

Damping of sound waves in superfluid nucleon-hyperon matter of neutron stars

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We consider sound waves in superfluid nucleon-hyperon matter of massive neutron-star cores. We calculate and analyze the speeds of sound modes and their damping times due to the shear viscosity and nonequilibrium weak processes of particle transformations. For that, we employ the dissipative relativistic hydrodynamics of a superfluid nucleon-hyperon mixture, formulated recently [1]. We demonstrate that the damping times of sound modes calculated using this hydrodynamics and the ordinary (nonsuperfluid) one, can differ from each other by several orders of magnitude.

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I. INTRODUCTION

In recent years there is a growing interest in studies of neutron-star pulsations. This is related to a number of reasons. First of all, the recently discovered high frequency oscillations of electromagnetic radiation during giant flares may be associated with the pulsations of neutron stars [2,3]. Second, the gravitational-wave detectors, which will be able to detect gravitational radiation from isolated pulsating neutron stars, are under construction [4–7].

For interpretation of the observations, it is important to have a well-developed theory of neutron-star pulsations. The construction of such a theory is complicated by the fact that the baryons in neutron-star cores can be in a superfluid state [8–13]. Thus, to study the pulsations, one has to use a hydrodynamics describing mixtures of superfluid liquids. There is a substantial body of literature, devoted to pulsations of superfluid neutron stars (see, e.g., Refs. [14–27]). All these papers deal with the nucleon $npe(\mu)$ matter composed of neutrons (n), protons (p), and electrons (e) with possible admixture of muons (μ). Most of them study pulsations at zero temperature (see, however, Refs. [16,17]).

In this paper we for the first time investigate dynamic properties of superfluid *nucleon-hyperon* matter in the cores of massive neutron stars, composed, in addition to neutrons, protons, electrons, and muons, of Λ and Σ^- hyperons (Λ and Σ , respectively). For that, we employ the relativistic hydrodynamics [1], describing a superfluid nucleon-hyperon mixture at *arbitrary* temperature. We study the simplest pulsations in such matter—sound modes, how they travel and how they damp. Within this simple example we demonstrate, in particular, that the characteristic damping times of pulsations, calculated self-consistently in the frame of superfluid hydrodynamics, can differ by several orders of magnitude from those calculated using the nonsuperfluid hydrodynamics (in the latter case the effects of superfluidity are taken into account only at calculating kinetic coefficients).

The paper is organized as follows. In Sec. II, we give an overview of the main reactions of particle mutual trans-

formations in the nucleon-hyperon matter. In Sec. III, we briefly discuss the relativistic dissipative hydrodynamics describing superfluid nucleon-hyperon mixture. In Sec. IV, we analyze the sound modes in such matter neglecting dissipation. In Sec. V, we calculate the damping times of sound modes due to the nonequilibrium reactions (1)–(4) [see Sec. II] and shear viscosity. Section IV presents a summary.

Throughout the paper, unless otherwise stated, we use the system of units in which the Planck constant \hbar , the speed of light c and the Boltzmann constant k_B equal unity, $\hbar = c = k_B = 1$.

II. THE MAIN PROCESSES OF PARTICLE TRANSFORMATIONS IN NUCLEON-HYPERON MATTER

The most effective weak processes in nucleon-hyperon matter are the following nonleptonic reactions [28–31]

$$n + n \leftrightarrow p + \Sigma^-, \quad (1)$$

$$n + p \leftrightarrow p + \Lambda, \quad (2)$$

$$n + n \leftrightarrow n + \Lambda, \quad (3)$$

$$n + \Lambda \leftrightarrow \Lambda + \Lambda. \quad (4)$$

The full thermodynamic equilibrium implies, in particular, the equilibrium with respect to these reactions,

$$\delta\mu_1 \equiv 2\mu_{n0} - \mu_{p0} - \mu_{\Sigma0} = 0, \quad (5)$$

$$\delta\mu_2 = \delta\mu_3 = \delta\mu_4 \equiv \mu_{n0} - \mu_{\Lambda0} = 0. \quad (6)$$

In this case the average number of direct reactions in unit volume per unit time is equal to the number of inverse reactions. In Eqs. (5) and (6) μ_{i0} are the chemical potentials of particle species $i = n, p, \Lambda$, and Σ , taken at equilibrium; $\delta\mu_m$ ($m = 1, \dots, 4$) are the disbalances of the chemical potentials for the reactions (1)–(4). In what follows, we mark the equilibrium values of thermodynamic quantities with the subscript 0. Accordingly, the thermody-

dynamic quantities without the subscript 0 (e.g., μ_i) refer to perturbed matter. Notice that the equilibrium conditions for the reactions (2)–(4) coincide.

Equations (5) and (6) do not hold in the perturbed matter ($\delta\mu_1 \neq 0$, $\delta\mu_2 = \delta\mu_3 = \delta\mu_4 \neq 0$), so that the numbers of direct and inverse reactions are not equal. The nonequilibrium reactions (1)–(4) act to return the system to the equilibrium state. This leads to dissipation of mechanical energy, accumulated in the matter.

Along with the reactions (1)–(4) there is a number of weak reactions with leptons. The leptonic reactions (e.g., the direct and modified Urca processes with electrons or muons) are much slower in comparison to the reactions (1)–(4). In the interesting range of parameters (temperatures and pulsation frequencies), they cannot influence substantially chemical composition of the perturbed matter and will be neglected in what follows. However, we assume that the unperturbed matter satisfies the equilibrium conditions with respect to these reactions,

$$\mu_{n0} = \mu_{p0} + \mu_{e0}, \quad \mu_{e0} = \mu_{\mu0}, \quad (7)$$

where μ_{e0} and $\mu_{\mu0}$ are the equilibrium chemical potentials for electrons and muons, respectively. In addition to the processes described above, there is a fast nonleptonic reaction due to the strong interaction of baryons,

$$n + \Lambda \leftrightarrow p + \Sigma^-. \quad (8)$$

We assume that the perturbed matter is always in equilibrium with respect to this reaction [1,30],

$$\delta\mu_{\text{fast}} \equiv \mu_n + \mu_\Lambda - \mu_p - \mu_\Sigma = 0. \quad (9)$$

It follows from the condition (9), that the chemical potential disbalances for all the four reactions (1)–(4) coincide,

$$\delta\mu \equiv \delta\mu_1 = \delta\mu_2 = \delta\mu_3 = \delta\mu_4. \quad (10)$$

Now the equilibrium conditions (5) and (6) can be rewritten as

$$\delta\mu = 0. \quad (11)$$

Below we denote the difference between the average number of direct and inverse reactions (1)–(4), occurring in the unit volume per unit time, by $\Delta\Gamma_1, \dots, \Delta\Gamma_4$, respectively. In this paper we consider small deviations from the chemical equilibrium, $\delta\mu \ll T$. In this case the quantities $\Delta\Gamma_m$ ($m = 1, \dots, 4$) can be expanded in powers of $\delta\mu$ and presented in the linear approximation as (see, e.g., [1,29,30])

$$\Delta\Gamma_m = \lambda_m \delta\mu, \quad (12)$$

where λ_m are the rates of the reactions (1)–(4), some functions of the number densities and temperature.

III. THE RELATIVISTIC HYDRODYNAMICS OF A SUPERFLUID NUCLEON-HYPERON MIXTURE

In this and subsequent sections, the subscripts i and k refer to baryons ($i, k = n, p, \Lambda, \Sigma$). The summation is assumed over repeated baryon indices i and k . The subscript l refers to leptons ($l = e, \mu$); the subscript j runs over all particle species ($j = n, p, \Lambda, \Sigma, e, \mu$); μ and ν are the space-time indices.

The relativistic hydrodynamics, describing a nucleon-hyperon mixture, composed of superfluid neutrons, protons, Λ and Σ^- hyperons, as well as normal electrons and muons, has been formulated in Ref. [1]. It is a direct generalization of the hydrodynamics of superfluid npe matter [17]. In this section we briefly discuss the main equations of this hydrodynamics.

A. The nondissipative hydrodynamics

Let us discuss first the nondissipative hydrodynamics of superfluid nucleon-hyperon matter. Equations of superfluid hydrodynamics include the continuity equations for particle species j

$$\partial_\mu j_{(j)}^\mu = 0, \quad (13)$$

where the particle four currents $j_{(j)}^\mu$ equal

$$j_{(i)}^\mu = n_i u^\mu + Y_{ik} w_{(k)}^\mu, \quad j_{(l)}^\mu = n_l u^\mu; \quad (14)$$

energy-momentum conservation law

$$\partial_\mu T^{\mu\nu} = 0, \quad (15)$$

where the energy-momentum tensor $T^{\mu\nu}$ equals

$$T^{\mu\nu} = (P + \varepsilon) u^\mu u^\nu + P \eta^{\mu\nu} + Y_{ik} [w_{(i)}^\mu w_{(k)}^\nu + \mu_i w_{(k)}^\mu u^\nu + \mu_k w_{(i)}^\nu u^\mu]; \quad (16)$$

the second law of thermodynamics

$$d\varepsilon = TdS + \mu_i dn_i + \mu_e dn_e + \mu_\mu dn_\mu + \frac{Y_{ik}}{2} d[w_{(i)}^\mu w_{(k)\mu}], \quad (17)$$

and a number of conditions for superfluid components, which are specified below (see Eqs. (20) and (21)). In Eqs. (13)–(17) $\eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ is the special relativistic metric; n_j is the number density of particle species j ; ε , T , and S are the energy density, temperature, and entropy density, respectively; P is the pressure, which is defined by the same formula as for nonsuperfluid matter [16,17]

$$P = -\varepsilon + \mu_i n_i + \mu_e n_e + \mu_\mu n_\mu + TS; \quad (18)$$

u^μ is the four velocity of the normal (nonsuperfluid) liquid component, normalized so that

$$u_\mu u^\mu = -1. \quad (19)$$

(We assume that all the nonsuperfluid components move with the same velocity u^μ .) Next, Y_{ik} is the symmetric relativistic entrainment matrix of nucleon-hyperon mixture, calculated in Ref. [32] for zero temperature and in Ref. [33] for arbitrary temperature. The important property is that if some particles (e.g., neutrons) are nonsuperfluid then the related elements of this matrix vanish, $Y_{nk} = Y_{kn} = 0$ ($k = n, p, \Lambda, \Sigma$). In the nonrelativistic limit this 4×4 matrix is expressed through the nonrelativistic entrainment matrix ρ_{ik} as [16,17] $Y_{ik} = \rho_{ik}/(m_i m_k)$, where m_i is the mass of a free baryon species i (the matrix ρ_{ik} is a generalization of the superfluid density to the case of mixtures, see, e.g., Refs. [34–36]). The motion of superfluid component of a species i is described by the four vector $w_{(i)}^\mu$, which meets the condition

$$u_\mu w_{(i)}^\mu = 0. \quad (20)$$

The potentiality of superfluid motion is expressed as

$$\partial^\nu [w_{(i)}^\mu + q_i A^\mu + \mu_i u^\mu] = \partial^\mu [w_{(i)}^\nu + q_i A^\nu + \mu_i u^\nu], \quad (21)$$

where q_i is the electric charge of particle species i ; A^μ is the four potential of the electromagnetic field. The potentiality condition (21) is equivalent to a statement that there is a scalar function ϕ_i , satisfying (see Refs. [1,17])

$$\partial^\mu \phi_i = w_{(i)}^\mu + q_i A^\mu + \mu_i u^\mu. \quad (22)$$

The scalar ϕ_i is related to the wave function phase of the Cooper-pair condensate Φ_i by the equality $\nabla \phi_i = \nabla \Phi_i/2$. In the nonrelativistic limit the spatial parts \mathbf{u} and $\mathbf{w}_{(i)}$ of the four vectors u^μ and $w_{(i)}^\mu$ transform into

$$\mathbf{u} = \mathbf{V}_q, \quad \mathbf{w}_{(i)} = m_i(\mathbf{V}_{s(i)} - \mathbf{V}_q), \quad (23)$$

where \mathbf{V}_q and $\mathbf{V}_{s(i)} = (\nabla \phi_i - q_i \mathbf{A})/m_i$ are, respectively, the normal and superfluid velocities of the nonrelativistic theory of superfluid liquids [37,38].

The hydrodynamics described above would be incomplete without an indication in what reference frame we define (measure) the main thermodynamic quantities (i.e., what frame is comoving). As was demonstrated in Ref. [17], the condition (20) dictates that the comoving is the frame where the four velocity u^μ equals $u^\mu = (1, 0, 0, 0)$. In this frame, the basic thermodynamic quantities ε , n_j , and $\mathbf{w}_{(i)}$ (or $\nabla \phi_i$) are defined by [see Eqs. (14), (16), and (20)]

$$j_j^0 = n_j, \quad (24)$$

$$\mathbf{j}_i = Y_{ik} \mathbf{w}_{(k)} = Y_{ik} (\nabla \phi_k - q_k \mathbf{A}), \quad (25)$$

$$T^{00} = \varepsilon. \quad (26)$$

All other thermodynamic quantities in nonequilibrium matter are the same functions of ε , n_j , and $\mathbf{w}_{(i)}$ (or, equiv-

alently, ε , n_j , and $w_{(i)}^\mu w_{(k)\mu}$) as in the full thermodynamic equilibrium.

B. Viscous dissipation

The main dissipative mechanisms in the pulsating nucleon-hyperon matter are the shear viscosity and effective bulk viscosity due to the nonequilibrium processes (1)–(4). These are the two mechanisms that will be analyzed in this paper.

The shear viscosity leads to an additional term in the energy-momentum tensor $T^{\mu\nu}$. It now takes the form

$$T^{\mu\nu} = (P + \varepsilon)u^\mu u^\nu + P\eta^{\mu\nu} + Y_{ik} [w_{(i)}^\mu w_{(k)}^\nu + \mu_i w_{(k)}^\mu u^\nu + \mu_k w_{(i)}^\nu u^\mu] + \tau_{\text{sh}}^{\mu\nu}, \quad (27)$$

where

$$\tau_{\text{sh}}^{\mu\nu} = -\eta H^{\mu\gamma} H^{\nu\delta} \left(\partial_\delta u_\gamma + \partial_\gamma u_\delta - \frac{2}{3} \eta_{\gamma\delta} \partial_\varepsilon u^\varepsilon \right); \quad (28)$$

$H^{\mu\nu} \equiv \eta^{\mu\nu} + u^\mu u^\nu$ is the projection matrix; η is the shear viscosity coefficient.

As has been already mentioned, the effect of the nonequilibrium processes (1)–(4) can be described in terms of the bulk viscosity formalism (for more details, see [1,29,30]). However, for the analysis of sound modes it is more convenient to take these processes into account explicitly, by introducing sources into the right-hand sides of the continuity Eq. (13),

$$\partial_\mu j_{(j)}^\mu = \Delta S_j, \quad (29)$$

similar to how it was done for normal (nonsuperfluid) npe matter in Ref. [39]. Here, ΔS_j is a number of particles of a species j , generated in the unit volume per unit time in the reactions (1)–(4). Since we neglect the leptonic reactions, $\Delta S_e = \Delta S_\mu = 0$.

As a result, the hydrodynamics describing a *viscous* superfluid nucleon-hyperon mixture differs from the non-dissipative hydrodynamics of Sec. III A only by the expression for the energy-momentum tensor [Eq. (27) instead of (16)] and by the continuity equations [Eq. (29) instead of (13)]. Using this hydrodynamics, one can derive the entropy generation equation (see also [1,17])

$$T \partial_\mu (S u^\mu) = \delta \mu \Delta \Gamma - \tau_{\text{sh}}^{\mu\nu} \partial_\mu u_\nu, \quad (30)$$

where we define

$$\Delta \Gamma \equiv \Delta \Gamma_1 + \Delta \Gamma_2 + \Delta \Gamma_3 + \Delta \Gamma_4 = \lambda \delta \mu, \quad (31)$$

with $\lambda \equiv \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ [see Eq. (12)].

IV. SOUND WAVES IN THE ABSENCE OF DISSIPATION

A. The main assumptions

We assume that all the hydrodynamic velocities equal zero in the equilibrium, $u^\mu = (1, 0, 0, 0)$ and $w_{(i)}^\mu = (0, 0, 0, 0)$ ($i = n, p, \Lambda, \Sigma$). [In principle, it is not necessary to have vanished all the components of the four vector $w_{(i)}^\mu$. It is well known, that even in thermodynamic equilibrium a motion is possible with nonzero superfluid velocities [37]. This means that generally, the spatial components of the four vector $w_{(i)}^\mu$ can be nonzero. As for the time component $w_{(i)}^0$, it vanishes in any case, as it follows from Eq. (20).]

We assume also that in the perturbed matter the quasi-neutrality condition holds

$$n_p = n_e + n_\mu + n_\Sigma, \quad (32)$$

in addition to the equilibrium condition (9) with respect to the fast reaction (8). Using Eq. (32) and the continuity Eq. (29) for protons, Σ^- hyperons, electrons, and muons, as well as the fact that $\Delta S_p - \Delta S_e - \Delta S_\mu - \Delta S_\Sigma = 0$, one obtains [1]

$$\partial_\mu [Y_{pk} w_{(k)}^\mu] = \partial_\mu [Y_{\Sigma k} w_{(k)}^\mu]. \quad (33)$$

In this paper we consider small deviations from the equilibrium state. Thus, we restrict ourselves to linear terms in perturbation. As follows from the normalization condition (19) and Eq. (20), in the linear approximation the time components u^0 and $w_{(i)}^0$ in the perturbed matter remain the same,

$$u^0 = 1, \quad w_{(i)}^0 = 0, \quad (34)$$

while their derivatives vanish

$$\partial_\mu u^0 = 0, \quad \partial_\mu w_{(i)}^0 = 0. \quad (35)$$

Using Eqs. (34) and (35), one gets from Eq. (33)

$$\text{div}[(Y_{pk} - Y_{\Sigma k})\mathbf{w}_{(k)}] = 0. \quad (36)$$

As we have already emphasized in Sec. III A, in the nonequilibrium matter any thermodynamic quantity (e.g., P or μ_j) is a function of the number densities n_j , temperature T , and scalars $w_{(i)}^\mu w_{(k)\mu}$ (we choose T instead of ε as an independent variable). In the linear approximation we can neglect the dependence on the quadratically small quantity $w_{(i)}^\mu w_{(k)\mu}$. Moreover, in the strongly degenerate nucleon-hyperon matter, the dependence of P and μ_j on T can also be neglected (see, e.g., [17,39,40]). Consequently, P and μ_j are functions of only n_j ($j = n, p, \Lambda, \Sigma, e, \mu$). These six number densities are related by the conditions (9) and (32), so that only $6 - 2 = 4$ of them are independent. Thus, the pressure P and the chemical potentials μ_j depend on some four number densities (or their functions). As the independent variables it is convenient to choose the num-

ber density of baryons $n_b = n_n + n_p + n_\Lambda + n_\Sigma$, the number density of hyperons $n_H = n_\Lambda + n_\Sigma$, and the quantities $n_{\Sigma n} = n_n + n_\Sigma$ and $y = n_e/n_\mu$ (see Ref. [1]). Notice that, in the thermodynamic equilibrium n_b , n_H , $n_{\Sigma n}$, and y are additionally constrained by the two conditions (7) and by the condition (11). In this case P and μ_j are functions of only one number density (e.g., n_b).

B. Equations governing the sound waves

In this section we derive the system of equations describing sound waves in the superfluid nucleon-hyperon matter neglecting dissipation due to the nonequilibrium reactions (1)–(4) and shear viscosity.

As we demonstrate below, the nonequilibrium reactions do not lead to dissipation in two limiting cases: (i) either in the limit of *slow* reactions, when the total rate λ of the reactions (1)–(4) is negligible, so that they cannot change the matter composition during the pulsations excited by a sound wave (i.e., $\Delta\Gamma_m = 0$, $m = 1, \dots, 4$); (ii) or in the limit of *fast* reactions, when the reaction rate is so high that the pulsating matter is always in equilibrium with respect to the reactions (1)–(4), so that the condition (11) is always satisfied. These are two cases that will be analyzed in what follows.

The system of linearized hydrodynamic equations, describing sound waves, consists of (i) the condition (36); (ii) momentum conservation law

$$\partial_t[(P_0 + \varepsilon_0)\mathbf{u} + \mu_{i0}Y_{ik}\mathbf{w}_{(k)}] = -\nabla P, \quad (37)$$

following from Eq. (15); and (iii) the four potentiality conditions for superfluid motion (for each baryon species)

$$\partial_t[\mu_{n0}\mathbf{u} + \mathbf{w}_{(n)}] = -\nabla\mu_n, \quad (38)$$

$$\partial_t[\mu_{\Lambda 0}\mathbf{u} + \mathbf{w}_{(\Lambda)}] = -\nabla\mu_\Lambda, \quad (39)$$

$$\partial_t[\mu_{p0}\mathbf{u} + \mathbf{w}_{(p)} + q_p\mathbf{A}] = -\nabla[\mu_p + q_p A^0], \quad (40)$$

$$\partial_t[\mu_{\Sigma 0}\mathbf{u} + \mathbf{w}_{(\Sigma)} + q_\Sigma\mathbf{A}] = -\nabla[\mu_\Sigma + q_\Sigma A^0]. \quad (41)$$

These conditions can be obtained from Eq. (21) with the help of Eq. (35). The vector potential \mathbf{A} and the scalar potential A^0 can be excluded from Eqs. (40) and (41) if one notices that $q_\Sigma = -q_p$ and takes a sum of these formulas. Subtracting then Eqs. (38) and (39) from the obtained sum and using the equilibrium condition (9), one gets

$$\partial_t[\mathbf{w}_{(\Sigma)} + \mathbf{w}_{(p)} - \mathbf{w}_{(\Lambda)} - \mathbf{w}_{(n)}] = 0. \quad (42)$$

In view of the condition (6), and the definition (10), it follows from Eqs. (38) and (39) that

$$\partial_t[\mathbf{w}_{(n)} - \mathbf{w}_{(\Lambda)}] = -\nabla\delta\mu. \quad (43)$$

Equations (36)–(38), (42), and (43) represent the five equations for five variables, \mathbf{u} and $\mathbf{w}_{(i)}$ ($i = n, p, \Lambda, \Sigma$). They completely describe the sound waves in the super-

fluid nucleon-hyperon matter under the condition that the pressure $P(n_b, n_H, n_{\Sigma n}, y)$, neutron chemical potential $\mu_n(n_b, n_H, n_{\Sigma n}, y)$, and chemical potential disbalance $\delta\mu(n_b, n_H, n_{\Sigma n}, y)$ are known as functions of \mathbf{u} and $\mathbf{w}_{(i)}$. Expanding these quantities in Taylor series near the equilibrium point and presenting P and μ_n as $P = P_0 + \delta P$ and $\mu_n = \mu_{n0} + \delta\mu_n$, one obtains

$$\delta P = \frac{\partial P}{\partial n_b} \delta n_b + \frac{\partial P}{\partial n_H} \delta n_H + \frac{\partial P}{\partial n_{\Sigma n}} \delta n_{\Sigma n} + \frac{\partial P}{\partial y} \delta y, \quad (44)$$

$$\delta\mu_n = \frac{\partial \mu_n}{\partial n_b} \delta n_b + \frac{\partial \mu_n}{\partial n_H} \delta n_H + \frac{\partial \mu_n}{\partial n_{\Sigma n}} \delta n_{\Sigma n} + \frac{\partial \mu_n}{\partial y} \delta y, \quad (45)$$

$$\delta\mu = \frac{\partial \delta\mu}{\partial n_b} \delta n_b + \frac{\partial \delta\mu}{\partial n_H} \delta n_H + \frac{\partial \delta\mu}{\partial n_{\Sigma n}} \delta n_{\Sigma n} + \frac{\partial \delta\mu}{\partial y} \delta y, \quad (46)$$

where in the last equation we take into account that $\delta\mu = 0$ in the equilibrium state [see Eq. (11)]. In Eqs. (44)–(46) δn_b , δn_H , $\delta n_{\Sigma n}$, and δy are the deviations of the quantities n_b , n_H , $n_{\Sigma n}$, and y from their equilibrium values n_{b0} , n_{H0} , $n_{\Sigma n0}$, and y_0 , respectively. In the appendix these deviations are expressed through the velocities \mathbf{u} and $\mathbf{w}_{(i)}$ in the limit of *slow* and *fast* reactions.

Assuming now, that the perturbations are harmonic ($\sim e^{i(\omega t + \mathbf{k} \cdot \mathbf{r})}$, where ω is the pulsation frequency and \mathbf{k} is the wave vector), the system of Eqs. (36)–(38), (42), and (43) can be rewritten as

$$(Y_{pk} - Y_{\Sigma k})\mathbf{w}_{(k)} = 0, \quad (47)$$

$$i\omega[(P_0 + \varepsilon_0)\mathbf{u} + \mu_{i0}Y_{ik}\mathbf{w}_{(k)}] = -ik\delta P, \quad (48)$$

$$i\omega[\mu_{n0}\mathbf{u} + \mathbf{w}_{(n)}] = -ik\delta\mu_n, \quad (49)$$

$$\mathbf{w}_{(\Lambda)} + \mathbf{w}_{(n)} = \mathbf{w}_{(\Sigma)} + \mathbf{w}_{(p)}, \quad (50)$$

$$i\omega[\mathbf{w}_{(n)} - \mathbf{w}_{(\Lambda)}] = -ik\delta\mu. \quad (51)$$

In the limit of *fast* reactions, when the condition (11) holds, Eq. (51) can be further simplified,

$$\mathbf{w}_{(n)} - \mathbf{w}_{(\Lambda)} = 0. \quad (52)$$

Equations (47)–(51) together with (44)–(46), as well as the expressions (A10)–(A12) for the case of *slow* reactions and (A13)–(A15) for the case of *fast* reactions, allow one to determine the speeds of sound modes.

C. Results for sound speeds

From the analysis of Eqs. (47)–(51) it is clear, that the vectors \mathbf{u} and $\mathbf{w}_{(i)}$ must be collinear with the wave vector \mathbf{k} : $\mathbf{u}, \mathbf{w}_{(i)} \parallel \mathbf{k}$. Thus, the system of Eqs. (47)–(51) has the form

$$\mathbf{A} \cdot \mathbf{x} = 0. \quad (53)$$

Here, \mathbf{x} is a vector that equals $\mathbf{x} = (u, w_{(n)}, w_{(p)}, w_{(\Lambda)}, w_{(\Sigma)})$, where $u \equiv \mathbf{u}\mathbf{k}/k$ and $w_{(i)} \equiv \mathbf{w}_{(i)}\mathbf{k}/k$; \mathbf{A} is a 5×5 matrix, which elements depend on the thermodynamic quantities (and their derivatives), on the relativistic entrainment matrix Y_{ik} , and on the frequency ω and the wave number k .

The system of Eqs. (53) has a nontrivial solution if $\det \mathbf{A} = 0$. This condition results in a cubic equation on the speed of sound $s \equiv \omega/k$ squared. Each of three solutions to this equation describes two sound waves, propagating with the same speed along and opposite to the wave vector \mathbf{k} . In principle, these solutions can be written out analytically, but here we do not present them because they are too lengthy.

In the case of *slow* reactions (when the condition $\lambda(\partial\delta\mu/\partial n_H - \partial\delta\mu/\partial n_{\Sigma n}) \ll \omega$ holds, see appendix), the three nonzero solutions to the cubic equation exist, corresponding to three different sound modes. In the case of *fast* reactions (when $\lambda(\partial\delta\mu/\partial n_H - \partial\delta\mu/\partial n_{\Sigma n}) \gg \omega$), one of the three roots of the cubic equation vanishes, hence the actual number of sound modes is two (this conclusion can be also drawn from the fact that in the limit of *fast* reactions one can use Eq. (52) instead of (51), which does not depend on \mathbf{k} and ω).

The number of independent sound modes in these two limiting cases can be easily understood from the following reasoning. When all baryon species are superfluid, there are five velocity fields in the matter, \mathbf{u} and $\mathbf{w}_{(i)}$. In the limit of *slow* reactions the velocities are related by Eqs. (47) and (50) [these equations follow from the quasineutrality condition (32) and the condition of the equilibrium with respect to the fast reaction (9), respectively]. Thus, in this limit we have three independent velocities ($5 - 2 = 3$). Correspondingly, the number of independent sound modes is also three. In the limit of *fast* reactions, there is a constraint (52) in addition to the conditions (47) and (50). This constraint is a consequence of the equilibrium condition (11) with respect to the reactions (1)–(4). Thus, the number of independent velocities (and sound modes) equals two ($5 - 3 = 2$).

Figure 1 illustrates the results of numerical calculation of the sound speeds s (in units of c) as functions of temperature T in the limit of *slow* reactions (left panel) and *fast* reactions (right panel). Here and below in this paper we use the *third* equation of state of Glendenning [41]. The figure is plotted for the baryon number density $n_b = 3n_0 = 0.48 \text{ fm}^{-3}$, where $n_0 = 0.16 \text{ fm}^{-3}$ is the nucleon number density in atomic nuclei. The critical temperatures T_{ci} ($i = n, p, \Lambda, \Sigma$) for transition of baryons to the superfluid state are poorly known. We take them to be $T_{cn} = 10^9 \text{ K}$, $T_{cp} = 2 \times 10^9 \text{ K}$, $T_{c\Lambda} = 3 \times 10^9 \text{ K}$, and $T_{c\Sigma} = 6 \times 10^9 \text{ K}$ in accordance with some theoretical predictions (see, e.g., [8,9,12,13]). The relativistic entrain-

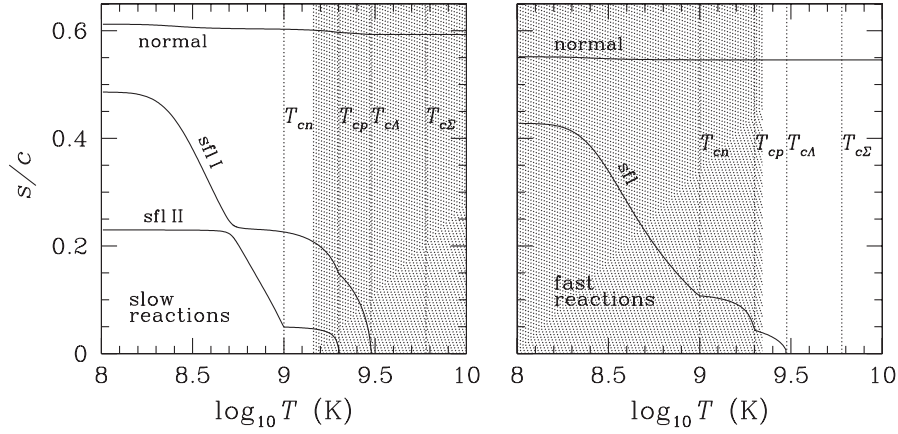


FIG. 1. Speed of sound modes s in units of c versus T at $n_b = 3n_{b0} = 0.48 \text{ fm}^{-3}$ for the *third* equation of state of Glendenning [41]. Pulsation frequency is $\omega = 10^4 \text{ s}^{-1}$. The left panel shows three sound modes (“normal,” “sfl I,” and “sfl II”) in the limit of *slow* reactions. The right panel shows two sound modes (“normal” and “sfl”) in the limit of *fast* reactions. Baryon critical temperatures are indicated by vertical dots. The range of T where the limit of *slow* (left panel) and *fast* (right panel) reactions is invalid (for $\omega = 10^4 \text{ s}^{-1}$) is shaded.

ment matrix Y_{ik} is taken from Ref. [33] (see also Ref. [32], where this matrix is calculated for $T = 0$).

The shaded region in the left panel of Fig. 1 corresponds to temperatures, where the limit of *slow* reactions is invalid. Rather conventionally, we define it by the inequality $\lambda(\partial\delta\mu/\partial n_H - \partial\delta\mu/\partial n_{\Sigma n}) > \omega/3$. Similarly, the shaded region in the right panel of Fig. 1 indicates the range of temperatures, where the limit of *fast* reactions is invalid. We define this region by the inequality $\lambda(\partial\delta\mu/\partial n_H - \partial\delta\mu/\partial n_{\Sigma n}) < 3\omega$.

Here and below we choose the frequency ω equal to $\omega = 10^4 \text{ s}^{-1}$. In fact, this value of ω is more appropriate for studies of pulsating neutron stars rather than sound waves, because it results in wavelengths of the order of stellar radius. However, we believe that the relatively simple analysis of sound waves with our choice of ω may provide some insight into the complex properties of global pulsations of neutron stars with superfluid nucleon-hyperon cores.

The rates of the reactions $\lambda_1, \dots, \lambda_4$, which constitute the quantity $\lambda = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$, are poorly known and depend essentially on the model of baryon-baryon interactions and on the many-body theory employed (see, e.g., [28–31, 42]). It is natural to expect that the rates λ_i are order-of-magnitude comparable [28]. In our numerical calculations we, following Ref. [30], take into account only the contribution to λ from the reactions (1) and (2) [i.e., we assume that $\lambda_3 = \lambda_4 = 0$]. The rates λ_{10} and λ_{20} for *nonsuperfluid* matter are taken from Ref. [30]. The baryon superfluidity suppresses the reaction rates, which can be presented as $\lambda_1 = \lambda_{10}R_1$ and $\lambda_2 = \lambda_{20}R_2$. The reduction factors $R_1 \leq 1$ and $R_2 \leq 1$ are calculated using formula (28) of Ref. [29]. Though the authors of Ref. [29] proposed the formula for the reaction (1), it remains valid for the reaction (2). In both cases the index i in that formula

enumerate the reacting particles [$i = n, n, p, \Sigma$ for the reaction (1) and $i = n, p, p, \Lambda$ for the reaction (2)].

As one can see from Fig. 1, at $T < T_{cn}$ (when all baryon species are superfluid) there are three sound modes in the limit of *slow* reactions and two in the limit of *fast* reactions. With further increasing temperature the number of sound modes decreases. As a result, at $T > T_{cA}$ there is only one mode in both limiting cases. At $T > T_{c\Sigma}$ this mode becomes the ordinary sound in nonsuperfluid nucleon-hyperon matter (in fact, it transforms into the ordinary sound already at $T > T_{cA}$, see the next paragraph). We term this sound mode normal; in the figure it is denoted as “normal”. Accordingly, the other modes are termed superfluid and denoted as “sfl I”, “sfl II” (see Fig. 1, left panel) and “sfl” (see Fig. 1, right panel). Notice that the speed of normal mode only weakly depends on T both in the limits of *slow* and *fast* reactions and practically coincides with the speed of sound of normal matter. On the contrary, the speeds of superfluid modes strongly depend on temperature and approach their asymptotic values only at $T \lesssim 10^8 \text{ K}$.

Let us explain, how the number of sound modes changes with T in the limit of *slow* reactions (Fig. 1, left panel). Limit of *fast* reactions can be considered in a similar way. As we have already discussed above, at $T < T_{cn}$ there are five velocities, \mathbf{u} and $\mathbf{w}_{(i)}$ ($i = n, p, \Lambda, \Sigma$). They are related by two conditions, (47) and (50). Thus, the number of independent velocities (and sound modes) equals three. At $T_{cn} < T < T_{cp}$ neutrons are normal, that is, there are only four velocities in the system, \mathbf{u} and $\mathbf{w}_{(i)}$ ($i = p, \Lambda, \Sigma$). These velocities are constrained by the only one condition (47) [the condition (50) is not applicable; it is valid only if all baryon species are superfluid]. Thus, the number of sound modes remains equal three. Then, at $T_{cp} < T < T_{cA}$ and at $T_{cA} < T < T_{c\Sigma}$ a motion with, respectively,

three (\mathbf{u} , \mathbf{w}_Λ , and \mathbf{w}_Σ) and two (\mathbf{u} and \mathbf{w}_Σ) velocities is possible. These velocities are related by the condition (47). Consequently, there are two sound modes in the range $T_{cp} < T < T_{c\Lambda}$ and one in the range $T_{c\Lambda} < T < T_{c\Sigma}$. In the latter case it follows from the condition (47) that $\mathbf{w}_\Sigma = 0$. Hence, the hydrodynamic equations are formally the same as those for normal liquid, and the only sound mode coincides with the ordinary sound in nonsuperfluid matter. Finally, at $T > T_{c\Sigma}$ all baryon species are normal and move with the same velocity \mathbf{u} . Since this velocity is not constrained, there is (as it should be) only one sound mode.

V. DAMPING OF SOUND WAVES

A. Damping times: General equations

In the previous section we analyzed the sound modes in the limit of *slow* and *fast* reactions neglecting dissipation. In this section we allow for a weak dissipation in these limiting cases, which is due to the shear viscosity and nonequilibrium processes (1)–(4). Our aim is the calculation of the characteristic damping times τ of sound waves (the so called e-folding times).

A few ways exist to calculate τ . For instance, one can explicitly take into account the shear viscosity in Eqs. (47)–(51) and the next (complex) terms in the expansion of δn_H and $\delta n_{\Sigma n}$ into series in powers of $\lambda(\partial\delta\mu/\partial n_H - \partial\delta\mu/\partial n_{\Sigma n})/\omega$ in the limit of *slow* reactions and in powers of $\omega/[\lambda(\partial\delta\mu/\partial n_H - \partial\delta\mu/\partial n_{\Sigma n})]$ in the limit of *fast* reactions (see appendix). Then, solving the system of Eqs. (44)–(51), one can find the small complex correction δs to the speed of sound s , which is related to the damping time τ by

$$\tau = \frac{i}{k\delta s}. \quad (54)$$

Another way to calculate τ is to use the effective bulk viscosity formalism [1], similar to how it was done in Ref. [17] for *npe* matter.

We calculated τ by both methods described above. However, here we present the most simple, third method of calculation. Of course, all three methods give the same results.

Let us define the characteristic damping time as

$$\tau \equiv -\frac{2E_{\text{puls}}}{\langle \dot{E}_{\text{puls}} \rangle}, \quad (55)$$

where E_{puls} is the mechanical energy (per unit volume) of pulsations; $\langle \dot{E}_{\text{puls}} \rangle$ is the rate of change of E_{puls} , averaged over the period $2\pi/\omega$. Here and below, angle brackets denote averaging over the pulsation period. The factor 2 in Eq. (55) appears because τ is the e-folding time for the hydrodynamic velocities rather than for energy. One can check that this definition of τ coincides with that given by Eq. (54).

The mechanical energy of a pulsating superfluid matter can be easily found if we notice that E_{puls} entirely transforms into its kinetic energy when the matter in the course of pulsations passes through the equilibrium point. Generally, the kinetic energy E_{kin} of the superfluid matter equals (we remind that $u \ll c$ and $w_{(i)}/\mu_i \ll c$)

$$E_{\text{kin}} = \frac{1}{2}\{(P + \varepsilon)\mathbf{u}^2 + Y_{ik}[\mu_i \mathbf{w}_{(k)}\mathbf{u} + \mu_k \mathbf{w}_{(i)}\mathbf{u} + \mathbf{w}_{(i)}\mathbf{w}_{(k)}]\}. \quad (56)$$

In the nonrelativistic limit and for two-component mixture this formula agrees with expression (7) of Ref. [34]. Since in a sound wave the vectors \mathbf{u} and $\mathbf{w}_{(i)}$ are collinear with \mathbf{k} and can be presented as $\mathbf{u} = \mathbf{u}_a \cos(\omega t + \mathbf{k}\mathbf{r})$ and $\mathbf{w}_{(i)} = \mathbf{w}_{(i)a} \cos(\omega t + \mathbf{k}\mathbf{r})$, one obtains for E_{puls} ($u_a \equiv \mathbf{u}_a \mathbf{k}/k$ and $w_{(i)a} \equiv \mathbf{w}_{(i)a} \mathbf{k}/k$)

$$E_{\text{puls}} = \frac{1}{2}\{(P_0 + \varepsilon_0)u_a^2 + Y_{ik}[\mu_{i0}w_{(k)a}u_a + \mu_{k0}w_{(i)a}u_a + w_{(i)a}w_{(k)a}]\}. \quad (57)$$

Here, we used the fact that at the equilibrium point the quantities \mathbf{u} and $\mathbf{w}_{(i)}$ are equal to their amplitudes \mathbf{u}_a and $\mathbf{w}_{(i)a}$, respectively.

To find $\langle \dot{E}_{\text{puls}} \rangle$ let us notice that all this energy goes into heat. Thus, using Eqs. (30) and (31), one can write (see also Refs. [39,43])

$$\langle \dot{E}_{\text{puls}} \rangle = -\langle T\partial_\mu(Su^\mu) \rangle = -\langle \lambda(\delta\mu)^2 \rangle + \langle \tau_{\text{sh}}^{\mu\nu} \partial_\mu u_\nu \rangle. \quad (58)$$

It follows from this equation together with Eq. (55) that the characteristic damping time τ_{bulk} due to the nonequilibrium reactions (1)–(4) equals

$$\tau_{\text{bulk}} = \frac{2E_{\text{puls}}}{\langle \lambda(\delta\mu)^2 \rangle}, \quad (59)$$

while the characteristic damping time τ_{sh} due to the shear viscosity is

$$\tau_{\text{sh}} = -\frac{2E_{\text{puls}}}{\langle \tau_{\text{sh}}^{\mu\nu} \partial_\mu u_\nu \rangle}. \quad (60)$$

As a consequence of Eq. (A9) [see appendix], in the limit of *slow* reactions

$$\langle \lambda(\delta\mu)^2 \rangle = \frac{\lambda}{2s^2} \left(\frac{\partial\delta\mu}{\partial n_b} J_{ba} + \frac{\partial\delta\mu}{\partial n_H} J_{Ha} + \frac{\partial\delta\mu}{\partial n_{\Sigma n}} J_{\Sigma na} \right)^2, \quad (61)$$

while in the limit of *fast* reactions

$$\langle \lambda(\delta\mu)^2 \rangle = \frac{\omega^2}{2s^2 \lambda (\partial\delta\mu/\partial n_H - \partial\delta\mu/\partial n_{\Sigma n})^2} \times \left(\frac{\partial\delta\mu}{\partial n_b} J_{ba} + \frac{\partial\delta\mu}{\partial n_H} J_{Ha} + \frac{\partial\delta\mu}{\partial n_{\Sigma n}} J_{\Sigma na} \right)^2. \quad (62)$$

In Eqs. (61) and (62) $J_{ba} \equiv n_b u_a + \sum_i Y_{ik} w_{(k)a}$, $J_{Ha} \equiv n_H u_a + Y_{\Sigma k} w_{(k)a} + Y_{\Lambda k} w_{(k)a}$, and $J_{\Sigma na} \equiv n_{\Sigma n} u_a + Y_{\Sigma k} w_{(k)a} + Y_{nk} w_{(k)a}$; s is the speed of sound calculated in the previous section neglecting dissipation; the factor $1/2$ is a result of the averaging over the pulsation period.

The dissipation rate of the mechanical energy due to the shear viscosity is the same in both limits,

$$\langle \tau_{\text{sh}}^{\mu\nu} \partial_\mu u_\nu \rangle = -\frac{2}{3} \eta \frac{\omega^2}{s^2} u_a^2. \quad (63)$$

To obtain this formula we used Eq. (28) for $\tau_{\text{sh}}^{\mu\nu}$. We see that $\langle \tau_{\text{sh}}^{\mu\nu} \partial_\mu u_\nu \rangle$ is formally given by the same expression as in the case of nonsuperfluid matter.

As follows from Eqs. (57), (58), and (61)–(63), E_{puls} and $\langle \dot{E}_{\text{puls}} \rangle$ are the functions of the amplitudes u_a and $w_{(i)a}$. Using the system of linear nondissipative Eqs. (44)–(51), the amplitudes $w_{(i)a}$ can be expressed through u_a and presented in the form

$$w_{(i)a} = \alpha_i(s) u_a, \quad (64)$$

where $\alpha_i(s)$ are some coefficients depending on s ; they differ for each sound mode. In view of Eq. (64), one gets from Eqs. (57), (58), and (61)–(63) that $E_{\text{puls}} \sim u_a^2$ and $\langle \dot{E}_{\text{puls}} \rangle \sim u_a^2$. Hence, τ is independent of u_a [see Eq. (55)].

B. Results for damping times

We numerically calculate the coefficients $\alpha_i(s)$ from Eqs. (44)–(51) and thus determine the characteristic damping times τ of sound modes in the limit of *slow* and *fast* reactions. Figs. 2–5 illustrate the results of our calculations. These figures are plotted assuming the same microphysics input (the baryon number density, the critical baryon temperatures, etc.) as in Fig. 1.

Figure 2 shows the characteristic damping times τ_{sh} due to the shear viscosity as a function of temperature T in the limit of *slow* reactions (left panel) and *fast* reactions (right panel). To calculate τ_{sh} it is necessary to know the shear viscosity coefficient η of superfluid nucleon-hyperon matter. This coefficient has not been considered in the literature so far. For definiteness, we take for η the shear viscosity of electrons and muons $\eta_{e\mu} = \eta_e + \eta_\mu$ from Ref. [44]. In this reference it is shown that (for $npe\mu$ matter) the contribution of $\eta_{e\mu}$ to the total shear viscosity η is dominant. Notice, however, that this result was obtained under assumption that only protons are possibly superfluid (neutrons were treated as normal). When plotting Fig. 2 we used the coefficient $\eta_{e\mu}$ calculated for *nonsuperfluid* matter from Eq. (37) of Ref. [44]. The effects of baryon superfluidity on τ_{sh} are illustrated in the next figure.

By the solid curves we show τ_{sh} calculated for each sound mode by means of Eq. (60). Dashes demonstrate the characteristic damping times $\tau_{\text{nfh-sh}}$ calculated using the simplified model, the hydrodynamics of normal (nonsuperfluid) liquid. In this case, there is only one mode in both limits; the corresponding curves are marked with “nfh,” which is the abbreviation of “normal fluid hydrodynamics.” It is straightforward to demonstrate that $\tau_{\text{nfh-sh}} = 3(P_0 + \varepsilon_0)s^2/(2\omega^2\eta)$ (see, e.g., Ref. [43]). As it should be, τ_{sh} for normal mode (in the figure it is marked “normal”) coincides with $\tau_{\text{nfh-sh}}$ at $T > T_{c\Lambda}$ (see Sec. IV C).

It follows from Fig. 2 that τ_{sh} , calculated in the frame of relativistic hydrodynamics of superfluid mixtures, can strongly (by several orders of magnitude) differ from $\tau_{\text{nfh-sh}}$. It is interesting that the maximum deviation of τ_{sh} from $\tau_{\text{nfh-sh}}$ at low temperatures ($T \lesssim 3 \times 10^8$ K) is observed for *normal* mode (Fig. 2, left panel), though it is analogous to the usual sound in nonsuperfluid matter.

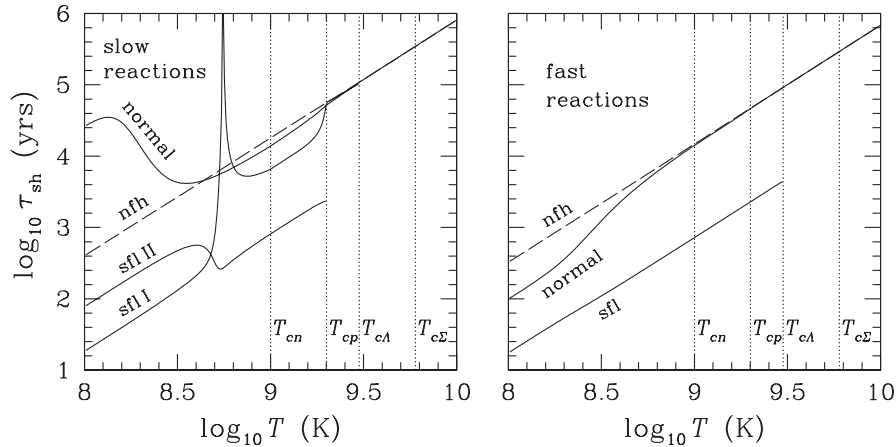


FIG. 2. The characteristic damping times τ_{sh} due to the shear viscosity versus T in the limit of *slow* reactions (left panel) and *fast* reactions (right panel). The dashed curves (marked with “nfh”) in both panels are obtained using the nonsuperfluid hydrodynamics, see the text. Other notations are the same as in Fig. 1.

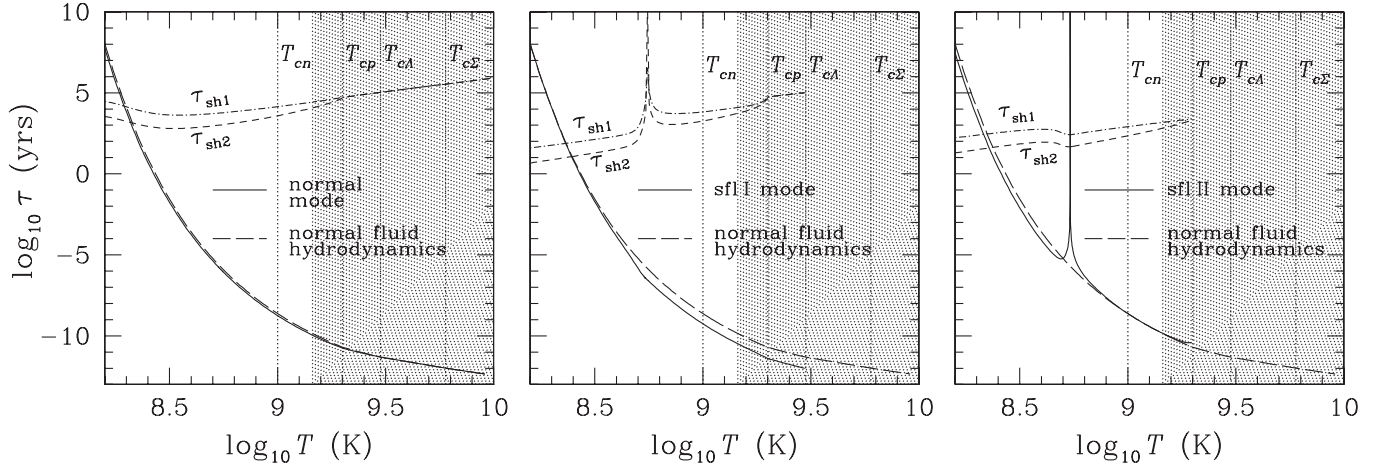


FIG. 3. The characteristic damping times τ versus T for “normal” mode (left panel), “sfl I” mode (middle panel), and “sfl II” mode (right panel) in the limit of *slow* reactions. The solid curves demonstrate τ_{bulk} calculated from Eq. (59); the long-dashed curve (the same in all panels) describes the characteristic damping times $\tau_{\text{nfh-bulk}}$, calculated using the normal fluid hydrodynamics. The dotted-dashed and short-dashed curves show damping times τ_{sh1} and τ_{sh2} due to the shear viscosity (see the text for more details). Other notations are the same as in Figs. 1 and 2.

At temperature $T \approx 5.55 \times 10^8$ K the characteristic damping time τ_{sh} for one of the superfluid modes (“sfl I”) becomes infinite. This is because at such temperature and for this mode the hydrodynamic motions occur in such a way that the normal component is always at rest ($u_a = 0$), while the superfluid components pulsate around it. Mathematically, this means that the coefficients $\alpha_i(s)$ in Eq. (64) are infinite. It follows then from Eqs. (57), (60), and (63) that dissipation due to the shear viscosity is absent.

Figure 3 presents the dependence of the characteristic damping times on T in the limit of *slow* reactions. Three panels correspond to three modes (from left to right: “normal,” “sfl I,” and “sfl II”). The damping times τ_{bulk} , calculated for each mode using Eq. (59), are shown by solid curves. For comparison, by long dashes we show the characteristic damping times $\tau_{\text{nfh-bulk}}$ due to the nonequilibrium reactions (1)–(4), which are obtained using the

hydrodynamics of *nonsuperfluid* liquid. When plotting the long-dashed curve superfluidity of baryons was taken into account only at calculating the total reaction rate λ . Because there is only one sound mode in the nonsuperfluid hydrodynamics, this curve is the same in all three panels.

One can see from the figure that for the normal mode the solid and long-dashed curves practically (on the logarithmic scale) coincide; they differ by a factor of 2 or less. On the contrary, τ_{bulk} for superfluid modes can differ from $\tau_{\text{nfh-bulk}}$ by orders of magnitude.

Notice that at $T \approx 5.41 \times 10^8$ K the damping time for the superfluid mode “sfl II” becomes infinite. The point is at such temperature and for this mode the condition $\delta\mu = 0$ is always preserved during the pulsations. The denominator in Eq. (59) is then vanished and τ_{bulk} tends to infinity.

By dotted dashes and short dashes in Fig. 3 we show, respectively, the characteristic damping times τ_{sh1} and τ_{sh2} due to the shear viscosity. The times τ_{sh1} are the same as in

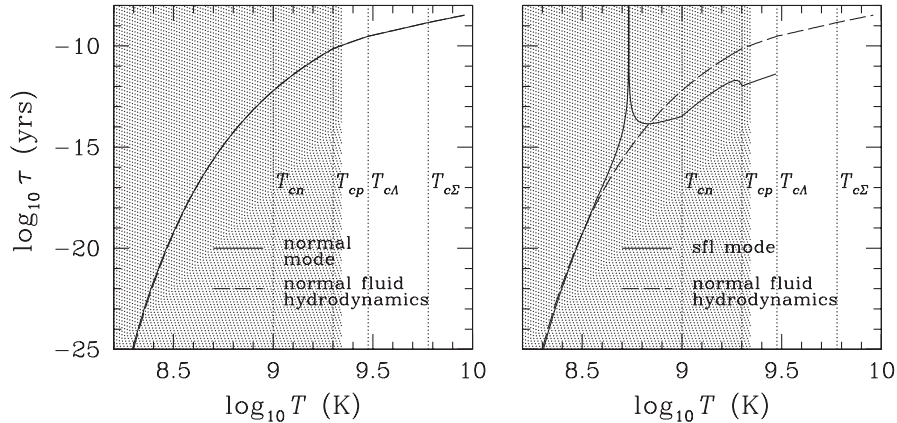


FIG. 4. The same as in Fig. 3 but in the limit of *fast* reactions. The damping times $\tau_{\text{sh}} \gg \tau_{\text{bulk}}$ and are not shown.

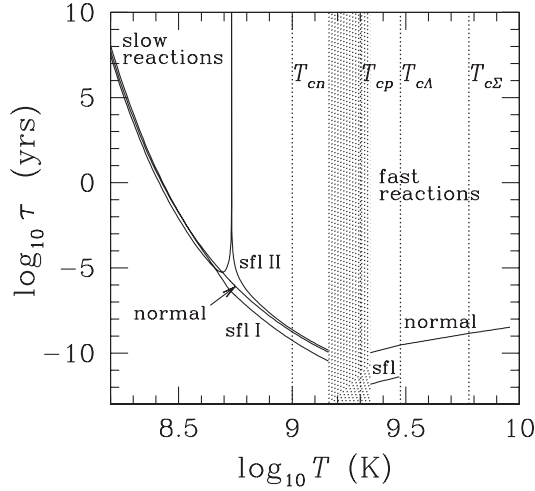


FIG. 5. Characteristic damping times τ_{bulk} versus T in the limit of *slow* ($T \lesssim 1.4 \times 10^9$ K) and *fast* ($T \gtrsim 2.3 \times 10^9$ K) reactions. The region of T where the rate of nonequilibrium reactions is intermediate (i.e., where both limits are invalid), is shaded.

Fig. 2 (left panel); they are plotted for the shear viscosity $\eta = \eta_{e\mu}$ of nonsuperfluid matter [see Eq. (37) of Ref. [44]]. To obtain the dependence $\tau_{\text{sh}2}(T)$ we assume that $\eta = \eta_{e\mu}$ as before, but additionally take into account the reduction of $\eta_{e\mu}$ by proton superfluidity. The reduction factor was calculated from Eq. (83) of Ref. [44]. It is worth noting that this formula is obtained for nucleon $npe\mu$ matter and does not imply the superfluidity of other baryon species except for protons. Thus, the dependence $\tau_{\text{sh}2}(T)$ only qualitatively describes possible effect of baryon superfluidity on τ_{sh} .

It follows from the analysis of Fig. 3, that at high enough temperatures $T \gtrsim 3 \times 10^8$ K, the dissipation due to the shear viscosity is negligible in comparison to that due to the nonequilibrium processes (1)–(4). Moreover, the threshold density, at which $\tau_{\text{bulk}} \approx \tau_{\text{sh}}$, only weakly depends on η .

Figure 4 is analogous to Fig. 3 but is plotted for the limit of *fast* reactions. The solid curve in the left panel demonstrates the dependence $\tau_{\text{bulk}}(T)$ for the normal mode (“normal”) and in the right panel—for the superfluid mode (“sfl”). The long-dashed curves are the same in both panels and describe the damping times $\tau_{\text{nfh-bulk}}(T)$. In the limit of *fast* reactions the damping times due to the shear viscosity are much greater than τ_{bulk} and are not shown.

As in the limit of *slow* reactions, τ_{bulk} for the normal mode does not differ substantially from $\tau_{\text{nfh-bulk}}$ (no more than by a factor of 2). However, at high enough temperatures τ_{bulk} for superfluid mode is 2 orders of magnitude smaller than $\tau_{\text{nfh-bulk}}$.

The analysis of Fig. 4 shows that the damping time for superfluid mode becomes infinite at the same temperature $T \approx 5.41 \times 10^8$ K as in the limit of *slow* reactions (the corresponding peak is in the shaded region where the limit

of *fast* reactions is not applicable for the chosen pulsation frequency $\omega = 10^4 \text{ s}^{-1}$). It is easy to understand why this peak falls on the same temperature. As we already discussed above, the infinite damping time τ_{bulk} means that the condition $\delta\mu = 0$ holds in the pulsating matter (at given temperature). In this case, the nondissipative Eqs. (47)–(51) in the limits of *slow* and *fast* reactions coincide. Thus, the functional dependence of the amplitudes $w_{(i)a}$ on u_a is the same in both limits (see Sec. VA). Consequently, as follows from Eqs. (61) and (62), if $\delta\mu = 0$ in one limit, then it vanishes in the other limit.

Figure 5 is a combined plot, showing the characteristic damping times τ_{bulk} as functions of T for the limits of *slow* and *fast* reactions. The shaded region (at $T \sim 2 \times 10^9$ K) corresponds to intermediate temperatures at which both limits are inapplicable. In this region the pulsation energy dissipates on a time scale of order of the pulsation period. Notice that at $T \gtrsim 3 \times 10^8$ K the characteristic damping time of pulsations $\tau = \tau_{\text{bulk}} + \tau_{\text{sh}}$ practically coincides with τ_{bulk} , since at such temperatures the contribution of the shear viscosity to dissipation is negligible.

VI. SUMMARY

In Ref. [1] the relativistic dissipative hydrodynamics was suggested to describe superfluid nucleon-hyperon matter in the cores of massive neutron stars. Using this hydrodynamics, we analyze the sound waves, which are the simplest example of pulsations in such matter.

We demonstrate that in the limit of *slow* reactions (1)–(4) [when the composition of pulsating matter is practically unaffected by these reactions] there are *three* sound modes: one normal and two superfluid. In the opposite limit of *fast* reactions [when the pulsating matter is nearly at equilibrium with respect to the reactions (1)–(4)], only *two* sound modes exist: the normal one and the superfluid one. In the intermediate case the sound waves cannot propagate because they are damped on a time scale of order of the pulsation period.

The speed of normal sound mode in both limits is practically independent of temperature and coincides with the sound speed for nonsuperfluid matter. This mode turns into the ordinary sound at $T > T_{cA}$. On the contrary, the speeds of superfluid modes strongly depend on temperature and vanish before the transition of matter to the normal state.

We analyze also the characteristic damping times τ of sound modes (Figs. 2–5). We allow for the two main dissipative mechanisms: damping due to the shear viscosity and due to the nonequilibrium reactions (1)–(4) [these reactions generate the effective bulk viscosity]. We demonstrate that (i) the damping times τ for normal and superfluid modes can differ from each other by orders of magnitude; (ii) at $T \gtrsim 3 \times 10^8$ K the damping due to the nonequilibrium reactions (1)–(4) is the dominant mechanism of dissipation; this result is nearly insensitive to an actual value of the shear viscosity coefficient η .

In addition, we compare τ with the damping time τ_{nfh} , calculated using the ordinary nonsuperfluid hydrodynamics, but taking into account the effects of superfluidity on the shear viscosity and on the rates of the reactions (1)–(4). We show that (iii) τ approximately (up to a factor of 2) coincides with τ_{nfh} only for the normal mode and under the condition that the shear viscosity can be neglected (i.e., $T \gtrsim 3 \times 10^8$ K). In other cases (for the superfluid modes and for the normal mode at $T < 3 \times 10^8$ K) τ can differ from τ_{nfh} by several orders of magnitude.

The results listed above are obtained from the analysis of sound waves in the superfluid nucleon-hyperon matter. However, they can serve as an indication that the effects, related to difference between the superfluid and normal fluid hydrodynamics, can also be very important in studies of global pulsations of superfluid neutron stars, essentially modifying their damping times.

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APPENDIX

Let us calculate the quantities $\delta n_b = n_b - n_{b0}$, $\delta n_H = n_H - n_{H0}$, $\delta n_{\Sigma n} = n_{\Sigma n} - n_{\Sigma n0}$, and $\delta y = y - y_0$, entering Eqs. (44)–(46) for δP , $\delta \mu_n$, and $\delta \mu$, respectively. For that, we make use of the continuity Eqs. (29), assuming the perturbations are harmonic. In the linear approximation the continuity equations for leptons (electrons and muons) have the form ($l = e, \mu$)

$$i\omega \delta n_l + ik n_{l0} \mathbf{u} = 0, \quad (\text{A1})$$

where we put $\Delta S_l = 0$, because the leptonic reactions are slow. From these equations it follows that

$$\delta y = 0. \quad (\text{A2})$$

The continuity equations for baryons, hyperons, and Σ^- -hyperons with neutrons can also be obtained from Eq. (29)

$$i\omega \delta n_b + ik \mathbf{J}_b = 0, \quad (\text{A3})$$

$$i\omega \delta n_H + ik \mathbf{J}_H = \lambda \delta \mu, \quad (\text{A4})$$

$$i\omega \delta n_{\Sigma n} + ik \mathbf{J}_{\Sigma n} = -\lambda \delta \mu. \quad (\text{A5})$$

Here, we used Eq. (31), and introduced the notations $\mathbf{J}_b \equiv n_b \mathbf{u} + \sum_i Y_{ik} \mathbf{w}_{(k)}$; $\mathbf{J}_H \equiv n_H \mathbf{u} + Y_{\Sigma k} \mathbf{w}_{(k)} + Y_{\Lambda k} \mathbf{w}_{(k)}$; and $\mathbf{J}_{\Sigma n} \equiv n_{\Sigma n} \mathbf{u} + Y_{\Sigma k} \mathbf{w}_{(k)} + Y_{nk} \mathbf{w}_{(k)}$. Solving now the system (A3)–(A5) taking into account Eqs. (46) and (A2), one gets for δn_b , δn_H , and $\delta n_{\Sigma n}$

$$\delta n_b = -\frac{k \mathbf{J}_b}{\omega}, \quad (\text{A6})$$

$$\delta n_H = \frac{k}{\omega} \frac{i\omega \mathbf{J}_H + \lambda [\mathbf{J}_b \partial \delta \mu / \partial n_b + (\mathbf{J}_H + \mathbf{J}_{\Sigma n}) \partial \delta \mu / \partial n_{\Sigma n}]}{\lambda (\partial \delta \mu / \partial n_H - \partial \delta \mu / \partial n_{\Sigma n}) - i\omega}, \quad (\text{A7})$$

$$\delta n_{\Sigma n} = \frac{k}{\omega} \frac{i\omega \mathbf{J}_{\Sigma n} - \lambda [\mathbf{J}_b \partial \delta \mu / \partial n_b + (\mathbf{J}_H + \mathbf{J}_{\Sigma n}) \partial \delta \mu / \partial n_H]}{\lambda (\partial \delta \mu / \partial n_H - \partial \delta \mu / \partial n_{\Sigma n}) - i\omega}. \quad (\text{A8})$$

Using these equalities, Eq. (46) for $\delta \mu$ can be rewritten as

$$\delta \mu = -\frac{k}{i\lambda (\partial \delta \mu / \partial n_H - \partial \delta \mu / \partial n_{\Sigma n}) + \omega} \left(\frac{\partial \delta \mu}{\partial n_b} \mathbf{J}_b + \frac{\partial \delta \mu}{\partial n_H} \mathbf{J}_H + \frac{\partial \delta \mu}{\partial n_{\Sigma n}} \mathbf{J}_{\Sigma n} \right). \quad (\text{A9})$$

In the limit of *slow* reactions, when the total rate λ of the reactions (1)–(4) is small, that is, $\lambda (\partial \delta \mu / \partial n_H - \partial \delta \mu / \partial n_{\Sigma n}) \ll \omega$, one has

$$\delta n_b = -k \mathbf{J}_b / \omega, \quad (\text{A10})$$

$$\delta n_H = -k \mathbf{J}_H / \omega, \quad (\text{A11})$$

$$\delta n_{\Sigma n} = -k \mathbf{J}_{\Sigma n} / \omega. \quad (\text{A12})$$

In the limit of *fast* reactions, when $\lambda (\partial \delta \mu / \partial n_H - \partial \delta \mu / \partial n_{\Sigma n}) \gg \omega$, one obtains

$$\delta n_b = -\frac{k \mathbf{J}_b}{\omega}, \quad (\text{A13})$$

$$\delta n_H = \frac{k (\mathbf{J}_b \partial \delta \mu / \partial n_b + (\mathbf{J}_H + \mathbf{J}_{\Sigma n}) \partial \delta \mu / \partial n_{\Sigma n})}{\omega (\partial \delta \mu / \partial n_H - \partial \delta \mu / \partial n_{\Sigma n})}, \quad (\text{A14})$$

$$\delta n_{\Sigma n} = -\frac{k (\mathbf{J}_b \partial \delta \mu / \partial n_b + (\mathbf{J}_H + \mathbf{J}_{\Sigma n}) \partial \delta \mu / \partial n_H)}{\omega (\partial \delta \mu / \partial n_H - \partial \delta \mu / \partial n_{\Sigma n})}. \quad (\text{A15})$$

One sees, that in the both limits δn_b , δn_H , and $\delta n_{\Sigma n}$ are real valued and do not depend on the total reaction rate λ . Correspondingly, the sound speeds are also real valued, or, in other words, the dissipation is absent. The dissipation due to the weak nonequilibrium processes (1)–(4) can be

taken into account by considering the next (complex) terms in the expansion of δn_H and $\delta n_{\Sigma n}$ into series in powers of $\lambda(\partial\delta\mu/\partial n_H - \partial\delta\mu/\partial n_{\Sigma n})/\omega$ in the case of *slow* reactions and in powers of $\omega/[\lambda(\partial\delta\mu/\partial n_H - \partial\delta\mu/\partial n_{\Sigma n})]$ in the case of *fast* reactions.

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