constant $k_{\rm B}$ equal unity, $\hbar = c = k_{\rm B} = 1$.

II. THE CHOICE OF A HAMILTONIAN

Let us obtain a model Hamiltonian for a homogeneous system of relativistic, degenerate, strongly interacting baryons with superfluid currents. A natural way to obtain such a Hamiltonian is to use the framework of Landau Fermi-liquid theory, generalized to account for superfluidity by Larkin and Migdal [18] and Leggett [19]. Following Ref. [19], the Hamiltonian H describing a superfluid nucleon-hyperon mixture can be generally presented in the form

$$H - \sum_i \mu_i N_i = H_{\rm LF} + H_{\rm pairing}. \tag{3}$$

Here, N_i and μ_i are, respectively, the number density operator and the chemical potential for baryon species $i = n, p, \Lambda, \Sigma$; $H_{\rm LF}$ is the Fermi-liquid Hamiltonian for the mixture; $H_{\rm pairing}$ is the pairing Hamiltonian.

If our system were *nonrelativistic*, $H_{\rm LF}$ would be given by (see, e.g., [16,19])

$$
\begin{aligned}
H_{\rm LF} = & \sum_{\boldsymbol{p}si} \varepsilon_0^{(i)}(\boldsymbol{p}) \left(a_{\boldsymbol{p}}^{(i)\dagger} a_{\boldsymbol{p}}^{(i)} - \theta_{\boldsymbol{p}}^{(i)} \right) \\
& + \frac{1}{2} \sum_{\boldsymbol{p}\boldsymbol{p}'ss'ik} f^{ik}(\boldsymbol{p},\boldsymbol{p}') \left(a_{\boldsymbol{p}}^{(i)\dagger} a_{\boldsymbol{p}}^{(i)} - \theta_{\boldsymbol{p}}^{(i)} \right) \left(a_{\boldsymbol{p}'}^{(k)\dagger} a_{\boldsymbol{p}'}^{(k)} - \theta_{\boldsymbol{p}'}^{(k)} \right).
\end{aligned}
\tag{4}
$$

Here $\boldsymbol{p}$ and $\boldsymbol{p}'$ are the particle momenta; s and s' are the spin indices; $a_{\boldsymbol{p}}^{(i)} \equiv a_{\boldsymbol{p}s}^{(i)} = a_{\boldsymbol{p}\uparrow}^{(i)}$ or $a_{\boldsymbol{p}\downarrow}^{(i)}$ is the annihilation operator of a Landau quasiparticle (not the Bogoliubov excitation) of a species i in a state $(\boldsymbol{p}s)$. We restrict ourselves to a spin-unpolarized nucleon matter. This allows us to simplify the notations by suppressing the spin indices, whenever possible. Furthermore, $\theta_{\boldsymbol{p}}^{(i)} = \theta(p_{{\rm F}i} - |\boldsymbol{p}|)$, where $\theta(x)$ is the step function; $\varepsilon_0^{(i)}(\boldsymbol{p}) = v_{{\rm F}i}(|\boldsymbol{p}| - p_{{\rm F}i})$, where $v_{{\rm F}i} = p_{{\rm F}i}/m_i^*$ and $p_{{\rm F}i}$ are, respectively, the Fermi-velocity and Fermi-momentum with m_i^* being the effective mass. Finally, $f^{ik}(\boldsymbol{p},\boldsymbol{p}')$ is the spin-averaged Landau quasiparticle interaction (we disregard the spin-dependence of this interaction since it does not affect our results). In the vicinity of the Fermi surface the arguments of the function $f^{ik}(\boldsymbol{p},\boldsymbol{p}')$ can be approximately put equal to $p \approx p_{{\rm F}i}$ and $p' \approx p_{{\rm F}k}$, while the function itself can be expanded into Legendre polynomials $P_l(\cos\theta)$,

$$f^{ik}(\boldsymbol{p},\boldsymbol{p}') = \sum_l f_l^{ik} P_l(\cos\theta), \tag{5}$$

where θ is the angle between $\boldsymbol{p}$ and $\boldsymbol{p}'$ and f_l^{ik} are the (symmetric) Landau parameters, $f_l^{ik} = f_l^{ki}$.

The Landau theory was extended to relativistic Fermi-liquids by Baym and Chin [20]. These authors showed that the formal structure of the theory remains practically the same as in the nonrelativistic case. In particular, the expression for the variation of the energy density has the same form. Thus, we assume that the Hamiltonian describing the *relativistic* mixture is still given by Eq. (4). We emphasize that this is our *assumption*; it would be interesting to derive this Hamiltonian from the microscopic theory in analogy to what was done by Leggett [19] for the case of nonrelativistic Fermi-liquid.

The relativistic effects affect only those properties of the theory which are related to transformation of various quantities from one frame to another. For example, as it is demonstrated in Ref. [20], the transformation law for the Landau quasiparticle interaction $f^{ik}(\boldsymbol{p},\boldsymbol{p}')$ is more complex than that in the nonrelativistic theory. Also, in the relativistic theory the expression for the effective mass m_i^* as a function of Landau parameters f_1^{ik} should be modified [2,20],

$$\frac{\mu_i}{m_i^*} = 1 - \sum_k \frac{\mu_k G_{ik}}{n_i}. \tag{6}$$

To reproduce the nonrelativistic result, one has to replace everywhere in this equation μ_i with the mass of a free particle m_i. In Eq. (6) n_i is the number density of particle species i,

$$n_i = \frac{p_{{\rm F}i}^3}{3\pi^2} \tag{7}$$

and the symmetric matrix G_{ik} is defined by

$$G_{ik} \equiv \frac{1}{9\pi^4} p_{{\rm F}i}^2 p_{{\rm F}k}^2 f_1^{ik}. \tag{8}$$

Let us obtain now an expression for the relativistic Hamiltonian $H_{\rm pairing}$, describing pairing between the Landau quasiparticles. The pairing in relativistic systems was studied in the literature mainly with application to quark matter (see, e.g., the early review [21] and a recent review [22] and references therein). However, there are also several papers that explore the importance of relativistic effects in terrestrial superconducting materials (see, e.g., Refs. [23–30]).

We start with the analysis of a pairing Hamiltonian for a mixture of relativistic *noninteracting* baryons. In other words, we neglect for a moment the Landau quasiparticle interaction $f^{ik}(\boldsymbol{p},\boldsymbol{p}')$ in Eq. (4) for $H_{\rm LF}$. As it is demonstrated in Refs. [23,24,28,29], in the mean-field approximation the relativistic analog of the well-known nonrelativistic Hamiltonian, responsible for $^1S_0$ pairing of particles, has the form

$$
\begin{aligned}
H_{\rm pairing} = & \frac{1}{2} \sum_i \int {\rm d}^3 r\, {\rm d}^3 r'\, \Delta^{(i)}(\boldsymbol{r},\boldsymbol{r}') \left[\overline{\Psi}^{(i)}(\boldsymbol{r}) \gamma^5 \Psi_{\rm C}^{(i)}(\boldsymbol{r}') \right. \\
& \left. - \frac{1}{2} \langle | \overline{\Psi}^{(i)}(\boldsymbol{r}) \gamma^5 \Psi_{\rm C}^{(i)}(\boldsymbol{r}') | \rangle \right] + {\rm H.c.}
\end{aligned}
\tag{9}
$$

Here $\Delta^{(i)}(\boldsymbol{r},\boldsymbol{r}') = \Delta^{(i)}(\boldsymbol{r}',\boldsymbol{r})$ is the superfluid order parameter; $\Psi^{(i)}(\boldsymbol{r})$ is the relativistic fermion field operator and $\Psi_C^{(i)}(\boldsymbol{r}) = i\gamma^2\gamma^0[\overline{\Psi}^{(i)}(\boldsymbol{r})]^T$ is the charge conjugate field (see, e.g., the textbook [31] for details). Furthermore, $\overline{\Psi}^{(i)}(\boldsymbol{r}) = \Psi^{(i)\dagger}(\boldsymbol{r})\gamma^0$; $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, where γ^l are the Dirac matrices ($l = 0, 1, 2,$ and 3). The second term in square brackets in Eq. (9) is presented to avoid double counting in the expression for the system energy. Terms of this kind is a typical feature of the mean-field approach. Similar terms appear in the mean-field formulation of the nonrelativistic pairing Hamiltonian (see, e.g., a review [32]).

The Hamiltonian (9) can be simplified if we take into account that for a homogeneous system with superfluid currents the order parameter $\Delta^{(i)}(\boldsymbol{r},\boldsymbol{r}')$ can be written in the form (see, e.g., [21])

$$\begin{aligned}\Delta^{(i)}(\boldsymbol{r},\boldsymbol{r}') &= \Delta^{(i)}(\boldsymbol{r}-\boldsymbol{r}')\,e^{i\boldsymbol{Q}_i(\boldsymbol{r}+\boldsymbol{r}')}\\ &= \sum_{\boldsymbol{p}}\Delta_{\boldsymbol{p}}^{(i)}\,e^{i\boldsymbol{p}(\boldsymbol{r}-\boldsymbol{r}')}\,e^{i\boldsymbol{Q}_i(\boldsymbol{r}+\boldsymbol{r}')}.\end{aligned} \quad (10)$$

Here $\Delta_{\boldsymbol{p}}^{(i)}$ is the Fourier component of the order parameter. It is even in $\boldsymbol{p}$,

$$\Delta_{\boldsymbol{p}}^{(i)} = \Delta_{-\boldsymbol{p}}^{(i)}, \quad (11)$$

since $\Delta^{(i)}(\boldsymbol{r},\boldsymbol{r}')$ is a symmetric function of $\boldsymbol{r}$ and $\boldsymbol{r}'$. Furthermore, $2\boldsymbol{Q}_i$ is the momentum of a Cooper pair in condensate. Below in this paper we always assume that $Q_i/p_{Fk} \ll 1$ and restrict ourselves to a *linear* approximation in Q_i/p_{Fk}.

Substituting Eq. (10) into Eq. (9), one gets, after some algebra [with an accuracy up to quadratic terms in Q_i/p_{Fk}],

$$\begin{aligned}H_{\text{pairing}} = -\sum_{\boldsymbol{p}i}\Delta_{\boldsymbol{p}}^{(i)}&\left[a_{\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)\dagger}a_{-\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)\dagger}-\frac{1}{2}\langle|a_{\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)\dagger}a_{-\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)\dagger}|\rangle\right.\\ &\left.+c_{\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)\dagger}c_{-\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)\dagger}-\frac{1}{2}\langle|c_{\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)\dagger}c_{-\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)\dagger}|\rangle\right]+\text{H.c.}\end{aligned}$$
$$(12)$$

The analogous derivation of a Hamiltonian for a homogeneous system is explained in detail in Ref. [28] for the case of $\boldsymbol{Q}_i = 0$. In Eq. (12) $c_{\boldsymbol{p}}^{(i)\dagger}$ is the creation operator for *antibaryons* of a species i. Generally, because H_{pairing} does not contain 'interference' terms (e.g, of the form $a_{\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)\dagger}c_{\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)\dagger}$), one may diagonalize the Hamiltonian $H_{\text{LF}} + H_{\text{pairing}}$ by simply performing the standard Bogoliubov transformation *separately* for baryons and antibaryons (this procedure is very clearly described in Ref. [28]; in this case the species index i in Eq. (4) for H_{LF} should also run over antibaryons). However, since the conditions in the neutron-star cores are such that the population of antibaryons is negligible, one can neglect them and rewrite H_{pairing} in its final form

$$\begin{aligned}H_{\text{pairing}} = -\sum_{\boldsymbol{p}i}\Delta_{\boldsymbol{p}}^{(i)}&\left[a_{\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)\dagger}a_{-\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)\dagger}\right.\\ &\left.-\frac{1}{2}\langle|a_{\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)\dagger}a_{-\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)\dagger}|\rangle\right]+\text{H.c.}\end{aligned} \quad (13)$$

We see that the Hamiltonian H_{pairing} for a homogeneous system of noninteracting relativistic baryons is given by the

same equation as in the nonrelativistic limit. Moreover, it coincides [16,19] with the pairing Hamiltonian describing *strongly interacting* nonrelativistic Fermi-liquid [the operators in Eq. (13) refer then to Landau quasiparticles rather than to real particles]. Thus, it is reasonable to assume, that H_{pairing} for interacting relativistic baryon mixture is also given by Eq. (13). All calculations below are made under such assumption.

In the Appendix we give further arguments supporting the form of the expression (13) for H_{pairing}. Namely, we apply the consideration given in this section to a specific case of relativistic mean-field model in which baryon interactions are mediated by various types of meson fields. We show that for this model the homogeneous Hamiltonian (13) can be directly calculated from the general expression (9).

In view of Eq. (11), in the linear approximation the superfluid order parameter $\Delta_{\boldsymbol{p}}^{(i)}$, entering Eq. (13), is the same function of $\boldsymbol{p}$ as in the system without superfluid currents. It can be chosen *real*, just as for the system with $\boldsymbol{Q}_i = 0$ (e.g., [33]). In this case $\Delta_{\boldsymbol{p}}^{(i)}$ is the energy gap in the dispersion relation for Bogoliubov quasiparticles (see below).

III. THE RELATIVISTIC ENTRAINMENT MATRIX AT FINITE T

A. General equations

It follows from Eqs. (4) and (13) that the relativistic expressions for H_{LF} and H_{pairing} are essentially the same as in the nonrelativistic case. Correspondingly, the further consideration is quite similar to that in Ref. [16]. Thus, here we only briefly sketch the main results, referring the reader to Ref. [16] for more details.

Introducing the Bogoliubov operators $b_{\boldsymbol{p}s}^{(i)}$

$$a_{\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)} = u_{\boldsymbol{p}}^{(i)}b_{\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)} + v_{\boldsymbol{p}}^{(i)}b_{-\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)\dagger}, \quad (14)$$

$$a_{\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)} = u_{\boldsymbol{p}}^{(i)}b_{\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)} - v_{\boldsymbol{p}}^{(i)}b_{-\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)\dagger}, \quad (15)$$

where $u_{\boldsymbol{p}}^{(i)}$ and $v_{\boldsymbol{p}}^{(i)}$ are even functions of $\boldsymbol{p}$ related by the normalization condition

$$u_{\boldsymbol{p}}^{(i)2} + v_{\boldsymbol{p}}^{(i)2} = 1, \quad (16)$$

one obtains for the system energy density E

$$\begin{aligned}E - \sum_i\mu_in_i = &\sum_{\boldsymbol{p}si}\varepsilon_0^{(i)}(\boldsymbol{p}+\boldsymbol{Q}_i)\left(\mathcal{N}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}-\theta_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}\right)\\ &+\frac{1}{2}\sum_{\boldsymbol{p}\boldsymbol{p}'ss'ik}f^{ik}(\boldsymbol{p}+\boldsymbol{Q}_i,\boldsymbol{p}'+\boldsymbol{Q}_k)\\ &\times\left(\mathcal{N}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}-\theta_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}\right)\left(\mathcal{N}_{\boldsymbol{p}'+\boldsymbol{Q}_k}^{(k)}-\theta_{\boldsymbol{p}'+\boldsymbol{Q}_k}^{(k)}\right)\\ &-\sum_{\boldsymbol{p}i}\Delta_{\boldsymbol{p}}^{(i)}u_{\boldsymbol{p}}^{(i)}v_{\boldsymbol{p}}^{(i)}\left(1-\mathcal{F}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}-\mathcal{F}_{-\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}\right).\end{aligned}$$
$$(17)$$

Here $\mathcal{N}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}$ and $\mathcal{F}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}$ are the distribution functions for Landau quasiparticles and Bogoliubov excitations with

momentum $(\boldsymbol{p}+\boldsymbol{Q}_i)$, respectively,

$$
\begin{aligned}
\mathcal{N}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)} &= \langle|a_{\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)\dagger}a_{\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)}|\rangle = \langle|a_{\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)\dagger}a_{\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)}|\rangle \\
&= v_{\boldsymbol{p}}^{(i)2} + u_{\boldsymbol{p}}^{(i)2}\mathcal{F}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)} - v_{\boldsymbol{p}}^{(i)2}\mathcal{F}_{-\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}, \quad (18)
\end{aligned}
$$

$$
\mathcal{F}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)} = \langle|b_{\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)\dagger}b_{\boldsymbol{p}+\boldsymbol{Q}_i\uparrow}^{(i)}|\rangle = \langle|b_{\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)\dagger}b_{\boldsymbol{p}+\boldsymbol{Q}_i\downarrow}^{(i)}|\rangle. \quad (19)
$$

Notice that, since the parameters $u_{\boldsymbol{p}}^{(i)}$ and $v_{\boldsymbol{p}}^{(i)}$ are even functions of $\boldsymbol{p}$, they do not depend on $\boldsymbol{Q}_i$ in the linear approximation.

The entropy of the system is given by the standard combinatorial expression

$$
S = -\sum_{\boldsymbol{p}si}\left[(1-\mathcal{F}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)})\ln(1-\mathcal{F}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}) + \mathcal{F}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}\ln\mathcal{F}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}\right].
$$

$$(20)$$

Using Eqs. (16)–(20) and minimizing the thermodynamic potential $F = E - \sum_i \mu_i n_i - TS$ with respect to $\mathcal{F}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}$ and $u_{\boldsymbol{p}}^{(i)}$, one gets

$$
\mathcal{F}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)} = \frac{1}{1+e^{\mathfrak{E}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}/T}}, \quad (21)
$$

$$
\begin{aligned}
\mathfrak{E}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)} &= \frac{1}{2}\left(H_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)} - H_{-\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}\right) \\
&\quad + \sqrt{\frac{1}{4}\left(H_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)} + H_{-\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}\right)^2 + \Delta_{\boldsymbol{p}}^{(i)2}}, \quad (22)
\end{aligned}
$$

$$
u_{\boldsymbol{p}}^{(i)2} = \frac{1}{2}\left(1 + \frac{H_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)} + H_{-\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}}{2\mathfrak{E}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)} + H_{-\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)} - H_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}}\right). \quad (23)
$$

Here $\mathfrak{E}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}$ is the energy of a Bogoliubov excitation of a species i with momentum $\boldsymbol{p}+\boldsymbol{Q}_i$; $H_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}$ is the energy of a Landau quasiparticle in normal (nonsuperfluid) matter,

$$
\begin{aligned}
H_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)} &= \varepsilon_0^{(i)}(\boldsymbol{p}+\boldsymbol{Q}_i) + \sum_{\boldsymbol{p}'s'k}f^{ik}(\boldsymbol{p}+\boldsymbol{Q}_i, \boldsymbol{p}'+\boldsymbol{Q}_k) \\
&\quad \times \left(\mathcal{N}_{\boldsymbol{p}'+\boldsymbol{Q}_k}^{(k)} - \theta_{\boldsymbol{p}'+\boldsymbol{Q}_k}^{(k)}\right). \quad (24)
\end{aligned}
$$

Since $Q_i \ll p_{Fk}$, it can be expanded in powers of $\boldsymbol{Q}_k$,

$$
H_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)} = \varepsilon^{(i)}(\boldsymbol{p}) + \Delta H_{\boldsymbol{p}}^{(i)}, \quad (25)
$$

where $\varepsilon^{(i)}(\boldsymbol{p}) \approx v_{Fi}(|\boldsymbol{p}| - p_{Fi})$ is the quasiparticle energy in the absence of superfluid currents and $\Delta H_{\boldsymbol{p}}^{(i)}$ is a small current-dependent term. In the linear approximation $\Delta H_{\boldsymbol{p}}^{(i)}$ can be generally written as

$$
\Delta H_{\boldsymbol{p}}^{(i)} = \sum_k \gamma_{ik}\, \boldsymbol{p}\, \boldsymbol{Q}_k, \quad (26)
$$

where γ_{ik} is a 4×4 matrix to be determined below (notice, that the definition of γ_{ik} differs by a factor of m_k from that used in Ref. [16]). Substituting Eqs. (22) and (25) into Eq. (23) one verifies that $u_{\boldsymbol{p}}^{(i)}$ (and $v_{\boldsymbol{p}}^{(i)}$) is indeed independent of $\boldsymbol{Q}_k$ in the linear approximation.

The distribution function $\mathcal{N}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}$ can also be expanded in powers of $\boldsymbol{Q}_i$. Using Eqs. (18) and (21)–(23), one obtains

$$
\mathcal{N}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)} = n_{\boldsymbol{p}}^{(i)} + \frac{\partial \mathfrak{f}_{\boldsymbol{p}}^{(i)}}{\partial E_{\boldsymbol{p}}^{(i)}}\Delta H_{\boldsymbol{p}}^{(i)}. \quad (27)
$$

Here $E_{\boldsymbol{p}}^{(i)}$, $\mathfrak{f}_{\boldsymbol{p}}^{(i)}$, and $n_{\boldsymbol{p}}^{(i)}$ denote, respectively, the quantities $\mathfrak{E}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}$, $\mathcal{F}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}$, and $\mathcal{N}_{\boldsymbol{p}+\boldsymbol{Q}_i}^{(i)}$, in the absence of superfluid currents. They are given by the following well-known expressions:

$$
E_{\boldsymbol{p}}^{(i)} = \sqrt{\varepsilon^{(i)2}(\boldsymbol{p}) + \Delta_{\boldsymbol{p}}^{(i)2}}, \quad (28)
$$

$$
\mathfrak{f}_{\boldsymbol{p}}^{(i)} = \frac{1}{1+e^{E_{\boldsymbol{p}}^{(i)}/T}}, \quad (29)
$$

$$
n_{\boldsymbol{p}}^{(i)} = v_{\boldsymbol{p}}^{(i)2} + \left(u_{\boldsymbol{p}}^{(i)2} - v_{\boldsymbol{p}}^{(i)2}\right)\mathfrak{f}_{\boldsymbol{p}}^{(i)}. \quad (30)
$$

B. Calculation of the matrix γ_{ik}

Let us calculate the matrix γ_{ik}. Substituting Eqs. (25) and (27) into Eq. (24), one obtains, with the accuracy to the terms linear in $\boldsymbol{Q}_i$,

$$
\Delta H_{\boldsymbol{p}}^{(i)} = \frac{\boldsymbol{p}\,\boldsymbol{Q}_i}{m_i^*} + \sum_{\boldsymbol{p}'s'k}f^{ik}(\boldsymbol{p},\boldsymbol{p}')\left\{\frac{\partial\mathfrak{f}_{\boldsymbol{p}'}^{(k)}}{\partial E_{\boldsymbol{p}'}^{(k)}}\,\Delta H_{\boldsymbol{p}'}^{(k)} - \frac{\partial\theta_{\boldsymbol{p}'}^{(k)}}{\partial\boldsymbol{p}'}\,\boldsymbol{Q}_k\right\}. \quad (31)
$$

The functions in curly brackets have sharp maximum near the Fermi surface of particle species k (at $p' \sim p_{Fk}$), so that the integrals in Eq. (31) can be approximately calculated and presented in the form

$$
\sum_{\boldsymbol{p}'s'}f^{ik}(\boldsymbol{p},\boldsymbol{p}')\frac{\partial\mathfrak{f}_{\boldsymbol{p}'}^{(k)}}{\partial E_{\boldsymbol{p}'}^{(k)}}\,\Delta H_{\boldsymbol{p}'}^{(k)} = -\frac{G_{ik}}{n_i}m_k^*\Phi_k\,\Delta H_{\boldsymbol{p}}^{(k)}, \quad (32)
$$

$$
\sum_{\boldsymbol{p}'s'}f^{ik}(\boldsymbol{p},\boldsymbol{p}')\frac{\partial\theta_{\boldsymbol{p}'}^{(k)}}{\partial\boldsymbol{p}'}\,\boldsymbol{Q}_k = -\frac{G_{ik}}{n_i}\,\boldsymbol{p}\,\boldsymbol{Q}_k. \quad (33)
$$

Here the matrix G_{ik} is defined in Eq. (8) while the function Φ_i is given by

$$
\Phi_i = -\frac{\pi^2}{m_i^*p_{Fi}}\sum_{\boldsymbol{p}s}\frac{\partial\mathfrak{f}_{\boldsymbol{p}}^{(i)}}{\partial E_{\boldsymbol{p}}^{(i)}}. \quad (34)
$$

It changes from $\Phi_i = 0$ at $T = 0$ to $\Phi_i = 1$ at $T \geqslant T_{ci}$ (we remind that T_{ci} is the critical temperature for transition of particle species i to the superfluid state). Assuming that the energy gap $\Delta_{\boldsymbol{p}}^{(i)}$ does not depend on the momentum $\boldsymbol{p}$, this function was calculated numerically and fitted in Ref. [34]. The fitting formula and further details concerning Φ_i can also be found in Ref. [16].

Using Eqs. (26), (32), and (33) and equating prefactors in front of the same $\boldsymbol{Q}_k$ in Eq. (31), one obtains the following system of 4×4 linear equations for the matrix γ_{ik}:

$$
\gamma_{ik} = \frac{\delta_{ik}}{m_i^*} + \frac{G_{ik}}{n_i} - \sum_l\frac{G_{il}}{n_i}m_l^*\,\Phi_l\,\gamma_{lk}. \quad (35)
$$

Notice that this matrix is *not* symmetric, $\gamma_{ik} \neq \gamma_{ki}$. In Eq. (35) for each k we have four equations (with $i = n, p, \Lambda$, and Σ). These equations can be decoupled from the whole system and solved separately. Thus, there are in fact four independent systems of four linear equations. Their solutions can be immediately found in a number of limiting cases:

(i) For very low temperatures, when all $\Phi_i = 0$,

$$\gamma_{ik} = \frac{\delta_{ik}}{m_i^*} + \frac{G_{ik}}{n_i}. \qquad (36)$$

(ii) For nonsuperfluid matter, when all $\Phi_i = 1$,

$$\gamma_{ik} = \frac{\delta_{ik}}{m_i^*}. \qquad (37)$$

(iii) If the species in the mixture do not interact with each other, that is $f_l^{ik} = 0$ for $i \neq k$, one has [see Eqs. (6) and (8)]

$$\gamma_{ik} = \frac{n_i \delta_{ik}}{\mu_i(n_i + G_{ii}m_i^* \Phi_i)}. \qquad (38)$$

(iv) If $\Phi_i = 0$ for all species except for the species j (i.e., $\Phi_j \neq 0$), then

$$\gamma_{ik} = \frac{\delta_{ik}}{m_i^*} + \frac{G_{ik}}{n_i} - \frac{G_{ij}(G_{jk}m_j^* + n_j \delta_{jk})\Phi_j}{n_i(n_j + G_{jj}m_j^* \Phi_j)}. \quad (39)$$

(v) Finally, for any nonsuperfluid species j (for which $\Phi_j = 1$), the matrix elements γ_{ij} related to this species, are given by the same Eq. (37) as in the case of a completely nonsuperfluid mixture,

$$\gamma_{ij} = \frac{\delta_{ij}}{m_i^*}. \qquad (40)$$

The general solution for the matrix γ_{ik} in case of four particle species is rather lengthy, but can be easily obtained from Eq. (35) and, for example, incorporated into a computer code. Here we present the solution for an important case of a mixture of two particle species (e.g., neutrons and protons)

$$\gamma_{ii} = \frac{(n_i + G_{ii}m_i^*)(n_k + G_{kk}m_k^* \Phi_k) - G_{ik}^2 m_i^* m_k^* \Phi_k}{m_i^* S}, \quad (41)$$

$$\gamma_{ik} = \frac{G_{ik}n_k(1 - \Phi_k)}{S}, \qquad (42)$$

where

$$S = (n_i + G_{ii}m_i^* \Phi_i)(n_k + G_{kk}m_k^* \Phi_k) - G_{ik}^2 m_i^* m_k^* \Phi_i \Phi_k. \qquad (43)$$

In Eqs. (41)–(43) indices i and k refer to different particle species, $i \neq k$. The presented solution agrees with that given in Ref. [16] for the nonrelativistic case (we remind the reader that the definition of γ_{ik} in Ref. [16] slightly differs from ours, see the comment to Eq. (26)). However, in contrast to Ref. [16], in Eqs. (41)–(43) one should use the *relativistic* expression for the effective masses given by Eq. (6).

C. Calculation of the matrix Y_{ik}

To find the relativistic entrainment matrix Y_{ik} let us calculate the particle current density $\boldsymbol{j}_i$. For that, we make use of the fact that, as was emphasized by Leggett [19], the expression for $\boldsymbol{j}_i$ has the same form as in the case of usual non-superfluid Fermi-liquid (see also Refs. [2,13–16]).

In other words,

$$\boldsymbol{j}_i = \sum_{ps} \frac{\partial H_{\boldsymbol{p}+\boldsymbol{\varrho}_i}^{(i)}}{\partial \boldsymbol{p}} \, \mathcal{N}_{\boldsymbol{p}+\boldsymbol{\varrho}_i}^{(i)}. \qquad (44)$$

Substituting expressions (25) and (27) into this equation and performing a simple integration, one obtains Eq. (1) with

$$Y_{ik} = n_i \, \gamma_{ik} \, (1 - \Phi_i). \qquad (45)$$

One may check that this matrix is symmetric, $Y_{ik} = Y_{ki}$. At zero temperature this equation can be simplified with the help of Eq. (36), so that one reproduces the result of GKH09,

$$Y_{ik} = \frac{n_i}{m_i^*}\delta_{ik} + G_{ik}. \qquad (46)$$

Notice that, as follows from Eq. (6), in this limit the matrix Y_{ik} satisfies the sum rule (see GKH09),

$$\sum_k \mu_k Y_{ik} = n_i. \qquad (47)$$

In the nonrelativistic limit the matrix Y_{ik} is related to the nonrelativistic entrainment matrix ρ_{ik} by the equation (see, e.g., [11])

$$\rho_{ik} = m_i m_k Y_{ik}. \qquad (48)$$

Using this equation and Eq. (45), one may reproduce the various limiting formulas for ρ_{ik}, presented in the literature [13,16,19].

Our results are illustrated in Fig. 1. The normalized symmetric matrix Y_{ik}/Y is shown as a function of temperature T for the baryon number density $n_b = 3n_0 = 0.48$ fm^{-3}. To plot the figure we employed the *third* equation of state of Glendenning [35]. The Landau parameters f_1^{ik} of nucleon-hyperon matter were calculated for this equation of state in GKH09. The normalization constant Y is taken to be $Y = 3n_0/\mu_n(3n_0) = 2.48 \times 10^{41}$ erg^{-1} cm^{-3}, where $n_0 = 0.16$ fm^{-3} is the normal nuclear density and $\mu_n(3n_0) = 1.94 \times 10^{-3}$ erg is the neutron chemical potential at $n_b = 3n_0$. We choose the baryon critical temperatures T_{ci} ($i = n, p, \Lambda, \Sigma$) equal to $T_{cn} = 5 \times 10^8$ K, $T_{cp} = 2 \times 10^9$ K, $T_{c\Lambda} = 3 \times 10^9$ K, and $T_{c\Sigma} = 6 \times 10^9$ K.

On the left panel we plot the 'nucleon' matrix elements; the 'hyperon' matrix elements are shown on the right panel. In agreement with Eq. (45), the matrix elements, involving a non-superfluid species, vanish. For instance, if neutrons are normal (i.e., $T > T_{cn}$), then $\Phi_n = 1$ and $Y_{ni} = Y_{in} = 0$. At $T \lesssim 10^8$ K all the matrix elements approach their asymptotic zero-temperature values [see Eq. (46)]. As follows from the figure, the nondiagonal matrix elements are essentially smaller than the diagonal ones and are all negative for the chosen equation of state.

IV. SUMMARY

The present paper is a continuation of GKH09, where the relativistic entrainment matrix Y_{ik} of nucleon-hyperon mixture was calculated for the case of zero temperature. Here we extend the results of GKH09 to finite temperatures. For that, we employ the relativistic Landau Fermi-liquid theory [20], generalized to allow for superfluidity of baryons [18,19].
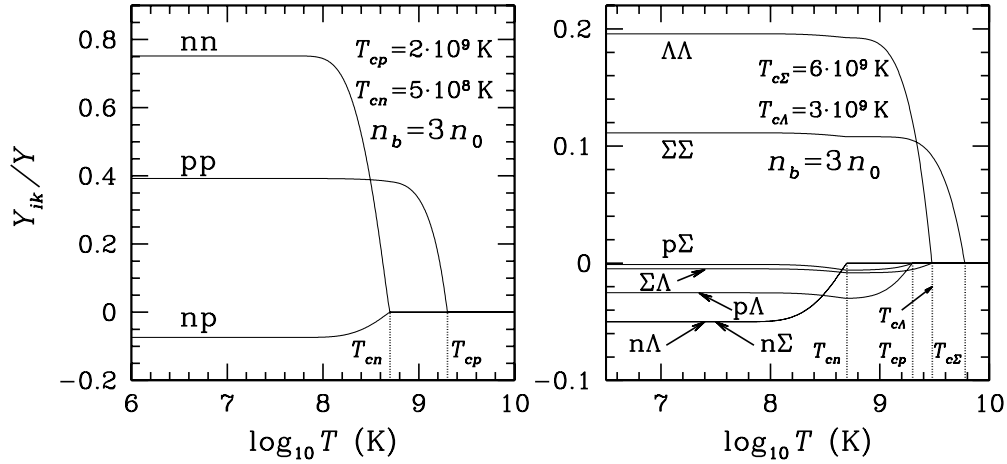
FIG. 1. Normalized symmetric matrix Y_{ik}/Y as a function of T for the third equation of state of Glendenning [35]. The normalization constant $Y = 3n_0/\mu_n(3n_0) = 2.48 \times 10^{41}$ erg^{-1} cm^{-3}. Solid lines show the elements of the matrix Y_{ik}/Y; each curve is marked by the corresponding symbol ik ($i, k = n, p, \Lambda, \Sigma$). Vertical dotted lines indicate baryon critical temperatures.

We demonstrate that, as in the case of nonrelativistic neutron-proton mixture [16], the matrix Y_{ik} is expressed through the Landau parameters f_1^{ik} and the function of temperature Φ_i. The Landau parameters f_1^{ik} for the relativistic nucleon-hyperon matter were derived in GKH09. The quantity $\Phi_i(T)$ is the universal function of T under assumption that the baryon energy gap $\Delta_{\boldsymbol{p}}^{(i)}$ is independent of momentum $\boldsymbol{p}$. In the latter case the fitting formula for Φ_i can be found in Refs. [16,34].

Our consideration of Y_{ik} differs from the previous calculations of the entrainment matrix, available in the neutron-star literature [13–16], by the following:

(i) we consider the problem in a fully relativistic framework;

(ii) we allow for the presence of two hyperon species (Λ- and Σ^--hyperons), in addition to neutrons and protons.

It should be noted that our results can be easily extended to describe any number of baryon species (not necessarily four). The main problem then will be to determine the Landau parameters for these species.

The calculated relativistic entrainment matrix Y_{ik} is an essential ingredient in hydrodynamics of superfluid mixtures [36]. It can be important for studying the pulsations of massive neutron stars with superfluid nucleon-hyperon cores (see Ref. [37] for an example of such study). Also, since the matrix Y_{ik} enters the expression (21) for the equilibrium distribution function of Bogoliubov excitations, it can influence various kinetic properties of superfluid baryon matter, for example, the shear viscosity. The related problems will be analyzed in a separate publication.

APPENDIX

The general approach of Sec. II can be illustrated with a specific example of a relativistic mean-field model in which interactions between baryons are mediated by meson fields. For definiteness, following GKH09, we consider σ-ω-ρ version of the mean-field model with self-interactions of scalar σ-field. However, our consideration remains essentially unaffected if one allows for additional meson fields (e.g., δ-meson or hidden strangeness σ^* and ϕ-meson fields).

In GKH09 we reformulated the σ-ω-ρ mean-field model in terms of the relativistic Landau Fermi-liquid theory. In particular, we calculated the Landau quasiparticle interaction function $f^{ik}(\boldsymbol{p}, \boldsymbol{p}')$. Thus, we determined the Hamiltonian H_{LF} for this model [see Eq. (4)]. Let us now turn to the Hamiltonian H_{pairing}. It is still given by Eq. (9) with

$$\Psi^{(i)}(\boldsymbol{r}) = \sum_{ps} \frac{1}{\sqrt{2\mathcal{E}_i}} \left(a_{ps}^{(i)} u^{(i)}(\boldsymbol{p}, s) e^{i\boldsymbol{pr}} + c_{ps}^{(i)\dagger} v^{(i)}(\boldsymbol{p}, s) e^{-i\boldsymbol{pr}} \right), \tag{A1}$$

$$\Psi_{\mathrm{C}}^{(i)}(\boldsymbol{r}) = \sum_{ps} \frac{i}{\sqrt{2\mathcal{E}_i}} \left(a_{ps}^{(i)\dagger} v^{(i)}(\boldsymbol{p}, s) e^{-i\boldsymbol{pr}} + c_{ps}^{(i)} u^{(i)}(\boldsymbol{p}, s) e^{i\boldsymbol{pr}} \right). \tag{A2}$$

The bispinors $u^{(i)}(\boldsymbol{p}, s)$ and $v^{(i)}(\boldsymbol{p}, s)$ correspond, respectively, to particles and antiparticles. For the mean-field model they can be written out explicitly (see, e.g., Ref. [38]),

$$u^{(i)}(\boldsymbol{p}, s) = \begin{pmatrix} \sqrt{\mathcal{E}_i + M_i^*}\, w_s \\ \sqrt{\mathcal{E}_i - M_i^*}\, (\boldsymbol{n\sigma}) w_s, \end{pmatrix}$$

$$v^{(i)}(\boldsymbol{p}, s) = \begin{pmatrix} \sqrt{\mathcal{E}_i - M_i^*}\, (\boldsymbol{n\sigma}) w_{-s}' \\ \sqrt{\mathcal{E}_i + M_i^*}\, w_{-s}'. \end{pmatrix} \tag{A3}$$

In Eqs. (A1)–(A3) $\mathcal{E}_i = \sqrt{\boldsymbol{P}^2 + M_i^{*2}}$; unit vector **n** is directed along $\boldsymbol{P}$; $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector composed of Pauli matrices; w_s and w_s' are the spinors which are defined as in Ref. [39] (see §23 of this reference). Furthermore,

$$\boldsymbol{P} = \boldsymbol{p} - g_{\omega i}\boldsymbol{\omega} - g_{\rho i} I_{3i}\boldsymbol{\rho}_3, \qquad (A4)$$

$$M_i^* = m_i - g_{\sigma i}\sigma, \qquad (A5)$$

where $\boldsymbol{\omega}$, $\boldsymbol{\rho}_3$, and σ are the meson fields that are generated by baryon currents and densities; $g_{\omega i}$, $g_{\rho i}$, and $g_{\sigma i}$ are the coupling constants; and I_{3i} is the isospin projection for baryon species i (for more details, see, e.g., GKH09).

Substituting now field operators (A1) and (A2) into the expression (9) for H_{pairing}, one verifies, that the homogeneous pairing Hamiltonian for the mean-field σ-ω-ρ model is given by the same expression as was obtained in Sec. II [see Eq. (12) of this section or Eq. (13), if we neglect antibaryons]. This particular example presents an additional argument supporting the conjecture that the general expression for H_{pairing} in the relativistic Landau Fermi-liquid theory, is also of the form (13).

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