

Vacuum polarization and cyclotron resonance effects on radiative transfer and plasma deceleration in subcritical X-ray pulsars

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ABSTRACT

We investigate the spectrum and polarization of radiation emerging from a subcritical X-ray pulsar using self-consistent radiation hydrodynamic simulations of an accretion channel in a strong magnetic field. The polarized radiative transfer in the channel above the hotspot is simulated for the two normal modes, taking into account resonant Compton scattering in a strongly magnetized plasma and the effects of vacuum polarization. We show that the deceleration of the accreting matter in the subcritical regime is mainly governed by resonant scattering. Our simulations provide the velocity profiles of the plasma flow and demonstrate that vacuum polarization dominates over plasma birefringence, enhancing both the cyclotron spectral feature and the radiative deceleration of the plasma. We also find that the energy of the cyclotron feature increases with accretion luminosity, indicating a positive correlation consistent with previous observational results and theoretical interpretation.

Key words: accretion, accretion discs – radiative transfer – stars: neutron – X-rays: binaries.

1 INTRODUCTION

X-ray pulsars (XRP) are strongly magnetized accreting neutron stars (NSs) in close binary systems (see A. Mushtukov & S. Tsygankov 2024, for review). A typical field strength at the surface of an XRP is $B \sim 10^{12}$ – 10^{13} G. In such systems, matter from the companion star falls on to the NS surface through an accretion channel along the field lines around magnetic poles. The kinetic energy of the accreting matter that hits the NS surface is mostly converted into X-ray radiation. Because of a misalignment of the magnetic and rotation axes of an NS, a remote observer detects a pulsating signal.

X-ray luminosities L_X of the XRP range over several orders of magnitude. XRP powering some ultra-luminous X-ray sources can reach luminosities of $\gtrsim 10^{41}$ erg s^{−1} (see e.g. G. L. Israel et al. 2017). At the other extreme, some XRP are observed at $L_X \sim 10^{33}$ erg s^{−1}, when accreting NSs enter the propeller regime (A. F. Illarionov & R. A. Sunyaev 1975) – emitting X-rays from cooling polar caps (S. S. Tsygankov et al. 2016; H. Xiao et al. 2025) – or, alternatively, when they sustain low-level accretion from a cold recombined disc (S. S. Tsygankov et al. 2017, 2019b) or from a hot quasi-static envelope around the NS magnetosphere (N. I. Shakura et al. 2012, 2015).

The XRP spectra usually resemble a power-law continuum with quasi-exponential cut-off at high energies (e.g. P. Mészáros

1992; A. Mushtukov & S. Tsygankov 2024, and references therein). Spectra of many XRP reveal cyclotron absorption lines superposed on the continuum (sometimes with one or a few harmonics), whose positions can vary with varying L_X (R. Staubert et al. 2019, and references therein). Recently, due to the Imaging X-ray Polarimetry Explorer (IXPE; M. C. Weisskopf et al. 2022), it has become also possible to measure the polarization of the X-ray radiation of the XRP in the band of 2–8 keV (V. Doroshenko et al. 2022; S. S. Tsygankov et al. 2022). The linear polarization degree of the XRP turned out to be within 20 per cent (see J. Poutanen, S. S. Tsygankov & S. V. Forsblom 2024 for review), much below the expectations based on the previous theoretical studies of X-ray polarization in hot magnetized plasmas (e.g. A. D. Kaminker, G. G. Pavlov & I. A. Shibanov 1983; P. Mészáros & W. Nagel 1985a, b), NS atmospheres (R. Taverna et al. 2020) and XRP accretion columns (I. Caiazzo & J. Heyl 2021a, b), although compatible with the predictions based on the model of a condensed NS surface (A. Potekhin, W. C. G. Ho & G. Chabrier 2016; R. Taverna et al. 2020).

A variety of formation mechanisms and geometries of accretion flows have been discussed in the literature (e.g. A. Mushtukov & S. Tsygankov 2024, and references therein). The higher the mass accretion rate $\dot{M}$, the higher the luminosity. There is a critical value of luminosity, at which radiation pressure becomes sufficient to stop the plasma flow above the NS surface (K. Davidson 1973). XRP with accretion rates above this value are called supercritical, while XRP with lower accretion rates are called subcritical. In the supercritical XRP, a

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radiation-dominated shock in the accretion channel is expected (M. M. Basko & R. A. Sunyaev 1976). Below the shock front, there is a slow moving dense hot plasma zone that is called accretion column, where the observed radiation of the supercritical XRP originates.

The mechanism of the radiation production of the subcritical XRP is less certain. At relatively low accretion rates $\dot{M} \lesssim (10^{15}\text{--}10^{16}) \text{ g s}^{-1}$ two fundamentally different accretion regimes have been discussed. One of them is determined by Coulomb braking of the accreting plasma in a relatively thin layer near the NS surface (e.g. A. A. Mushtukov et al. 2021; E. Sokolova-Lapa et al. 2021, and references therein). This process leads to heating of the deceleration layer and to the formation of hot atmosphere regions around the magnetic poles, from which the observed X-ray radiation emerges. In the other regime (e.g. S. L. Shapiro & E. E. Salpeter 1975; S. H. Langer & S. Rappaport 1982; A. M. Bykov & A. M. Krassilchtchikov 2004), a collisionless shock wave appears, which plays the crucial role in the accretion channel structure and radiation formation. In this paper, we will focus on the first regime. So far it has been assumed that the radiation that is formed in the atmosphere around the magnetic poles freely escapes to infinity. In this paper, we study the influence of radiative transfer in the accretion channel above the hotspot on the spectra and polarization of the outgoing radiation of the subcritical XRP in the shockless regime.

The strong magnetic fields affect the formation of XRP radiation due to a number of physical effects. First, a strongly magnetized plasma is a birefringent medium (e.g. V. L. Ginzburg 1970). Under the conditions established by Y. N. Gnedin & G. G. Pavlov (1974), electromagnetic waves can be represented in terms of two normal modes (NMs) with separate refractive indices and absorption coefficients.

Secondly, strong magnetic fields change the wavefunctions of charged particles, their motion transverse to the field being quantized in the Landau levels (see e.g. A. Sokolov & I. M. Ternov 1986), which strongly affects interaction of matter with radiation. Various radiation processes in strong magnetic fields under different conditions were studied by many authors (see e.g. A. K. Harding & D. Lai 2006, for review and references). Under the conditions characteristic of the XRP, considered in this work, the most important of these processes is Compton scattering, which was treated in detail by R. Sina (1996) and A. A. Mushtukov, D. I. Nagirner & J. Poutanen (2016).

Thirdly, formation of the XRP radiation can be strongly affected by vacuum polarization, which is a specific effect of quantum electrodynamics (QED) arising from interactions of virtual electron–positron pairs with a magnetic field (see e.g. G. G. Pavlov & Y. N. Gnedin 1984; J. Heyl & I. Caiazzo 2018, and references therein). This effect is important at sufficiently strong magnetic fields and low plasma densities, where the virtual pairs compete with real plasma particles in their influence on the electromagnetic waves propagation.

A theoretical modelling of an XRP can be divided in two main stages. At the first stage, the hydrodynamic structure of the accretion channel is determined. At the second stage, XRP radiation properties are calculated.

The simulations of an accretion column structure were mostly performed for supercritical pulsars with a radiation-dominated shock wave. Both stationary (K. Davidson 1973; H. Inoue 1975; S. L. Shapiro & E. E. Salpeter 1975; M. M. Basko & R. A. Sunyaev 1976; Y. M. Wang & J. Frank 1981; S. H. Langer & S. Rappaport 1982; A. Braun & R. Z. Yahel 1984; J. G. Kirk 1984,

1985; P. A. Becker 1998; K. A. Postnov et al. 2015; A. A. Mushtukov et al. 2015b; B. F. West, K. D. Wolfram & P. A. Becker 2017a; M. I. Gornostaev 2021) and non-stationary (R. I. Klein & J. Arons 1989; A. M. Bykov & A. M. Krassilchtchikov 2004; T. Kawashima et al. 2016; P. Abolmasov & G. Lipunova 2023; L. Zhang, O. Blaes & Y.-F. Jiang 2022, 2023, 2025; X. Sheng et al. 2023) regimes were explored, using (semi)analytical and numerical models. Most of the works did not take into account the effect of the resonant scattering of photons on the deceleration of accreting plasma. A. A. Mushtukov et al. (2015b) calculated the structure of an accretion column using fully relativistic resonant cross-sections, but neglecting plasma motion. X. Sheng et al. (2023) and L. Zhang, O. Blaes & Y.-F. Jiang (2025) performed full radiation hydrodynamical calculations of an XRP accretion channel using an approximate treatment of the influence of a magnetic field on the opacity in the case of supercritical pulsars.

Hydrodynamical simulations of the subcritical accretion channel were carried out mainly in connection with studies of possible collisionless shock waves in such channels (S. L. Shapiro & E. E. Salpeter 1975; S. H. Langer & S. Rappaport 1982; A. Braun & R. Z. Yahel 1984; A. M. Bykov & A. M. Krassilchtchikov 2004). In particular, A. M. Bykov & A. M. Krassilchtchikov (2004) considered in detail a number of resonant processes and non-stationary plasma evolution. However, their model did not include the recoil effect in the Compton scattering, which plays an important role in formation of accretion channel structure and outgoing spectrum (P. A. Becker & M. T. Wolff 2007). In a previous work (I. D. Markozov, A. D. Kaminker & A. Y. Potekhin 2023, hereafter Paper I), we constructed a model of an accretion channel taking resonant scattering into account in a strong magnetic field and calculated polarized radiation transfer. In Paper I, we used the electron plasma approximation, which did not take into account the damping, ion cyclotron motion, and vacuum polarization effects in the dielectric tensor. Besides, we considered only the case of a completely filled accretion channel. A disc accretion scenario allows formation of a hollow channel, only partially filled with plasma.

Other theoretical studies of the XRP rely on radiation transfer simulations under predetermined physical conditions in the radiation subsystem. There are two prevailing techniques of solving the radiation transfer equations: the Feautrier method (e.g. D. Mihalas 1978; P. Mészáros 1992) and Monte Carlo method (e.g. G. I. Marchuk et al. 1980; U. M. Noebauer & S. A. Sim 2019). W. Nagel (1981a, b) was the first to employ the Feautrier technique to study the XRP spectra, beaming, and cyclotron line formation with Comptonization. O. Nishimura (2008, 2015) employed this technique to show that taking into account the dependence of the magnetic field, temperature, and density on the height of the accretion channel leads to broad and shallow line profiles, similar to those observed in the real XRP spectra. The Monte Carlo technique has been used to model XRP spectra since 1970s (e.g. R. Z. Yahel 1979). This technique makes it possible to include the complex microphysics related to processes in a strong magnetic field. R. A. Araya & A. K. Harding (1999) used the latter technique to include fully relativistic Compton scattering cross-sections with a few cyclotron resonances. They found that a photon multiplication associated with relaxation of excited electron states makes the cyclotron lines broader and shallower even in the uniform medium. They also showed that the first cyclotron harmonic can be deeper than the fundamental one at certain angles of the line of sight. G. Schönherr et al. (2007) and F. W.

Schwarm et al. (2017a, b) used the model by R. A. Araya & A. K. Harding (1999) in Monte Carlo simulations of radiation in the accreting plasma. S. Kumar, S. Bala & D. Bhattacharya (2022) included up to 10 resonances and the general relativity (GR) effects to calculate observed pulsars spectra. M. A. Garasev, E. V. Derishev & V. V. Kocharovsky (2011) and M. A. Garasev et al. (2016) studied the effects of frequency redistribution of the cyclotron resonant photons in the atmospheres of magnetized compact stars and the impact of these effects on the spectral line shape and on dynamics of resonant photon transfer. Recently, P. Fotiadis et al. (2026) simulated radiative transfer in an accretion channel of subcritical XRP, using the polarization-averaged resonant Compton scattering cross-section in a strong magnetic field, and obtained spectra of outgoing radiation with a pronounced cyclotron absorption line, which depends on the accretion luminosity and angles of observation. In particular, they demonstrated the increase of the centroid photon energy of this line with increasing the luminosity in agreement with observations.

Most of the above-mentioned works treated a hydrodynamical structure of the radiating region as predetermined, without consideration of its consistency with a radiative transfer. Coupling of the plasma magnetohydrodynamics with the radiative transfer has been included in numerical simulations of accretion columns of supercritical XRP in several recent papers (B. F. West et al. 2017a, b; L. Zhang et al. 2022; X. Sheng et al. 2023; L. Zhang et al. 2023, 2025). Two of them (X. Sheng et al. 2023; L. Zhang et al. 2025) have approximately taken into account a resonant scattering in the strong magnetic field.

In this paper, we present a self-consistent calculation of the hydrodynamical structure of the accretion channels of the subcritical XRP together with their radiation, going beyond the electron plasma approximation by taking into account the ion cyclotron resonance and damping effects on the dielectric tensor, as well as the vacuum-polarization effects, which constitute the main advance over Paper I. The importance of vacuum polarization in the simulations of subcritical pulsar radiation was shown in the recent paper by E. Sokolova-Lapa et al. (2023), which was focused on the hot polar cap radiation. Here, we demonstrate that radiative transfer in the accretion channel with allowance for the vacuum polarization has a substantial influence on the radiation of the subcritical XRP. In addition, we consider not only the filled accretion channel, as in Paper I, but also the case of a hollow accretion channel, which is considered as more realistic in the case of a disc accretion (e.g. M. M. Basko & R. A. Sunyaev 1976; A. Mushtukov & S. Tsygankov 2024).

The paper is organized as follows. In Section 2, we introduce definitions of the main physical quantities and their relationships used in further calculations. In particular, we describe polarization characteristics of the NMs in the strongly magnetized plasma (Section 2.1), a set of cross-sections for Compton scattering of the polarized photons (Section 2.2), and the general system of hydrodynamic and radiation transfer equations (Section 2.3). In Section 3, we describe the employed numerical schemes: the basic relationships used at the hydrodynamic substep (Section 3.1.1), the algorithm of calculations at the radiation substep (Section 3.1.2), the energy-momentum exchange (Section 3.1.3), and boundary conditions (Section 3.2). In Section 4, we discuss the results of numerical calculations, including the velocity distributions of plasma flow along the accretion channel (Section 4.1), spectra of the outgoing radiation (Section 4.2), dependence of the cyclotron feature on the accretion rate (Section 4.3), and linear polarization

degree (Section 4.4). The summary and discussion are given in Section 5. In Appendix A, we give explicit expressions for the resonant cross-sections in NMs. Appendix B presents examples of the code verification.

2 PHYSICS INPUT

2.1 Radiation in a magnetized plasma

Propagation of electromagnetic waves in magnetized plasmas has been comprehensively described in the monograph by V. L. Ginzburg (1970). The waves propagate in the form of two NMs at photon circular frequencies ω , which lie sufficiently far from resonances and are much larger than the electron plasma frequency $\omega_{pe} = (4\pi e^2 n_e / m_e)^{1/2}$, where m_e is the electron mass, n_e is the electron number density, and e is the elementary charge. The NM whose electric vector oscillates predominantly in directions close to the $\mathbf{k}$ - $\mathbf{B}$ plane, where $\mathbf{k}$ is the wave vector, is called ordinary (O-mode), and the NM whose electric vector oscillates predominantly in the perpendicular plane is called extraordinary (X-mode). They have different polarization vectors $\mathbf{e}_j$ and different absorption and scattering cross-sections, which depend on the angle θ_B between $\mathbf{k}$ and $\mathbf{B}$. The two modes interact with one another through scattering. J. Ventura (1979) performed an analysis of the polarization modes in application to the NSs from the physics point of view. Y. N. Gnedin & G. G. Pavlov (1974) formulated the radiative transfer problem in terms of these modes and specified the conditions of their applicability. They showed that in the conditions typical for strongly magnetized NSs, except narrow frequency ranges near resonances, a strong Faraday depolarization occurs, which allows one to consider specific intensities of the two NMs instead of the four components of the Stokes vector.

2.1.1 Dielectric tensor

It is convenient to describe a monochromatic electromagnetic wave in the complex vector representation, in which the electric field vector of a plane electromagnetic wave is written as $\mathcal{E}(\omega) e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$, where $\mathbf{r}$ is the radius vector and t is the time. In the following, we will omit the common factor $e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$ for brevity. In this representation and in the Cartesian coordinate system (xyz) with the z -axis along $\mathbf{B}$, the dielectric tensor of a plasma has the form (e.g. V. L. Ginzburg 1970)

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{\perp} & i\varepsilon_{\wedge} & 0 \\ -i\varepsilon_{\wedge} & \varepsilon_{\perp} & 0 \\ 0 & 0 & \varepsilon_{\parallel} \end{pmatrix}, \quad (1)$$

where¹

$$\varepsilon_{\parallel} \approx 1 - \frac{\omega_{pe}^2}{\omega^2 + i\omega\nu_{\parallel}(\omega)} \quad (2)$$

and $\nu_{\parallel}(\omega)$ is an effective damping frequency. Here, and hereinafter we use the cold plasma approximation (that is, neglect thermal motion effects on the dielectric tensor) and assume that

¹Here, we have taken into account that $\omega_{pi} \ll \omega_{pe}$. The plus sign at the damping factor in the denominator corresponds to the chosen form $e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}$ of a monochromatic wave; it is opposite to the sign in V. L. Ginzburg (1970) according to his convention $e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}$.

damping frequencies are small compared to ω . For the other components of ϵ , we will use the approximation suggested by D. Lai & W. C. G. Ho (2003):

$$\epsilon_{\pm} \pm \epsilon_{\lambda} \approx 1 - \frac{\omega_{pe}^2(1 + i\nu_{ri}/\omega) + \omega_{pi}^2(1 + i\nu_{re}/\omega)}{(\omega \pm \omega_{B,e} + i\nu_{re})(\omega \mp \omega_{B,i} + i\nu_{ri}) + i\omega\nu_{ei}^{\perp}(\omega)}, \quad (3)$$

where ω_{pe} is the electron plasma frequency defined above, $\omega_{pi} = (4\pi Ze^2 n_e/m_i)^{1/2}$ is the ion plasma frequency, m_i is the ion mass, and Z is its effective charge number,

$$\omega_{B,e} = \frac{eB}{m_e c}, \quad \omega_{B,i} = \frac{ZeB}{m_i c} \quad (4)$$

are the electron and ion cyclotron frequencies,

$$\nu_{re} = \frac{2}{3} \frac{e^2}{m_e c^3} \omega^2, \quad \nu_{ri} = \frac{2}{3} \frac{Z^2 e^2}{m_i c^3} \omega^2 \quad (5)$$

are natural radiative half-widths of the electron and ion quantum states and $\nu_{ei}^{\perp}(\omega)$ is an effective damping frequency due to the electron-ion collisions. Since $\nu_{ri} \ll \nu_{re}$, the damping frequency in equation (2) is $\nu_{\parallel} \approx \nu_{ei}^{\parallel} + \nu_{re}$. At the high temperatures typical for the XRPs, ν_{ei} is mainly determined by the free-free transitions. In this case, it is customary to write (e.g. G. G. Pavlov & A. N. Panov 1976)

$$\nu_{ei}^{\parallel,\perp} = \frac{4}{3} \sqrt{\frac{2\pi}{m_e k_B T}} \frac{n_e e^4}{\hbar \omega} (1 - e^{-\hbar\omega/k_B T}) \Lambda_{\parallel,\perp}^{\text{ff}}(\omega), \quad (6)$$

where n_e is the electron number density, k_B is the Boltzmann constant, and $\Lambda_{\parallel,\perp}^{\text{ff}}(\omega)$ is a Coulomb logarithm. By definition, the Coulomb logarithm is related to the thermally averaged Gaunt factor $g_{\parallel,\perp}^{\text{ff}}$ as $\Lambda_{\parallel,\perp}^{\text{ff}} = (\pi/\sqrt{3})g_{\parallel,\perp}^{\text{ff}}$; symbols $\parallel$ and $\perp$ correspond to photon polarization along and transverse to $\mathbf{B}$, respectively.

The most general explicit expressions for $\Lambda_{\parallel,\perp}^{\text{ff}}$ with allowance for quantizing magnetic fields and ion motions are given in A. Y. Potekhin (2010); they generalize and correct earlier results presented by different authors for specific particular cases (e.g. G. G. Pavlov & A. N. Panov 1976 for $\omega \gg \omega_{B,i}$ or P. Mészáros 1992 for $\omega_{B,i} \ll \omega \ll \omega_{B,e}$).

The above results have been obtained within the framework of the so-called elementary theory, which is adequate at $\omega \gg \omega_{pe}$ (V. L. Ginzburg 1970).

2.1.2 Vacuum polarization

The influence of the vacuum polarization on the NS radiation was first evaluated by R. Novick et al. (1977), whose results were revisited and corrected by Y. N. Gnedin, G. G. Pavlov & Y. A. Shibano (1978); the effect was studied in detail by G. G. Pavlov & Y. N. Gnedin (1984). In the linear approximation for the electromagnetic waves, the dielectric tensor with allowance for both the magnetized plasma and vacuum polarization can be written as

$$\epsilon' = \epsilon + 4\pi\chi_v, \quad \chi_v = \frac{1}{4\pi} \text{diag}(a_v, a_v, a_v + q_v), \quad (7)$$

where ϵ is given by equation (1), χ_v is the polarizability of vacuum, $\text{diag}(\dots)$ denotes the diagonal matrix, a_v and q_v are specified below. Magnetic permeability of the QED vacuum μ_v is determined by

$$\mu_v^{-1} = \text{diag}(1 + a_v, 1 + a_v, 1 + a_v + m_v). \quad (8)$$

In these equations, a_v , q_v and m_v are scalar functions of B , which can be approximated by expressions (A. Y. Potekhin et al. 2004)

$$a_v = -\frac{2\alpha_f}{9\pi} \ln \left(1 + \frac{b^2}{5} \frac{1 + 0.25487 b^{3/4}}{1 + 0.75 b^{5/4}} \right), \quad (9)$$

$$q_v = \frac{7\alpha_f}{45\pi} b^2 \frac{1 + 1.2 b}{1 + 1.33 b + 0.56 b^2}, \quad (10)$$

$$m_v = -\frac{\alpha_f}{3\pi} \frac{b^2}{3.75 + 2.7 b^{5/4} + b^2}, \quad (11)$$

where $\alpha_f \equiv e^2/\hbar c \approx 1/137$ is the fine-structure constant. Equations (9)–(11) agree with accurate analytical (J. S. Heyl & L. Hernquist 1997) and numerical (K. Kohri & S. Yamada 2002) results at any B . They also reproduce the asymptotes derived by S. L. Adler (1971) in the limit of small parameter $b \equiv B/B_{\text{QED}}$, where $B_{\text{QED}} \equiv m_e^2 c^3 / e \hbar = 4.414 \times 10^{13}$ G is the natural magnetic field unit in the QED: $a_v = -2\delta_v$, $q_v = 7\delta_v$, and $m_v = -4\delta_v$, where $\delta_v \equiv (\alpha_f/45\pi)b^2 \approx 2.65 \times 10^{-8} B_{12}^2$ and $B_{12} \equiv B/10^{12}$ G. These low- b asymptotes are quite accurate in the case of $b \sim 0.1$ considered below.

The regions of predominant importance of either the plasma effects or the vacuum polarization effects are separated by the vacuum resonance energy E_{vac} (see W. C. G. Ho & D. Lai 2003). The concept of a vacuum resonance was introduced in the theory by Y. N. Gnedin et al. (1978). If the photon energy $E \equiv \hbar\omega$ is small compared with E_{vac} , the dielectric tensor is determined by plasma, while at $E \gg E_{\text{vac}}$ it is determined by vacuum polarization. At $b \ll 1$,

$$E_{\text{vac}} \approx \left(\frac{15\pi}{\alpha_f} \right)^{1/2} \frac{\omega_{pe}}{\omega_{B,e}} m_e c^2 \approx 10^2 \left(\frac{Z}{A} \right)^{1/2} \frac{\sqrt{\rho_1}}{B_{12}} \text{ keV}, \quad (12)$$

where $\rho_1 \equiv \rho/(1 \text{ g cm}^{-3})$, ρ is the plasma mass density, and A is the ion mass number. One can write $E_{\text{vac}} \approx E_B/\sqrt{W}$, where $E_B \equiv \hbar\omega_{B,e} \approx 11.6 B_{12}$ keV is the electron cyclotron energy and

$$W \simeq (3 \times 10^{28} \text{ cm}^{-3}/n_e) b^4, \quad (13)$$

is the vacuum polarization parameter (G. G. Pavlov & Y. A. Shibano (1979); A. D. Kaminker, G. G. Pavlov & Y. A. Shibano (1982)).

2.1.3 Polarization vectors of the normal modes

Let us consider the right-handed Cartesian coordinate system (xyz) with the z-axis directed along $\mathbf{k}$ and the x-axis perpendicular to the $(\mathbf{k}-\mathbf{B})$ plane.²

In the NM approximation, the complex electric vectors can be numbered by the polarization index j , which takes two values for the two polarization modes. We will use the transverse approximation $\mathcal{E} \cdot \mathbf{k} = 0$. Then

$$\mathcal{E}_j = \mathcal{E}_j \mathbf{e}_j = \mathcal{E}_{j,x} \hat{\mathbf{e}}_x + \mathcal{E}_{j,y} \hat{\mathbf{e}}_y, \quad (14)$$

where $\mathcal{E}_j$ determines the NM amplitude, $\mathbf{e}_j$ is the complex unit vector of polarization, $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ are the unit vectors along the x- and y-axes, $\mathcal{E}_{j,x}$ and $\mathcal{E}_{j,y}$ are complex electric field amplitude components along these axes. Following W. C. G. Ho & D. Lai

²Note that this coordinate system is rotated by the angle θ_B relative to the one used in Sections 2.1.1 and 2.1.2 (see e.g. V. L. Ginzburg 1970, section 10, for the transformations of ϵ between the coordinate systems).

(2001, 2003), we adopt the values $j = 1$ and 2 for the X- and O-modes, respectively, and define the dimensionless quantities

$$u_e = \frac{\omega_{B,e}^2}{\omega^2}, \quad u_i = \frac{\omega_{B,i}^2}{\omega^2}, \quad v_e = \frac{\omega_{pe}^2}{\omega^2}, \quad v_i = \frac{\omega_{pi}^2}{\omega^2}. \quad (15)$$

In the following, we will assume that $v_e \ll 1$ (which also implies that $v_i \ll 1$). It is also convenient to introduce the complex ellipticity parameter (e.g. W. C. G. Ho & D. Lai 2003)³

$$K_j = i\varepsilon_{j,y}/\varepsilon_{j,x} = ie_{j,y}/e_{j,x}, \quad (16)$$

where $e_{j,y} = \mathbf{e}_j \cdot \hat{\mathbf{e}}_y$ and $e_{j,x} = \mathbf{e}_j \cdot \hat{\mathbf{e}}_x$ are projections of $\mathbf{e}_j$ on the y and x axes.

In Paper I, we neglected the vacuum polarization, collisions between plasma particles and ion cyclotron oscillations in a magnetic field. In such approximations,

$$K_j \approx \frac{-2 \cos \theta_B}{\sqrt{u_e \sin^2 \theta_B - (-1)^j \sqrt{u_e \sin^4 \theta_B + 4 \cos^2 \theta_B}}}, \quad (17)$$

where θ_B is the angle between $\mathbf{k}$ and $\mathbf{B}$, and we have neglected v_e in the factors $(1 - v_e)$ (cf. equation 11.26 of V. L. Ginzburg 1970).

In the present calculations, we rely on a more general expression for the complex ellipticity parameter, which takes into account the finite ion mass, damping at the resonances and vacuum polarization (W. C. G. Ho & D. Lai 2003):

$$K_j = \beta \left\{ 1 + (-1)^j \left[1 + \frac{1}{\beta^2} + \frac{m_v}{1 + a_v} \frac{\sin^2 \theta_B}{\beta^2} \right]^{1/2} \right\}, \quad (18)$$

where

$$\beta = \frac{\varepsilon'_{\parallel} - \varepsilon'_{\perp} + \varepsilon'_{\wedge}/\varepsilon'_{\perp} + \varepsilon'_{\parallel} m_v/(1 + a_v)}{2 \varepsilon_{\wedge}} \frac{\varepsilon'_{\perp} \sin^2 \theta_B}{\varepsilon'_{\parallel} \cos \theta_B}, \quad (19)$$

$\varepsilon'_{\perp} = \varepsilon_{\perp} + a_v$ and $\varepsilon'_{\parallel} = \varepsilon_{\parallel} + a_v + q_v$. The parameter β is sometimes also called polarization parameter (e.g. D. Lai & W. C. G. Ho 2003). We use the transverse approximation ($\mathbf{e}_j \cdot \mathbf{k} = 0$) and therefore neglect the z -components of the polarization vectors $e_{j,z} = \mathbf{e}_j \cdot \hat{\mathbf{e}}_z$ (see the explicit expression for $K_{j,z} = -ie_{j,z}/e_{j,x} \propto v_e \ll 1$ in W. C. G. Ho & D. Lai 2003). At $B \ll B_{\text{QED}}$, using the relations $v_i \ll v_e \ll 1$, $\sqrt{u_i} \ll \sqrt{u_e}$, and $|u_e - 1| \gg v_e$, one can approximate equation (19) by the expression (G. G. Pavlov & Y. A. Shibano 1979; A. D. Kaminker et al. 1982)

$$\beta = \sqrt{u_e} \frac{\sin^2 \theta_B}{2 \cos \theta_B} \left(1 - W \frac{u_e - 1}{u_e^2} \right), \quad (20)$$

where W is given by equation (13).

Fig. 1 demonstrates the typical behaviour of the parameter β as a function of photon energy. It crosses zero near the electron and ion cyclotron resonances and near the point $E = E_B/\sqrt{W}$; the values of β away from these points are large. In contrast, β decreases with growing E at $E \gg E_{B,i}$ as $\beta \propto E_B/E$, if the vacuum polarization is neglected. We also see that the approximation (20) is excellent at $E \gg E_{B,i}$, but becomes inapplicable at smaller energies, where the cyclotron motion of ions becomes important. The dependence of K_j on E and ρ reflects the dependence of β according to equation (18): since the vacuum parameter m_v

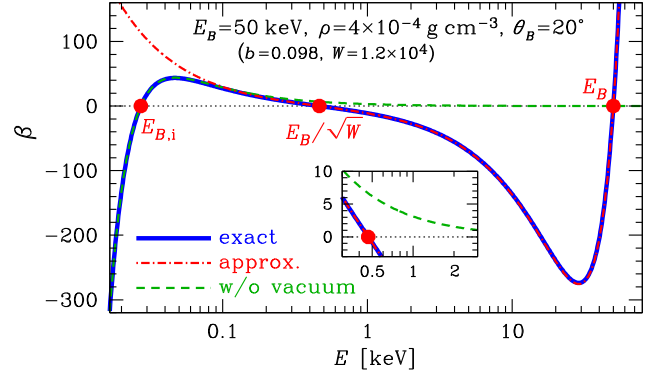


Figure 1. Parameter β according to equations (19) (solid curve) and (20) (dot-dashed curve) as functions of photon energy E at magnetic field $B = 4.32 \times 10^{12}$ G, mass density $\rho = 4 \times 10^{-4}$ g cm $^{-3}$, and angle $\theta_B = \pi/9$. The dashed curve shows this parameter without account of the QED vacuum, i.e. according to equation (19) with $\varepsilon'_{\parallel,\perp,\wedge}$ replaced by $\varepsilon_{\parallel,\perp,\wedge}$. Heavy dots mark the energy values $E_{B,i}$, $E_B/\sqrt{W}$, and E_B , which are approximations to the values of E at which β changes its sign. The inset shows a zoom to $E \sim E_B/\sqrt{W}$.

is small, $K_2 \approx 2\beta$ whenever $\beta^2 \gg 1$, $K_2 \approx \text{sign}\beta$ if $\beta^2 \ll 1$, and $K_1 \approx -K_2^{-1}$. It means that away from the resonances, where $\beta^2 \gg 1$, the NM polarization is almost linear (as first noticed by G. G. Pavlov & Y. A. Shibano 1979).

Instead of the X- and O-modes with the ellipticities K_j , which are determined by relation $|K_2| > |K_1|$ and given by equation (18), one can refer to the ‘plus’ and ‘minus’ modes (D. Lai & W. C. G. Ho 2002; W. C. G. Ho & D. Lai 2003), whose complex ellipticities are given by

$$K_{\pm} = \beta \pm \sqrt{\beta^2 + 1 + m_v \sin^2 \theta_B/(1 + a_v)}. \quad (21)$$

At $\beta > 0$, the plus-mode is the O-mode and the minus-mode is the X-mode, and vice versa at $\beta < 0$.

Under the considered conditions, typical for the XRPs, the plasma and damping frequencies are small compared to ω . Then one can neglect the anti-Hermitian part of the dielectric tensor at ω outside the Doppler cores of cyclotron resonances (see G. G. Pavlov & Y. A. Shibano 1979; G. G. Pavlov, I. A. Shibano & D. G. Yakovlev 1980). Thus, the damping frequencies serve only to remove divergencies near the resonances by replacing the factors $(\omega - \omega_{B,e})$ and $(\omega - \omega_{B,i})$ by $(\omega - \omega_{B,e} + i\nu_{\text{eff},e})$ and $(\omega - \omega_{B,i} + i\nu_{\text{eff},i})$ at $\omega \approx \omega_{B,e}$ and $\omega \approx \omega_{B,i}$, respectively, where $\nu_{\text{eff},e}$ and $\nu_{\text{eff},i}$ are effective cumulative damping frequencies, which can be found from equation (3). Since $\omega_{B,i} \ll \omega_{B,e}$ and $\nu_{ri} \ll \omega_{B,e}$, the denominator in equation (3) for $\varepsilon_{\perp} - \varepsilon_{\wedge}$ at the electron cyclotron resonance ($\omega \approx \omega_{B,e}$) approaches $\omega_{B,e}(\omega - \omega_{B,e} + i\nu_{re} + i\nu_{ei}^{\perp})$, which means that

$$\nu_{\text{eff},e} \approx \nu_{re} + \nu_{ei}^{\perp}(\omega_{B,e}). \quad (22)$$

In the same way, at the ion cyclotron resonance ($\omega \approx \omega_{B,i}$) the denominator in equation (3) for $\varepsilon_{\perp} + \varepsilon_{\wedge}$ can be approximated by $\omega_{B,e}(\omega - \omega_{B,i} + i\nu_{ri}) + i\omega_{B,i}\nu_{ei}^{\perp}$, so that⁴

$$\nu_{\text{eff},i} \approx \nu_{ri} + \frac{\omega_{B,i}}{\omega_{B,e}} \nu_{ei}^{\perp}(\omega_{B,i}) = \nu_{ri} + \frac{m_e}{m_i} \nu_{ei}^{\perp}(\omega_{B,i}). \quad (23)$$

⁴The factor (m_e/m_i) in equation (23), which also appears in equation (53) of A. Y. Potekhin & G. Chabrier (2003), was missed by D. Lai & W. C. G. Ho (2003) and M. Adelsberg & D. Lai (2006).

³The choice of sign in equation (16) is opposite to that in A. A. Mushtukov et al. (2022), in order to make our definition of K_j equivalent to that by W. C. G. Ho & D. Lai (2003), taking into account that the coordinate system with the x -axis lying in the $(\mathbf{k}-\mathbf{B})$ plane was chosen by these authors, while we have chosen the y -axis lying in this plane. Thus, defined parameters K_j are equivalent to $-iK_j$ in section 11 of V. L. Ginzburg (1970).

For estimations, it is convenient to rewrite equations (5) and (6) in the form (D. Lai & W. C. G. Ho 2003)

$$\frac{\nu_{re}}{\omega} \approx 9.5 \times 10^{-6} E_1, \quad \frac{\nu_{ri}}{\omega} \approx 5.2 \times 10^{-9} \frac{Z^2}{A} E_1, \quad (24)$$

$$\frac{\nu_{ei}}{\omega} \approx 9.2 \times 10^{-5} \frac{Z^2 \rho_1}{AT^{1/2} E_1^2} (1 - e^{-E/k_B T}) g_{\parallel, \perp}^{\text{ff}}, \quad (25)$$

where $E_1 = E/(1 \text{ keV})$ and $T_6 = T/(10^6 \text{ K}) \approx 11.6 k_B T/(1 \text{ keV})$.

In the following, we consider the hydrogen plasma and set $Z = A = 1$. The Gaunt factor $g_{\parallel, \perp}^{\text{ff}}$ depends on E , ρ , T and B in a non-trivial way, but typically remains in the range ~ 0.1 – 10 (see e.g. A. Y. Potekhin 2010). Thus, we see from equations (24) and (25) that for the plasma parameters considered in this paper ($E_1 \gtrsim 1$, $\rho_1 \lesssim 10^{-2}$, $T_6 \gg 10$) we have $\nu_{re} \gg \nu_{ei}^{\parallel, \perp}$ and $\nu_{ri} \gg (m_e/m_i) \nu_{ei}^{\parallel, \perp}$. Then, according to equations (22) and (23), $\nu_{\text{eff}, e} \approx \nu_{re}$ and $\nu_{\text{eff}, i} \approx \nu_{ri}$, therefore we neglect the free-free damping.

As follows directly from the definition (16), in the transverse approximation the polarization vectors can be written in the form

$$\mathbf{e}_j = \frac{e^{i\phi_j}}{\sqrt{1 + |K_j|^2}} (\hat{\mathbf{e}}_x - iK_j \hat{\mathbf{e}}_y), \quad (26)$$

where ϕ_j is an arbitrary phase. If the vectors $\mathbf{e}_j$ are mutually orthogonal, $\mathbf{e}_1 \cdot \mathbf{e}_2^* = 0$ (the asterisk means complex conjugate), then equation (26) provides the relation

$$K_1 = -K_2^{-1}. \quad (27)$$

With allowance for the vacuum polarization, equation (27) is fulfilled only approximately, but with high accuracy for the parameter values considered here. In this case, the small inaccuracy of equation (27) is equivalent to the small inaccuracy of the NM orthogonality. In general case, the NM orthogonality was investigated by Y. N. Gnedin & G. G. Pavlov (1974), G. G. Pavlov & Y. A. Shibanov (1979), and G. G. Pavlov et al. (1980).

Let us write equation (26) in the matrix form by analogy with A. A. Mushtukov et al. (2022),

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{e}_1^\ell \\ \mathbf{e}_2^\ell \end{pmatrix}, \quad (28)$$

where $\mathbf{M}$ is composed of elements M_{jl} with the first index ($j = 1, 2$) numbering the complex polarization vectors $\mathbf{e}_j$ of the elliptical NMs and the second one ($l = 1, 2$) numbering the linear (indicated by the superscript ℓ) polarization vectors $\mathbf{e}_1^\ell = \hat{\mathbf{e}}_x$ and $\mathbf{e}_2^\ell = \hat{\mathbf{e}}_y$:

$$M_{j1} = \frac{e^{i\phi_j}}{\sqrt{1 + |K_j|^2}}, \quad M_{j2} = \frac{-iK_j e^{i\phi_j}}{\sqrt{1 + |K_j|^2}} \quad (j = 1, 2). \quad (29)$$

G. G. Pavlov & Y. A. Shibanov (1979) represented the parameter β (equation 19) as $\beta = q + ip$, where q and p are real, q determines the ellipticity and p is associated with collisional processes. Under the conditions considered in this paper, the inequality $q^2 \gg p^2$ is fulfilled and we may neglect the parameter p . Then $\mathbf{e}_1$ and $\mathbf{e}_2$ are mutually orthogonal (disregarding the above-mentioned minor inaccuracy) and using equation (27) we can write $\mathbf{M}$ in the form

$$\begin{aligned} \mathbf{M} &= \frac{1}{\sqrt{1 + K_1^2}} \begin{pmatrix} e^{i\phi_1} & -ie^{i\phi_1} K_1 \\ |K_1| e^{i\phi_2} & ie^{i\phi_2} \text{sign } K_1 \end{pmatrix} \\ &= \frac{1}{\sqrt{1 + K_2^2}} \begin{pmatrix} |K_2| e^{i\phi_1} & ie^{i\phi_1} \text{sign } K_2 \\ e^{i\phi_2} & -ie^{i\phi_2} K_2 \end{pmatrix}, \end{aligned} \quad (30)$$

which generalizes⁵ and corrects⁶ a similar matrix in A. A. Mushtukov et al. (2021, 2022). This matrix is unitary: $\mathbf{M}^{-1} = \mathbf{M}^\dagger$, where the dagger sign means the Hermitian conjugate.

2.2 Compton scattering in a strong magnetic field

Compton scattering in a strong magnetic field differs significantly from the field-free one. In this section, we outline cross-sections of the process by analogy with A. A. Mushtukov et al. (2022) and Paper I.

We will assume that an electron occupies the ground Landau level before and after scattering. This is generally a good approximation for the XRPs, because the cyclotron radiative de-excitation has a time-scale much shorter than collisional processes (e.g. P. Mészáros 1992).⁷ Then the energy and momentum conservation laws are

$$\sqrt{m_e^2 c^4 + p_i^2 c^2} + E_i = \sqrt{m_e^2 c^4 + p_f^2 c^2} + E_f, \quad (31)$$

$$E_i \cos \theta_i + p_i c = E_f \cos \theta_f + p_f c, \quad (32)$$

where p_i (p_f) is the projection of the electron momentum on $\mathbf{B}$, E_i (E_f) is the photon energy, and θ_i (θ_f) is the value of θ_B before (after) scattering.

The differential cross-section of photon scattering on an electron with $p_i = 0$ in a strong magnetic field can be written as (e.g. H. Herold 1979)

$$\frac{d\sigma_{\text{fi}}^0}{d\Omega_{\text{f}}^0}(E_i^0, \mathbf{\Omega}_i^0, \mathbf{\Omega}_f^0) = \frac{3}{32\pi} \sigma_T |a_{\text{fi}}|^2, \quad (33)$$

where E_i^0 is the initial photon energy, $\mathbf{\Omega}_{i,f}^0$ are the unit vectors along $\mathbf{k}$ before and after scattering, the superscript 0 indicates that they are measured in the electron rest frame before the scattering, σ_T is the Thomson cross-section, and a_{fi} is a complex scattering amplitude. Hereinafter we use the non-relativistic approximation with a single cyclotron resonance in a_{fi} , which is valid at $B \ll B_{\text{QED}}$ and $E_{B,i} \ll E_i^0 \ll m_e c^2$, where $E_{B,i} \equiv \hbar \omega_{B,i} = (m_e/m_i) E_B$. Then, the scattering amplitudes for linear polarizations orthogonal and coplanar to the $\mathbf{k}$ - $\mathbf{B}$ plane are (H. Herold 1979)

$$a_{11}^\ell = A_+ + A_-, \quad (34)$$

$$a_{22}^\ell = 2 \sin \theta_i \sin \theta_f + (A_+ + A_-) \cos \theta_i \cos \theta_f, \quad (35)$$

$$\begin{pmatrix} a_{12}^\ell \\ a_{21}^\ell \end{pmatrix} = \begin{pmatrix} -i \cos \theta_i \\ i \cos \theta_f \end{pmatrix} (A_+ - A_-), \quad (36)$$

$$\text{where } A_+ \equiv \frac{E_i e^{i(\varphi_i - \varphi_f)}}{E_i + E_B}, \quad A_- \equiv \frac{E_i e^{-i(\varphi_i - \varphi_f)}}{E_i - E_B + i\hbar \nu_{\text{eff}, e}}. \quad (37)$$

Here, the first and second subscripts relate to the final and initial states of a photon, respectively, $\varphi_{i,f}$ are azimuthal angles in the plane orthogonal to $\mathbf{B}$, the subscripts ‘1’ and ‘2’ indicate the

⁵The factors $e^{i\phi_j}$ are introduced for generality, but the choice of the phases ϕ_j does not affect any results. Indeed, the cross-sections do not depend on ϕ_j , as can be seen from equations (33) and (38). In the calculations, we set $\phi_1 = 0$ and $\phi_2 = \pi$.

⁶Equations (4)–(6) in A. A. Mushtukov et al. (2022) mistakenly referred to the non-normalized vectors $\mathcal{E}$ instead of polarization vectors $\mathbf{e}$ and contained erroneous signs, inconsistent with any choices of ϕ_1 and ϕ_2 , as well as equation (15) in A. A. Mushtukov et al. (2021).

⁷Note, however, that this approximation does not allow one to reproduce the broad and shallow shapes of the cyclotron lines obtained by R. A. Araya & A. K. Harding (1999), which are due to the multiple generation (spawning) of photons by electrons at higher Landau levels in the process of scattering.

linear polarizations $\mathbf{e}_1^\ell$ and $\mathbf{e}_2^\ell$, and $\nu_{\text{eff},e}$ is the effective damping frequency (equation 22); in our case $\nu_{\text{eff},e} \approx \nu_{\text{re}}$.

Let us compose a 2×2 matrix $\mathbf{a}_{\text{fi}}^\ell$ of the four scattering amplitudes for the linear polarizations $\mathbf{a}_{\text{fi}}^\ell$ in equations (34)–(36). To transform it to the analogous matrix $\mathbf{a}_{\text{fi}}$ of the amplitudes for the elliptical NM polarizations $\mathbf{a}_{\text{fi}}$, we convert the initial NM basis vectors $\mathbf{e}_j$ into the initial linearly polarized vectors $\mathbf{e}_l$ by applying the matrix $\mathbf{M}_i^{-1} = \mathbf{M}_i^\dagger$ and the final linearly polarized vectors $\mathbf{e}_{l'}$ into the final NM basis $\mathbf{e}_{j'}$ by using the matrix $\mathbf{M}_f$, which gives

$$\mathbf{a}_{\text{fi}} = \mathbf{M}_f \mathbf{a}_{\text{fi}}^\ell \mathbf{M}_i^\dagger. \quad (38)$$

The rest-frame differential cross-sections for NM scattering are given by equations (33)–(38) with $\mathbf{a}_{\text{fi}} = \mathbf{a}_{\text{fi}}^\ell$, where the subscripts j and j' enumerate the elliptical NMs of initial and final photon states, respectively. Explicit expressions for the resulting scattering amplitudes are given in Appendix A.

The differential cross-sections in an arbitrary reference frame, where the electron has a non-zero initial longitudinal momentum p_i , are given by the Lorentz transformation as

$$\frac{d\sigma_{\text{fi}}}{d\Omega_f}(p_i, E_i, \Omega_i, \Omega_f) = \frac{d\sigma_{\text{fi}}^0}{d\Omega_f^0}(E_i^0, \Omega_i^0, \Omega_f^0) \frac{1 - \beta_i^2}{(1 - \beta_i \cos \theta_i)^2}, \quad (39)$$

where

$$\beta_i = \frac{p_i}{\sqrt{m_e^2 c^2 + p_i^2}} \quad (40)$$

is the dimensionless electron velocity before the scattering and the arguments of $d\sigma_{\text{fi}}/d\Omega_f$ are related to their values in the electron rest frame by relativistic Doppler effect and aberration formulae (see e.g. L. D. Landau & E. M. Lifshitz 1975)

$$E_i^0 = E_i \frac{1 - \beta_i \cos \theta_i}{\sqrt{1 - \beta_i^2}}, \quad \cos \theta_{i,f}^0 = \frac{\cos \theta_{i,f} - \beta_i}{1 - \beta_i \cos \theta_{i,f}}. \quad (41)$$

Since the electrons in the plasma have different longitudinal momenta, a photon with a given energy and direction, scattered into another given direction, can have different energies, depending on p_i according to equations (31) and (32),

$$\hat{E}_f(p_i) = E_i + (m_e^2 c^4 + p_i^2 c^2)^{1/2} - [m_e^2 c^4 + (p_i c + E_i \cos \theta_i - E_f \cos \theta_f)^2]^{1/2}. \quad (42)$$

Therefore, a differential cross-section averaged over all electrons depends on both the initial and final photon energies, so that

$$\frac{d\sigma_{\text{fi}}}{dE_f d\Omega_f}([f_e]; E_i, \Omega_i, E_f, \Omega_f) = \int_{-\infty}^{+\infty} \frac{d\sigma_{\text{fi}}}{d\Omega_f}(p_i, E_i, \Omega_i, \Omega_f) \times \delta(E_f - \hat{E}_f(p_i)) f_e(p_i) dp_i \quad (43)$$

$$= \sum_{\hat{p}_i} \left[\frac{d\sigma_{\text{fi}}}{d\Omega_f}(p, E_i, \Omega_i, \Omega_f) f_e(p) \left| \frac{d\hat{E}_f(p)}{dp} \right|^{-1} \right]_{p=\hat{p}_i}, \quad (44)$$

where $f_e(p)$ is the distribution function of the longitudinal momenta of the electrons and $\hat{p}_i$ is a value of p_i which solves the system (31), (32) at given E_i, E_f, θ_i , and θ_f . The sum in equation (44) is over all real solutions $\hat{p}_i$; it is replaced by zero if they do not exist. We use a one-dimensional version of the Maxwell–Jüttner distribution of electron momenta $f_e(p)$ (e.g. G. Chacón-Acosta, L. Dagdug & H. A. Morales-Técotl 2010; also see appendix A in A. A. Mushtukov et al. 2022).

In the simulations, we mostly use a simplified averaging, taking the delta function out of the integral in equation (43):

$$\frac{d\sigma_{\text{fi}}}{dE_f d\Omega_f}([f_e]; E_i, \Omega_i, E_f, \Omega_f) \approx \delta(E_f - \hat{E}_f(p_{\text{bulk}})) \frac{d\sigma_{\text{fi}}}{d\Omega_f}, \quad (45)$$

where p_{bulk} is the bulk plasma momentum and

$$\frac{d\sigma_{\text{fi}}}{d\Omega_f} = \int_{-\infty}^{+\infty} \frac{d\sigma_{\text{fi}}}{d\Omega_f}(p_i, E_i, \Omega_i, \Omega_f) f_e(p_i) dp_i. \quad (46)$$

We have checked by a direct comparison that this simplification does not introduce any noticeable error in our results presented below.

The total scattering cross-section for a photon of a NM j is given by integration over the final energies and angles, as well as summation over the final modes,

$$\sigma_j([f_e]; E_i, \Omega_i) = \sum_{j'=1}^2 \sigma_{j'j}([f_e]; E_i, \Omega_i), \quad (47)$$

$$\sigma_{j'j}([f_e]; E_i, \Omega_i) = \int_{(4\pi)} \frac{d\sigma_{j'j}}{d\Omega_f}([f_e]; E_i, \Omega_i, \Omega_f) d\Omega_f. \quad (48)$$

In this work, we primarily use cross-sections in which the finite ion mass, damping and vacuum polarization effects are taken into account with the ellipticities given by equation (18). Fig. 2 shows examples of cross-sections that include these effects. For comparison, in the left panel we also show the cross-sections calculated with ellipticities K_j in the electron plasma approximation (17) that has been used in Paper I. A key difference of the more accurate cross-sections is the increased strength of the resonance in the O-mode, which becomes comparable with the resonance in the X-mode. This behaviour agrees with A. D. Kaminker et al. (1982, cf. their fig. 5) and is explained by the appearance of virtual positrons in the QED vacuum, which interact with radiation similarly to the electrons, but for the opposite rotation directions of the electric vector in the electromagnetic wave (circular minus-mode), so that the scattering cross-section of the O-mode becomes resonant due to the interaction with virtual positrons. Comparing the middle and right panels, we see that the density dependence of the cross-sections at $W \gg 1$ is insignificant. Since the ‘plasma approximation’ (17) omits density dependence, it means that the impact of vacuum polarization on the cross-sections is nearly the same for both considered densities.

In the following sections, we study the impact of vacuum polarization on the radiation transfer and hydrodynamics in the accretion channel of subcritical XRPCs.

2.3 Accretion channel hydrodynamics

Accreting plasma moves towards the NS surface along magnetic field lines and forms an *accretion channel* near the magnetic pole (see e.g. M. M. Basko & R. A. Sunyaev 1976). In the case where accreting matter is supplied by the disc, the base of the accretion channel has a ring-shaped cross-section with a void in the centre, while in the case of the wind accretion the channel is fully filled with accreting matter (see section 2 in M. M. Basko & R. A. Sunyaev 1976). We consider both cases and approximately represent an accretion channel as a hollow (with a ring-shaped cross-section) or filled cylinder (designated as *ring* or *circle* geometry, respectively), neglecting the curvature of the magnetic field.

Due to the current limitation of the employed numerical code (specified in Section 3.1.1 below), in this work we do not account for the GR effects and neglect the special relativity modifications of radiation hydrodynamics equations, which then read (e.g. J. I. Castor 2007, section 2.2):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (49)$$

$$\frac{\partial}{\partial t} \left[\rho \epsilon + \frac{\rho v^2}{2} \right] + \nabla \cdot \left[\rho \epsilon \mathbf{v} + \rho \frac{v^2}{2} \mathbf{v} + P \mathbf{v} \right] = Q_g + Q_{\text{rad}}, \quad (50)$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P = \mathbf{F}_{\text{rad}} + \mathbf{F}_g, \quad (51)$$

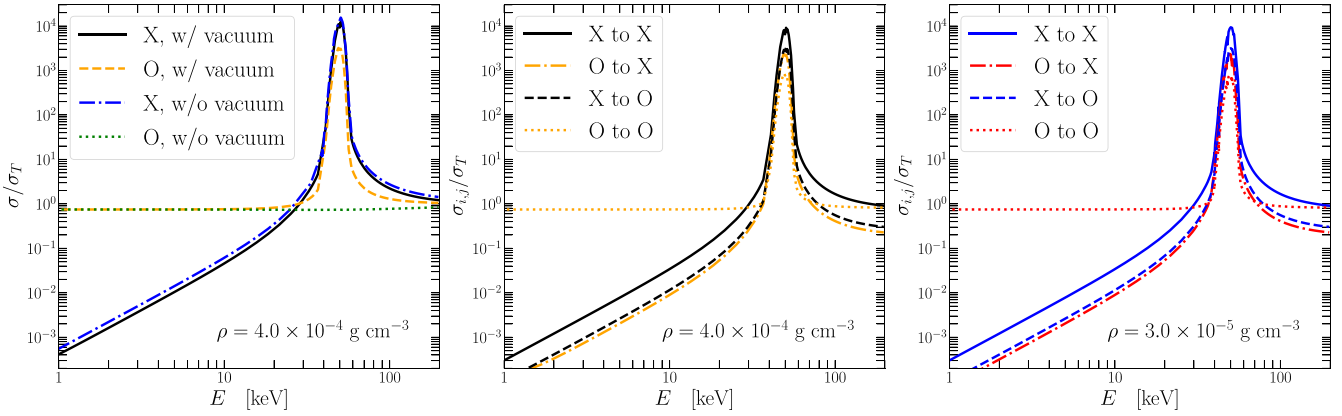


Figure 2. Scattering cross-sections as functions of photon energy in a hydrogen plasma with temperature $k_B T = 5$ keV and cyclotron energy $E_B = 50$ keV for radiation propagating at angle $\theta_B = \pi/3$ to the magnetic field. Left panel: the total cross-section (47) for the X-mode (solid and dot-dashed lines) and O-mode (dashed and dotted lines) at plasma mass density $\rho = 4 \times 10^{-4} \text{ g cm}^{-3}$ (the vacuum polarization parameter (13) $W \approx 1.2 \times 10^4$) taking vacuum polarization into account (solid and dashed lines) and without it (dot-dashed and dotted lines). Middle panel: cross-sections (48) of the scattering from X- to X-mode (solid line), from O- to X-mode (dot-dashed line), from X- to O-mode (dashed line) and from O- to O-mode (dotted line) with account for the vacuum polarization at the same parameters. Right panel: the same as in the middle panel but at $\rho = 3 \times 10^{-5} \text{ g cm}^{-3}$ ($W \approx 1.5 \times 10^5$).

where

$$Q_{\text{rad}} = \int_0^\infty dE \int_{(4\pi)} d\Omega (\kappa_E I_E - J_E), \quad (52)$$

$$\mathbf{F}_{\text{rad}} = \frac{1}{c} \int_0^\infty dE \int_{(4\pi)} d\Omega (\kappa_E I_E - J_E) \mathbf{n}. \quad (53)$$

Here, $\mathbf{v}$ is the bulk velocity of the plasma, ϵ the mass energy density, P the pressure, $Q_g = \rho \mathbf{v} \cdot \mathbf{g}$ the gravity power density, $\mathbf{F}_g = \rho \mathbf{g}$ the gravity force density, $\mathbf{g} = -(GM_{\text{NS}}/R_{\text{NS}}^2) \mathbf{n}$ the gravitational acceleration at the surface of an NS with mass M_{NS} and radius R_{NS} , $\mathbf{n}$ the outer normal to the surface, $\Omega = \mathbf{k}/|\mathbf{k}|$ the unit vector of radiation direction, I_E^j the specific intensity in the mode j normalized to the photon energy E , $I_E = \sum_{j=1}^2 I_E^j$, $\kappa_E^j = n_e \sigma_j$ the absorption coefficient in the mode j , $\kappa_E = \sum_{j=1}^2 \kappa_E^j$, J_E^j the emission coefficient in the mode j , and $J_E = \sum_{j=1}^2 J_E^j$, while Q_{rad} and $\mathbf{F}_{\text{rad}}$ describe, respectively, the energy and momentum interchange between radiation and plasma.

This system of equations is coupled to the stationary two-mode radiation transfer equation (e.g. A. Z. Dolginov, Y. N. Gnedin & N. A. Silant'ev 1995)

$$\begin{aligned} \Omega \cdot \nabla I_E^j &= J_E^j - \kappa_E^j I_E^j \\ &= \sum_{j'=1}^2 \int_0^\infty dE' \int_{(4\pi)} d\Omega' [R_{j \rightarrow j'}(E', \Omega' \rightarrow E, \Omega) I_{E'}^{j'}(\Omega') \\ &\quad - R_{j \rightarrow j'}(E, \Omega \rightarrow E', \Omega') I_E^j(\Omega)], \end{aligned} \quad (54)$$

where

$$R_{j \rightarrow j'}(E, \Omega \rightarrow E', \Omega') = n_e \frac{d\sigma_{jj'}}{dE' d\Omega'}([f_e]; E, \Omega, E', \Omega') \quad (55)$$

is the redistribution function of photons in the polarization state j , propagating with energy E in the direction Ω , into the polarization state j' with energy E' and propagation direction Ω' .

Deceleration of the accreting matter is determined by the radiative (integral) terms in equations (50) and (51). In this work, we neglect all radiation processes except scattering, hence the redistribution function $R_{j \rightarrow j'}(E, \Omega \rightarrow E', \Omega')$ takes only this process into account. Nevertheless it has a complex form, because the scattering cross-sections depend strongly on the photon energy, polarization, and propagation direction.

The system of radiation hydrodynamics equations (49)–(54) should be completed by the plasma equation of state. We assume that electrons and protons have the same temperature T and apply an ideal gas equation of state, which is a good approximation at the considered XRP conditions, because $k_B T \sim$ a few keV is much greater than the hydrogen binding energy at $B \lesssim 10^{13}$ G. Since electrons at $E_B \gg k_B T$ constitute a one-dimensional gas with the adiabatic index $\gamma = 3$, while protons at $E_{B,i} \ll k_B T$ constitute the usual three-dimensional non-relativistic gas with $\gamma = 5/3$, the total adiabatic index for a fully ionized hydrogen plasma is $\gamma = 2$. However, as mentioned in Paper I, the adiabatic index as well as the gas pressure itself is unimportant for our model calculations, because the accretion flow is mainly controlled by the radiation pressure and gravity, rather than by the plasma pressure.

3 NUMERICAL MODEL

3.1 Numerical scheme

To solve numerically the system of equations (49)–(54), we use time discretization and apply the operator splitting method (see e.g. R. J. LeVeque 2002). Each time-step is divided into three substeps, described below.

3.1.1 Hydrodynamical substep

At the first substep we solve the hydrodynamical system of equations (49)–(51), neglecting the radiative terms Q_{rad} and $\mathbf{F}_{\text{rad}}$. To solve this simplified system, we employ the open library VH-1 ('Virginia Hydrodynamics-1'; P. Colella & P. R. Woodward 1984).⁸ The numerical grid is organized as a cylinder of radius R_c and height H , divided into N_z slices of equal height ($N_z = 100$ in the examples shown in Section 4). The value of H in each simulation is chosen sufficiently large to exceed the height of the zone where radiation pressure is significant, so that at the height H

⁸<http://wonka.physics.ncsu.edu/pub/VH-1/>; it was also employed in the simulations of supercritical XRPs by T. Kawashima et al. (2016).

above the NS surface the calculated plasma velocity v almost coincides with the free-fall velocity. Although our model allows two-dimensional hydrodynamical calculations, in this work we neglect variations of all quantities in the hydrodynamic equations along the radius of the cylinder, because we have found in [Paper I](#) that the profiles of these quantities do not significantly vary along the radial coordinate. In the circle geometry there is only one cylindrical boundary with a radius R_c . In the ring geometry, there are external and internal cylindrical boundaries, the latter having the radius $R_c - d_w$, where d_w is the width of the cylindrical wall. Since the plasma moves along $\mathbf{B}$, the hydrodynamical problem is reduced to one dimension.

3.1.2 Radiative substep

At the second substep the system (54) of polarized radiation transfer equations is solved using the Monte Carlo method and the terms Q_{rad} (52) and $\mathbf{F}_{\text{rad}}$ (53) are computed. A detailed description of our Monte Carlo code can be found in A. A. Mushtukov et al. (2022). We summarize the main elements here.

In the simulations, individual seed energy packets ('photons') are injected into the accretion channel at its bottom, which is associated with a hot polar cap at the NS surface (details of their generation are described in Section 3.2). Each photon packet moves in the accretion channel and scatters on the electrons. Density and velocity profiles in the channel are taken from the previous substep. The probability of a scattering event is computed according to the cross-sections (47), which are interpolated from precalculated tables.

If a scattering has occurred, the probability that the photon goes into a certain direction is found from an interpolated cumulative distribution function $f_{i \rightarrow f}$ of the probability that a photon with energy E_i , polarization j_i , initial angle to the magnetic field θ_i and initial azimuthal angle φ_i scatters to the angles θ_f , φ_f and polarization j_f , the energy of the scattered photon being fixed to $E_f = \hat{E}_f(p_{\text{bulk}})$ according to equations (42) and (45).

The key ingredients of our Monte Carlo procedure are precalculated tables of scattering cross-sections $\sigma_{j'j}$ (equation 48), converted into the distribution functions $f_{i \rightarrow f}$, which depend on ρ , T , and E_B as parameters. We treat these parameters as constants during each simulation, with values typical for accretion channels of subcritical XRPCs. The tables are calculated for $p_{\text{bulk}} = 0$ (i.e. in the plasma reference frame) and used in the simulations together with the Lorentz transformation, analogous to equation (39). We work with the tables of $\sigma_{j'j}$ and $f_{i \rightarrow f}$ for all four combinations of the photon polarizations (j, j') before and after scattering. The probability of scattering from mode j into mode j' equals $\sigma_{j'j}/\sigma_j$.

As can be seen from equations (30) and (34)–(38), the scattering amplitude does not depend on the initial and final azimuthal angles separately, but only on their difference $\Delta\varphi = \varphi_f - \varphi_i$. Moreover, using equations (33)–(39) and (46), we can write the differential cross-section in the general form

$$\frac{d\sigma_{j'j}}{d\Omega_f} = \sigma_{d0} + \sigma_{d1} \cos(\Delta\varphi) + \sigma_{d2} \cos(2\Delta\varphi), \quad (56)$$

where the coefficients σ_{d0} , σ_{d1} , and σ_{d2} depend on $j_{i,f}$, E_i , and $\theta_{i,f}$, but not on $\varphi_{i,f}$. Therefore, the integral cross-section $\sigma_{j'j}$ (48) is independent of φ_i and the corresponding tables are two-dimensional: one entry of each table is the initial angle θ_i and the other is the initial photon energy E_i . The full range of angles $\theta_i \in [0, \pi]$ is divided in 100 equal steps. The range of initial

energies E_i is divided differently in three intervals with different arrays of E_i entries, adjusted empirically so as to reproduce with a sufficient accuracy the cross-sections near the resonance and at both its far wings. In the benchmark case of $k_B T = 5$ keV, we set the intervals $[0.01E_B, 0.5E_B]$, $[0.5E_B, 1.5E_B]$, and $[1.5E_B, 3E_B]$ with equidistant 250 table entries in $\log E_i$ in the first and last of these intervals, and with 500 entries in the middle, distributed with a condensation towards E_B . For lower and higher T , we use analogous arrays of table entries with smaller and larger widths of the central interval, respectively.

The cumulative distribution function of photon scattering to directions defined by inequalities $0 \leq \theta_f \leq \theta$ (with $0 \leq \theta \leq \pi$; regardless of φ_f) can be written using equation (56) as

$$f_{j'j}(E_i, \theta_i, \theta) = \frac{2\pi}{\sigma_{j'j}} \int_0^\theta \sigma_{d0,j'j}(E_i, \theta_i, \theta_f) \sin \theta_f d\theta_f. \quad (57)$$

At fixed polar angles $\theta_{i,f}$, the cumulative distribution function of photon azimuthal deflection by angle $\Delta\varphi \in [0, \varphi]$ (where $0 \leq \varphi < 2\pi$) can be obtained from equation (56) by integration over $\Delta\varphi$:

$$\tilde{f}_{j'j}(E_i, \theta_i, \theta_f, \varphi) = \frac{\sigma_{d0} \varphi + \sigma_{d1} \sin \varphi + \sigma_{d2} \sin(2\varphi)/2}{2\pi \sigma_{d0}}. \quad (58)$$

Accordingly, we determine the photon direction after scattering in two steps: first we pick the final polar angle θ_f from the cumulative distribution (57) and next we take $\Delta\varphi$ from the cumulative distribution (58). Since the φ -dependence is analytical, we can use tables of σ_{d0} , σ_{d1} , σ_{d2} , and f with entries only in E_i , θ_i , and θ_f (for each of the four combinations of j_i and j_f). The arrays of entries for E_i and θ_i are the same as above, and the array for θ_f is the same as for θ_i .

In [Paper I](#), we found that in the case of the circle geometry a typical scale of radiation penetration into the accretion channel is about 1–2 km for subcritical accretion regimes. This is ~ 10 –20 per cent of R_{NS} , therefore the variation of the magnetic fields can be about a few tens of percent at this scale. Such variation could affect the position and width of the cyclotron line, as noted by O. Nishimura (2008) for supercritical pulsars. However, in our case of subcritical pulsars, resonance photons have much smaller penetration depths in the large-gradient plasma velocity zone near the NS pole (see Section 4.1). Besides, in the case of the ring geometry, as we will see in Section 4, the total height of the radiation-penetration zone in the channel is only a few tens of metres. Therefore, the approximation of a constant E_B should not introduce large errors in our case. On the other hand, ρ and T , if determined self-consistently, are not constant in the accretion channel, but we apply this approximation for $f_{i \rightarrow f}$ to improve the numerical efficiency of the code.

3.1.3 Energy–momentum exchange

In Section 3.1.1, we neglected the integral terms Q_{rad} and $\mathbf{F}_{\text{rad}}$ in equations (50) and (51). Their contributions are restored at the third substep by solving the system of equations

$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial}{\partial t} \left(\rho \epsilon + \frac{\rho v^2}{2} \right) = Q_{\text{rad}}, \quad \frac{\partial \rho \mathbf{v}}{\partial t} = \mathbf{F}_{\text{rad}}, \quad (59)$$

which is equivalent to neglecting contributions of gravity and spatial inhomogeneities in equations (49)–(51). We solve the system of equations (59) using the explicit Euler method, which is appropriate in the case of subcritical pulsars (in the case of

supercritical pulsars it would require too short time-steps to be practical).

Instead of the integration in equations (52) and (53), we calculate the contributions for the energy and momentum exchange in the course of Monte Carlo simulations at the preceding radiation substep as follows. A simulated monochromatic packet of photons with $\hbar\omega = E$ carries a total energy E_p . If the photon energy changes in a scattering from E to E' , the energy of the packet changes to $E'_p = E_p E'/E$, where E' is calculated according to equation (42), assuming that $p_i = p_{\text{bulk}}$ (without thermal dispersion). The energy and momentum transfer from radiation to matter in a scattering event are, respectively, $\Delta E_p = E_p - E'_p$ and $\Delta p_p = (E_p \cos \theta_i - E'_p \cos \theta_f)/c$. Here, we have taken into account that only the longitudinal (along $\mathbf{B}$) momentum component is transferred to the plasma in a strongly quantizing magnetic field. At each time-step, ΔE_p and Δp_p are summed up as cumulative totals in arrays for the respective spatial grid cells, thus providing the total energy and momentum transferred from radiation to matter in each cell. Then Q_{rad} and $\mathbf{F}_{\text{rad}}$ are obtained by dividing these totals by the time-step and the cell volume.

3.2 Boundary conditions

We set the plasma inflow velocity at the top of the accretion channel v_{in} equal to the free-fall velocity $v_{\text{ff}} = \sqrt{2GM_{\text{NS}}/(R_{\text{NS}} + H)} \approx \sqrt{2GM_{\text{NS}}/R_{\text{NS}}}$. The plasma density at the top boundary equals $\rho = \dot{M}/v_{\text{in}}S_b$, where S_b is the channel cross-section ($S_b = \pi R_c^2$ in the circle geometry and $S_b = 2\pi R_c d_w - \pi d_w^2$ in the ring geometry). The inflow plasma temperature T_{in} is taken to be effectively zero (for numerical reasons we set $k_B T_{\text{in}} = 10^{-9}$ keV). Radiation can escape freely through the top and side boundaries of the channel.

The boundary conditions at the bottom of the channel are not so obvious. In our model, the bottom is placed at the NS surface. Therefore, the proper bottom boundary conditions are determined by the physics of the interaction of the accreting plasma and radiation with the NS surface or the atmosphere, which is a complex, not completely resolved problem. The details of plasma deceleration in the upper layers of a strongly magnetized NS and the role of collective phenomena in this process are not fully understood. The poorly known plasma behaviour in a strong magnetic field at the base of the accretion channel affects the energy fraction radiated into the channel through this surface and the characteristics of this radiation.

For example, collective modes in a two-temperature plasma can produce a shock wave in the accretion channel (G. S. Bisnovatyi-Kogan & A. M. Fridman 1969; S. L. Shapiro & E. E. Salpeter 1975; S. H. Langer & S. Rappaport 1982; A. M. Bykov & A. M. Krassilchchikov 2004). Besides, at sufficiently high accretion rates, thermal pressure can exceed the magnetic pressure in the lower part of the accretion column of a supercritical XRP, which results in a leakage of the plasma through the cylinder walls (M. M. Basko & R. A. Sunyaev 1976; P. Abolmasov & G. Lipunova 2023). We will not consider such situations, which are unlikely to occur in the subcritical XRPs.

We apply the zero-gradient conditions for hydrodynamical variables $\mathbf{v}$, ρ , and P at the bottom of the accretion channel and assume that the total (kinetic and radiation) energy passed through its lower boundary into the NS at a given time-step is fully re-emitted from the NS surface into the channel at the subsequent time-step.

The next non-trivial problem is setting up the spectrum, angular distribution and polarization of the radiation coming from the NS surface into the accretion channel. The seed energy packets are emitted with equal energies E_p . We assume that each packet is monochromatic and consists of E_p/E photons. We use the algorithm described in section 4.2.2 of U. M. Noebauer & S. A. Sim (2019) to generate the blackbody spectrum of the seed radiation at the bottom of the accretion channel. The effective temperature of this spectrum, T_b is determined by the energy balance condition: the energy emitted from the surface is equal to the sum of the kinetic energy of plasma at the surface and the energy of radiation that is carried into the surface by back-scattered photons.

In Paper I, we considered only non-polarized radiation with blackbody spectrum. In this work, to test possible effects of polarization of the surface radiation in a strong magnetic field, this model is supplemented by consideration of two limiting cases, where the surface radiation is generated entirely in one of the two NMs.

In addition, we also consider the possibility of using an arbitrary seed spectrum. For this purpose, we calculate in advance a table of a cumulative distribution function $F_{\text{ph}}(E)$, which gives the probability that a seed photon has an energy below a certain value E . The energy of a seed photon is generated according to this distribution as $E = F_{\text{ph}}^{-1}(\eta)$, where η is a random number generated with a uniform distribution in the interval (0,1) and F_{ph}^{-1} is the inverse function. The power of the seed surface radiation is determined by the same energy balance condition as described above for the blackbody model.

In this way, in addition to the initial blackbody radiation spectrum, we test photon distribution functions $F_{\text{ph}}(E)$ corresponding to the three best-fitting spectral models for the observed spectra of the subcritical XRP 1A 0535+262, presented by S. S. Tsygankov et al. (2019a) for three different accretion states of this XRP with luminosities $L_1 = 7 \times 10^{34}$ erg s $^{-1}$, $L_2 = 6 \times 10^{35}$ erg s $^{-1}$, and $L_3 = 2 \times 10^{36}$ erg s $^{-1}$, with the cumulative photon distribution functions $F_1(E)$, $F_2(E)$, and $F_3(E)$, respectively.

Obviously, using observed spectra as initial ones is not self-consistent, so we consider it only as a test of possible alternatives to using the blackbody initial spectrum. In particular, this test model includes cyclotron features (which are present in the spectra of XRP 1A 0535+262) into the seed photon distribution.

3.3 Remarks on mode conversion

As noticed by G. G. Pavlov & Y. A. Shibano (1979), a linear transformation of NMs can occur if NM photons propagate in an inhomogeneous medium. As noted in Section 2.1.3, the plus-mode is ordinary, if the polarization parameter β is positive, and it is extraordinary, if $\beta < 0$, and vice versa for the minus-mode. According to equation (20), the sign of β is the same as the sign of $\cos \theta_B$ at $E > E_B$, whereas at $E < E_B$ it is so only if the NM polarization is dominated by the plasma, but the sign is opposite if it is dominated by the QED vacuum.

Since the vacuum resonance energy E_{vac} decreases with decreasing density, a photon with a given energy $E < E_{\text{vac}}$ can encounter the vacuum resonance at a density ρ_{vac} where $E \approx E_{\text{vac}}$. If the density variation is sufficiently gentle, the polarization of a photon evolves adiabatically, preserving the electric vector rotation direction (plus-mode or minus-mode), but changing the direction of the major axis of the rotation ellipse relative to the magnetic field, which means a conversion between the O-mode

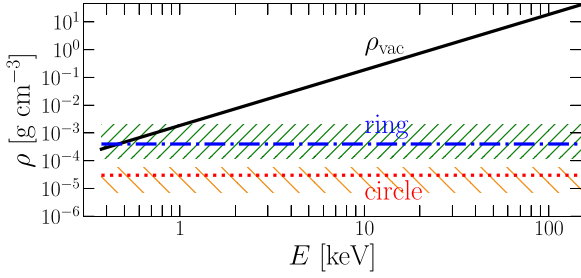


Figure 3. Dependence of the mass density ρ_{vac} , corresponding to the vacuum resonance, on photon energy (solid line) compared with the characteristic plasma densities considered in this work. The hatched bands cover the densities encountered in our numerical calculations in the cases of circle (lower band) and ring (upper band) geometries. The horizontal lines mark the densities $\rho = 4 \times 10^{-4}$ and $3 \times 10^{-5} \text{ g cm}^{-3}$, at which the cross-sections (Fig. 2) have been calculated.

and X-mode (D. Lai & W. C. G. Ho 2002). According to equation (12), $E = E_{\text{vac}}$ at $\rho_{\text{vac}} \approx 10^{-4} B_{12}^2 E_1^2 \text{ g cm}^{-3}$.

Although such densities are attained in our simulations at the lowest considered energies, as illustrated in Fig. 3, we neglect the mode conversion effect, because the accretion channel is almost transparent for radiation at the relevant photon energies. Let us demonstrate it with simple estimates.

In all simulations presented below, we take $E_B = 50 \text{ keV}$, which corresponds to $B = 4.319 \times 10^{12} \text{ G}$. Mass density can be calculated as $\rho = \dot{M}/(|v|S_b)$. The typical subcritical accretion rates considered in our work are $\dot{M} = (10^{15}\text{--}10^{16}) \text{ g s}^{-1}$. As we will see in Section 4, the typical plasma velocity in accretion channels of subcritical pulsars is about $|v| = 0.2\text{--}0.5c$, so we will use $|v| = 0.5c$ for estimates. A typical radius of an accretion channel is $R_c \sim 0.5 \text{ km}$ (see e.g. estimates in A. A. Mushtukov et al. 2015b, and references therein) and its thickness is $d_w \sim 10 \text{ m}$ in the case of ring geometry. Using these estimates, we obtain $\rho \lesssim 3 \times 10^{-4} \text{ g cm}^{-3}$ for the circle geometry and $\rho \lesssim 10^{-2} \text{ g cm}^{-3}$ for the ring geometry. In both cases, $E_{\text{vac}} \ll E_B$, therefore around the cyclotron line the NM polarization is dominated by vacuum.

Let us define an effective optical depth $\tau_{\text{eff}} = -\ln(N_{\text{ns}}/N_{\text{tot}})$, where N_{tot} is the total number of photons with energy E , injected in the accretion channel, and N_{ns} is the number of injected photons with the same energy which leave the accretion channel without scattering. Thus, τ_{eff} is the characteristic of the scattering region. Fig. 4 shows an example of $\tau_{\text{eff}}(E)$ from our calculations (discussed in more detail in the next section) at $\dot{M} = 2.5 \times 10^{15} \text{ g s}^{-1}$, $k_B T = 5 \text{ keV}$, and $E_B = 50 \text{ keV}$, for the ring geometry with $R_c = 0.5 \text{ km}$ and $d_w = 15 \text{ m}$. The height of the computational domain is $H = 40 \text{ m}$; the underlying tables for interpolation of cross-sections and photon redistribution functions have been calculated with account of the vacuum polarization, assuming $\rho = 4 \times 10^{-4} \text{ g cm}^{-3}$ (the vacuum polarization parameter $W \simeq 10^4$). Note that the maximum of $\tau_{\text{eff}}(E)$ is redshifted from E_B . This redshift is caused by the Doppler effect due to the bulk plasma velocity, discussed in more detail below. It can be seen from Fig. 4 that $\tau_{\text{eff}} \ll 1$, if $E \ll E_B$. Therefore, at $E \sim E_{\text{vac}} \simeq E_B/\sqrt{W}$ most of the photons pass through the accretion channel without scattering. Under this condition, the mode conversion can only change the sign of the polarization but not its magnitude. Moreover, it is clear that for an unpolarized initial radiation (which will be considered in all but one examples presented below) the mode conversion has no effect under the considered conditions.

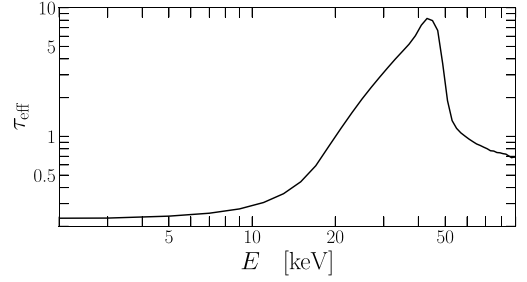


Figure 4. Effective optical depth, averaged over photon directions $0 < \theta_B < \theta_{\text{max}}$ as a function of photon energy for the ring-shaped base of the accretion channel with radius $R_c = 0.5 \text{ km}$ and thickness $d_w = 15 \text{ m}$ at the mass accretion rate $\dot{M} = 2.5 \times 10^{15} \text{ g s}^{-1}$, cyclotron energy $E_B = 50 \text{ keV}$, and electron temperature $k_B T = 5 \text{ keV}$. The scattering cross-sections have been computed at mass density $\rho = 4 \times 10^{-4} \text{ g cm}^{-3}$ and electron temperature $k_B T = 5 \text{ keV}$.

4 NUMERICAL RESULTS

In this section, we present results of numerical simulations. In all the simulations, we have adopted the surface gravity $g = 1.29 \times 10^{14} \text{ cm s}^{-2}$, which corresponds to $M_{\text{NS}} = 1.4 M_{\odot}$ and $R_{\text{NS}} = 12 \text{ km}$ in the non-GR approach used here (or to $M_{\text{NS}} = 1.4 M_{\odot}$ and $R_{\text{NS}} = 13.2 \text{ km}$ with the GR corrections). We will present steady-state results, to which the time-dependent solutions converge at the end of each simulation. In fact, our model does not allow us to trace the time evolution accurately, since we use the stationary radiative transfer equations (54) in order to accelerate the calculations. We have checked that our stationary solutions remain stable in the non-stationary calculations with the term $c^{-1} \partial I_E^j / \partial t$ added to the left-hand side of equation (54).

4.1 Plasma deceleration

In Figs 5–8, we consider the influence of different factors on plasma deceleration in an accretion channel. The higher the mass accretion rate, the higher the deceleration.

Fig. 5 displays bulk plasma velocity profiles in the circle geometry with the same radius and magnetic field, with and without taking the vacuum polarization effects into account, for accretion rates 10^{15} and $5 \times 10^{15} \text{ g s}^{-1}$. For comparison, the free-fall velocity is also shown. The height of the accretion channel is higher for the circle geometry than for the ring geometry. Accordingly, in this case the computational domain has been increased to $H = 2 \text{ km}$ and the interpolation tables have been calculated at a smaller density $\rho = 3 \times 10^{-5} \text{ g cm}^{-3}$. Although the dipole field B should vary by a factor ~ 1.6 along this height, we neglect this variation because of computational-resource limitations (the resulting inaccuracy is not expected to be significant, as argued in Section 3.1.2). For both accretion rates, the plasma deceleration in the accretion channel is stronger when the vacuum polarization is taken into account, although the difference is not large.

Fig. 6 provides a clearer illustration of the influence of the mass accretion rate on the deceleration of the accreting plasma. Compared with Fig. 5, the filled (circle geometry) accretion channel is replaced by the hollow one (i.e. ring geometry). In this case, it is sufficient to use a relatively small height $H = 35 \text{ m}$. The vacuum polarization effects are included in consideration. The plasma velocity is shown as a function of the height above the NS surface for accretion rates $\dot{M} = 10^{15}$, 2.5×10^{15} , and $5 \times 10^{15} \text{ g s}^{-1}$. The free-fall velocity is also shown as a reference line. We see

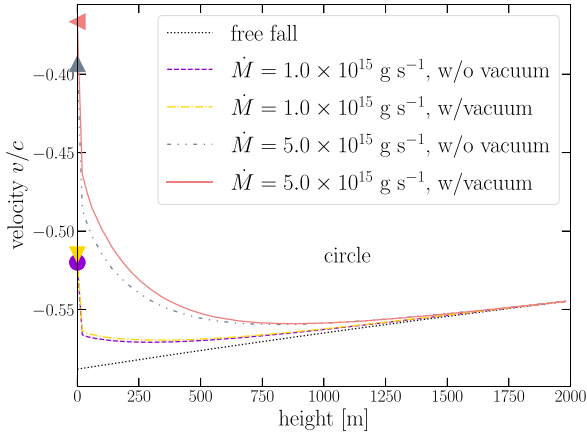


Figure 5. Plasma velocity v in units of c as a function of height above the hot polar cap of a NS emitting the blackbody radiation in the circle geometry with $R_c = 0.5$ km. The negative sign of v means motion towards the surface. The dotted line corresponds to the free-falling plasma; the dashed line corresponds to the accretion rate $\dot{M} = 10^{15} \text{ g s}^{-1}$ and electron plasma birefringence; the dash-dotted line corresponds to the same $\dot{M}$ with vacuum polarization birefringence; the dash-double-dot line and the solid line show the cases without and with the vacuum polarization effects, respectively, for $\dot{M} = 5 \times 10^{15} \text{ g s}^{-1}$. The symbols of respective colours at the left vertical axis visualize the v/c values attained at the NS surface (see Table 1). The boundary condition at the base of the accretion channel is the non-polarized blackbody radiation with the effective temperature T_b obtained from the energy balance condition (see section 3.2). The scattering cross-sections have been computed at $\rho = 3 \times 10^{-5} \text{ g cm}^{-3}$ and $k_B T = 5 \text{ keV}$.

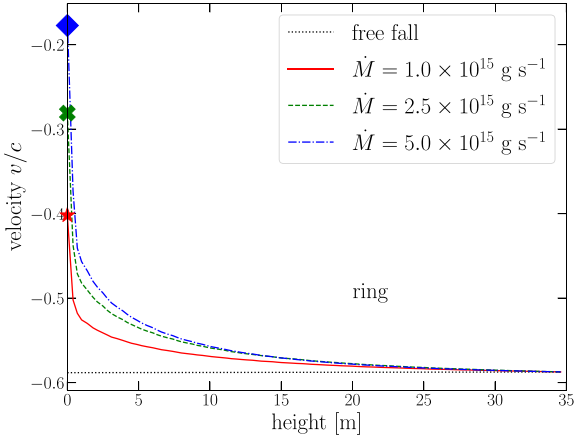


Figure 6. Same as in Fig. 5 but for the ring geometry with the same model setup as in Fig. 4. The solid line corresponds to $\dot{M} = 10^{15} \text{ g s}^{-1}$, the dashed one to $\dot{M} = 2.5 \times 10^{15} \text{ g s}^{-1}$, the dash-dotted line to $\dot{M} = 5 \times 10^{15} \text{ g s}^{-1}$, and the dotted one corresponds to the free fall. Vacuum polarization is taken into account.

that the increase of $\dot{M}$ leads to a noticeable enhancement of the radiative braking of the plasma.

Fig. 7 illustrates an influence of the radiative lower boundary conditions on the plasma deceleration. For this purpose, along with the blackbody radiation from the NS surface, we also consider the models F_1 , F_2 , and F_3 that fit the spectrum of XRP 1A 0535+262 in three different states (S. S. Tsygankov et al. 2019a), mentioned in Section 3.2. For comparison, the free-fall plasma velocity is also shown. In this set of simulations, the radius R_c

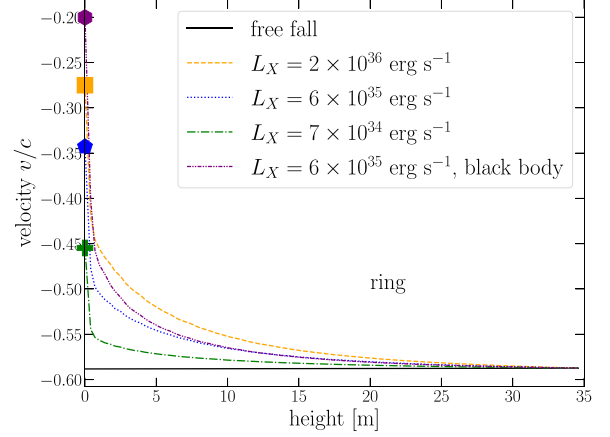


Figure 7. Same as in Fig. 6 but for complex (non-blackbody) spectra of radiation from the NS surface, taken to be the fits by S. S. Tsygankov et al. (2019a) to the observed spectra of XRP 1A 0535+262 for the states with X-ray luminosities $L_1 = 7 \times 10^{34} \text{ erg s}^{-1}$, $L_2 = 6 \times 10^{35} \text{ erg s}^{-1}$, and $L_3 = 2 \times 10^{36} \text{ erg s}^{-1}$, with R_c and d_w determined from equations (60) and (61). The dash-dotted, dotted and dashed lines correspond to the simulated spectra from the accretion channel for these three cases, respectively. For comparison, the dash-double-dot line corresponds to $L_X = 6 \times 10^{35} \text{ erg s}^{-1}$ with a blackbody spectrum from the base of the channel and the black solid line corresponds to the free fall.

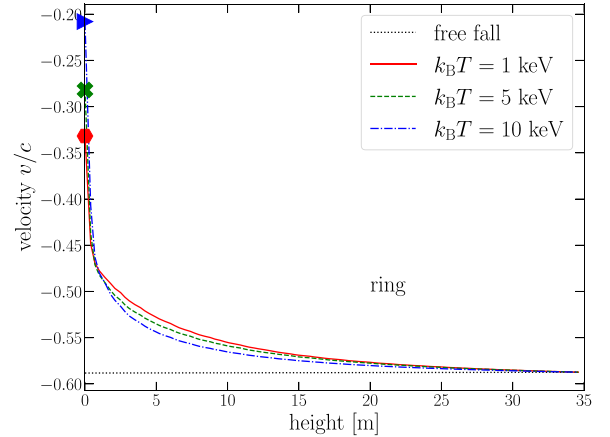


Figure 8. Same as in Fig. 6 at $\dot{M} = 2.5 \times 10^{15} \text{ g s}^{-1}$ but for different electron temperatures T used in the cross-section interpolation tables: $k_B T = 1 \text{ keV}$ (solid line), $k_B T = 5 \text{ keV}$ (dashed line), and $k_B T = 10 \text{ keV}$ (dash-dotted line).

and thickness d_w of the accretion channel have been determined self-consistently from the relations for disc accretion with a low rate (V. F. Suleimanov, G. V. Lipunova & N. I. Shakura 2007):

$$S_b = 6.6 \times 10^9 \Lambda^{-7/8} L_{39}^{2/5} B_{12}^{-1/2} \left(\frac{M}{M_\odot} \right)^{-13/20} R_6^{19/10} \text{ cm}^2, \quad (60)$$

$$2\pi \frac{R_c}{d_w} = 95 \Lambda^{-1/8} L_{39}^{-4/35} B_{12}^{-1/14} \left(\frac{M}{M_\odot} \right)^{71/140} R_6^{19/10}, \quad (61)$$

where $L_{39} = L_X / (10^{39} \text{ erg s}^{-1})$, $R_6 = R_{NS} / (10^6 \text{ cm})$, and Λ is a dimensionless correction parameter, which we set equal to 0.5, as appropriate for the ring geometry. In the calculations, we have set $E_B = 50 \text{ keV}$ and obtained the ring radius $R_c = 349 \text{ m}$ and width $d_w = 7 \text{ m}$ for the case F_1 ; $R_c = 474 \text{ m}$ and $d_w = 12 \text{ m}$ for F_2 ; $R_c = 563 \text{ m}$ and $d_w = 16 \text{ m}$ for F_3 . The height of the computational domain was $H = 35 \text{ m}$.

Table 1. Plasma velocity (in units of speed of light) at the NS surface for the accretion models considered in Figs 5–7: the circle or ring geometry of the accretion channel without (the ‘plasma’ model) or with (the ‘vacuum’ model) account of vacuum polarization effects. For Figs 5 and 6 (lines 2–8), the blackbody emission from the hotspot was assumed. For Fig. 7 (lines 9–12), the ring geometry with vacuum polarization effects is assumed and either a complex model of the initial spectrum at the surface (‘model boundary’, see the text; lines 9–11) or the blackbody initial spectrum (line 12) is used. The free-fall velocity is given in the first line for comparison.

Fig. no.	Model	$\dot{M}$ (10^{15} g s $^{-1}$)	L_X (10^{35} erg s $^{-1}$)	v/c
5–7	Free fall	Any	Any	–0.587
5	Circle, plasma	1.0	1.55	–0.520
5	Circle, vacuum	1.0	1.55	–0.515
5	Circle, plasma	5.0	7.74	–0.393
5	Circle, vacuum	5.0	7.74	–0.366
6	Ring, vacuum	1.0	1.55	–0.402
6	Ring, vacuum	2.5	3.87	–0.282
6	Ring, vacuum	5.0	7.74	–0.172
7	Model boundary	0.452	0.7	–0.453
7	Model boundary	3.88	6.0	–0.341
7	Model boundary	12.9	20	–0.271
7	Blackbody	3.88	6.0	–0.197

The strong differences in the deceleration curves are primarily caused by the different number of photons with energies close to the cyclotron resonance. These photons mainly control the deceleration of the accretion flow due to the resonant peak in the cross-sections. For a fixed luminosity, the presence of an absorption line in the seed spectrum causes a lack of resonance photons and suppresses the braking, as seen from a comparison of the curves in Fig. 7, obtained for $L = L_2$ with the blackbody and non-blackbody boundary conditions.

Note the large difference of height scales in Figs 5 and 7. It corresponds to much higher density in the ring-shaped geometry than in the circle geometry at similar mass accretion rates. In the circle geometry, the typical height values, at which the deceleration mostly occurs, are $z \sim R_c$ while in the ring geometry they are typically $z \sim d_w$, both values corresponding to approximately the same optical depth (cf. A. A. Mushtukov et al. 2015a).

The final values of the velocities at the NS surface for the model settings used in Figs 5–7 are listed in Table 1. We have verified the robustness of these values with respect to numerical code parameters (such as the mesh size). In these simulations, we used the tables of the scattering cross-sections calculated at typical densities $\rho = 3 \times 10^{-5}$ g cm $^{-3}$ for the circle geometry or $\rho = 4 \times 10^{-4}$ g cm $^{-3}$ for the ring geometry and at fixed electron temperature $k_B T = 5$ keV. Fig. 8 demonstrates the influence of the value of the latter parameter on the velocity profiles at fixed $\dot{M} = 2.5 \times 10^{15}$ g s $^{-1}$ in the ring geometry. Although this dependence is not very strong (when we vary $k_B T$ between 1 and 10 keV, the velocity at the NS surface varies within tens of per cent) we can see a stronger deceleration of the accretion flow near the NS surface at higher temperatures. It is explained by a larger Doppler broadening of the cyclotron resonance and accordingly a wider involvement of photons in the resonant scattering.

At high accretion rates (near the transition to the supercritical regime), we observe oscillations of the plasma density and velocity in the accretion channel, which could not be damped away in our calculation. These oscillations can be a sign of the incipient radiative-dominated shock, although we cannot exclude the possibility that they are an artefact due to the limited computational

time. In the figures, we have averaged plasma characteristics over the computational time in the cases where the oscillations are observed (the dash–dotted line in Fig. 6 and the dash–double-dot line in Fig. 7).

The increase of the deceleration with increasing temperature affects the cyclotron feature and polarization, which will be described in Sections 4.2–4.4. Therefore, the variation of T in the calculations of interpolation tables affect quantitatively the results of our calculations. At contrast, we have checked that a moderate variation of the plasma density ρ in these calculations does not noticeably affect the results, which justifies using a constant ρ in the tables.

4.2 Pulsar spectra

Let us consider how different factors affect the radiation from an accretion channel. First we consider the energy spectra for the sum of the two NMs.

Fig. 9 shows examples of calculated spectral flux per unit solid angle (equal to the specific intensity integrated over a surface perpendicular to $\mathbf{k}$), averaged over a small range $\delta\vartheta$ (we have set $\delta\vartheta = \pi/50$) around several chosen angles ϑ between $\mathbf{k}$ and $\mathbf{n}$ for outgoing photons:

$$J = \frac{1}{\delta\vartheta} \int_{\vartheta-\delta\vartheta/2}^{\vartheta+\delta\vartheta/2} \frac{dQ_E(\vartheta')}{2\pi \sin \vartheta' d\vartheta' dE dt} d\vartheta', \quad (62)$$

as functions of the photon energy E . Here, $dQ_E(\vartheta)$ is the energy of radiation emitted by the accretion channel into the interval at angles $\vartheta \pm d\vartheta/2$ and photon energies $E \pm dE/2$ during time interval dt . The interpolation tables have been calculated at $k_B T = 5$ keV and the typical densities $\rho = 3 \times 10^{-5}$ g cm $^{-3}$ for the circle geometry and 4×10^{-4} g cm $^{-3}$ for the ring geometry. To evaluate the spectral intensities and energy fluxes as functions of photon energy, the energy range from 0 to 200 keV has been divided into 100 equal intervals, and the wave packets, which were traced in the Monte Carlo simulations, have been counted separately in each interval to give the flux estimate at its centre.

The left panel (a) of Fig. 9 presents the spectra calculated in the simplified approximation (17), which does not take into account plasma collisions and vacuum polarization, for the circle geometry, as in Paper I. The middle (b) and right (c) panels show the results of calculations with using the more accurate expression (18) for the circle and ring geometry, respectively. The increase of numerical noise towards high energies, which is observed in this and subsequent figures, is related to the sharp decrease of the Wien tail of the spectrum, which results in an insufficient number of wave packets in the Monte Carlo simulations at such energies.

It can be seen that the spectra in Fig. 9 become progressively redshifted with decreasing ϑ . In panels (b) and (c), the absorption feature is also shifted to $E < E_B$. Qualitatively, this can be explained by the Doppler effect at the resonant scattering (in the electron’s rest frame) of softer photons ($E < E_B$) propagating in the direction opposite to the flow movement ($\vartheta \leq \pi/2$), and harder photons ($E > E_B$) propagating in the direction of the flow movement ($\vartheta > \pi/2$). The scattered photons predominantly leave their initial spectral distribution (in the NS frame), forming visible absorption features. Consequences of this effect for the beaming pattern are discussed in I. D. Markozov et al. (2026). When taking into account the vacuum polarization, the cyclotron

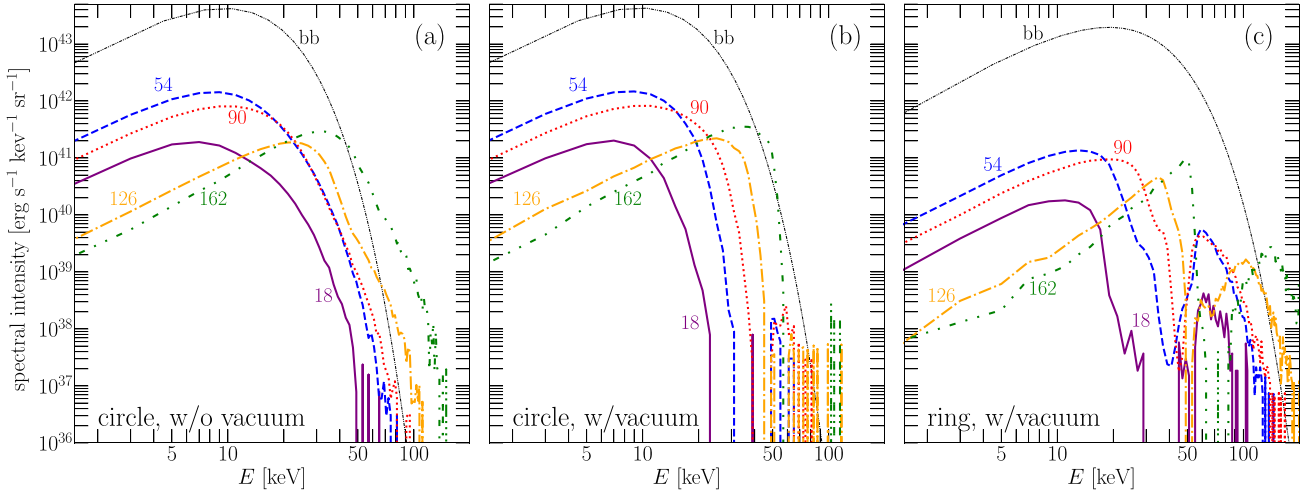


Figure 9. Integrated intensity (i.e. spectral flux per unit solid angle) of radiation outgoing from an accretion channel at different angles ϑ to the channel axis as a function of photon energy E for the mass accretion rate $\dot{M} = 5 \times 10^{15} \text{ g s}^{-1}$ at the electron cyclotron energy $E_B = 50 \text{ keV}$ for three models: (a) the circle geometry and the collisionless plasma approximation neglecting vacuum polarization, (b) the circle geometry and the more accurate description of polarization according to equation (18), including the vacuum polarization effect, and (c) the ring geometry with using the same description of the vacuum polarization effects, as in panel (b). The values of ϑ are marked near the curves: $\vartheta = 18^\circ$ (solid line), 54° (dashed line), 90° (dotted line), 126° (dot-dashed line), and 162° (double-dot-dashed line). The thin double-dot-dashed curve marked ‘bb’ represents the blackbody integrated intensity emitted from the hotspot at the effective surface temperature $k_B T_b = 3.4 \text{ keV}$ and 7.1 keV in the cases of circle (a, b) and ring (c) geometries, respectively. The model parameters are the same as in Fig. 5 for the circle geometry (panels a, b) and the same as in Fig. 6 for the ring geometry (panel c).

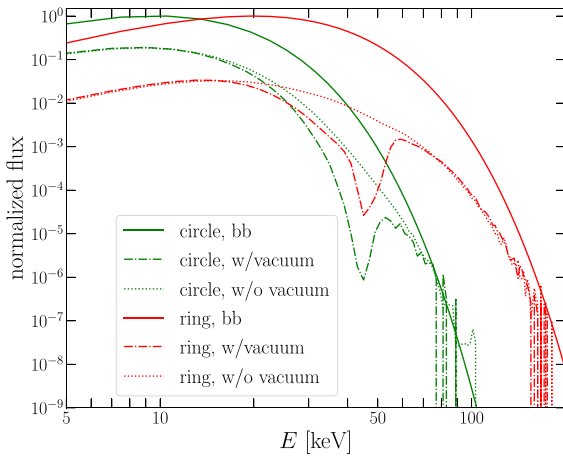


Figure 10. Radiative spectral flux into the upper hemisphere ($\vartheta \leq 90^\circ$) as a function of photon energy for the circle (green curves with maxima at lower E) and ring (red curves with maxima at higher E) geometries of the accretion channel with the same parameters as in Fig. 9, including the accretion rate $\dot{M} = 5 \times 10^{15} \text{ g s}^{-1}$. The cases with (dot-dashed lines) and without (dotted lines) account of the vacuum polarization are shown. For each channel geometry, the curves are normalized to the maximum of the initial blackbody spectral flux from the surface, shown by the solid line, with effective temperatures $k_B T_b = 3.4 \text{ keV}$ in the case of circle geometry and 7.1 keV in the case of ring geometry.

feature becomes more pronounced,⁹ although the specific form of the features depends heavily on the chosen parameters.

Fig. 10 highlights the difference between spectra for the circle and ring geometries. It shows the spectral flux into the upper

hemisphere ($\vartheta < 90^\circ$). The same numerical parameters and interpolation tables as in Fig. 9 have been used. The spectra obtained with taking the vacuum polarization into account are compared with the results obtained in the approximations of Paper I, neglecting vacuum polarization. In the latter case, however, the hydrodynamic structure (the density and velocity profiles) have been fixed the same as for the case where the vacuum polarization is taken into account, in order to highlight specifically the radiation transfer effects. We see that the vacuum polarization leads to the appearance of a strong cyclotron feature. This feature is located at a slightly higher energy for the ring geometry than for the circle geometry. In general, its position and shape depend heavily on parameters of the seed radiation, as well as on the mass accretion rate, as will be discussed in Section 4.3.

For each geometry, the spectral flux in Fig. 10 is normalized to the maximum of the flux emitted into the channel from the bottom, which is also shown in the figure. One can see that the integrated flux injected from the hotspot surface is much higher than the flux emitted into the upper hemisphere from the accretion channel. This is due to the energy losses to radiative deceleration of the plasma inside the channel as well as due to the energy carried back to the bottom of the channel by photons scattered into the lower hemisphere.

4.3 Dependence of cyclotron feature on luminosity

The position of the cyclotron absorption feature in the spectrum of some XRPs varies with X-ray luminosity (see R. Staubert et al. 2019; A. Mushtukov & S. Tsygankov 2024, and references therein). In supercritical XRPs (including V0332+53, see S. S. Tsygankov et al. 2006, and 4U 0115+63, see S. S. Tsygankov et al. 2007), where the luminosity $L_X \gtrsim 10^{37} \text{ erg s}^{-1}$, a cyclotron line centroid energy shows a negative correlation with L_X . In some subcritical XRPs (including Her X-1, see R. Staubert et al. 2007, and GX 304-1, see D. Klochov et al. 2012; R. E. Rothschild et al.

⁹The opposite result in Paper I was caused by an accidental numerical error.

2017), where $L_X \lesssim 10^{37} \text{ erg s}^{-1}$, a positive correlation between the line centroid energy and luminosity was reported. Recently, it was found that the negative correlation arises again at luminosity level well below the critical one (X. Chen et al. 2021; Q. C. Shui et al. 2024).

To explain the relation between the accretion luminosity and cyclotron line centroid energy, different authors have proposed models accounting for changes in the geometry of a line forming region and physical conditions there (see Section 4.4.3 in A. Mushtukov & S. Tsygankov 2024, and references therein). In the case of subcritical accretion, A. A. Mushtukov et al. (2015c) proposed a mechanism explaining the variation of the observed cyclotron-line energy with luminosity, attributing it to the Doppler effect within the accretion channel of a subcritical pulsar. The authors adopted an approximate plasma velocity profile dependent on the accretion rate and evaluated the magnetized Compton opacity averaged over the channel, together with the resulting centroid energy of the cyclotron feature. Their calculations showed that the cyclotron line centroid exhibits a systematic blueshift with increasing accretion rate.

However, the previous authors did not conduct a full radiation hydrodynamical modelling and used many simplifications. Therefore, it is of interest to verify the explanation discussed above using the present calculations of magnetic radiation transfer coupled to hydrodynamics. The results of the calculations show that the blueshift of the feature for directions opposite to the flow motion ($\vartheta < \pi/2$), at $E < E_B$, can be qualitatively explained by a decrease of the averaged Doppler effect with increasing the deceleration of the accretion flow under the influence of increasing radiation luminosity emerging from the base of the accretion channel.

We check the influence of the Doppler effect on the observed cyclotron energy by applying our code to the model of a complex initial spectrum, corresponding to the subcritical XRP 1A 0535+262 as fitted by S. S. Tsygankov et al. (2019a). Its accretion channel structures in the states F_1 , F_2 , and F_3 with luminosities $L_1 = 7 \times 10^{34} \text{ erg s}^{-1}$, $L_2 = 6 \times 10^{35} \text{ erg s}^{-1}$, and $L_3 = 2 \times 10^{36} \text{ erg s}^{-1}$ have been shown in Fig. 7. Fig. 11 shows the flux into the upper hemisphere as a function of photon energy, obtained by the Monte Carlo radiative transfer simulations for the models F_1 and F_3 . The dot-dashed line corresponds to $L_X = 7 \times 10^{34} \text{ erg s}^{-1}$, while the dashed line corresponds to $L_X = 2 \times 10^{36} \text{ erg s}^{-1}$. Other parameters of the simulations are $T = 5 \text{ keV}$, $\rho = 4 \times 10^{-4} \text{ g cm}^{-3}$, $R_c = 500 \text{ m}$, $d_w = 15 \text{ m}$, and $E_B = 50 \text{ keV}$. The initial model spectra of radiation from the hotspot (the seed spectra) are also shown in the figure. This seed radiation has been assumed unpolarized.

We see that the spectra outgoing from the accretion channel reveal sharp and deep cyclotron lines, unlike the shallow features in the seed spectra. The cyclotron line centroid energy is higher for the larger accretion rate.

Qualitatively similar results were obtained by P. Fotiadis et al. (2026), who performed Monte Carlo unpolarized radiative-transfer simulations of cyclotron-line formation in subcritical XRP in the electron plasma (without QED vacuum) approximation, using prescribed velocity and density profiles in the filled (circle geometry) accretion funnel. Here, we have extended the treatment by coupling the radiative transfer to the hydrodynamic structure of the flow and by including polarized normal-mode transfer together with vacuum-polarization effects. In particular, in Fig. 10 we observe no significant absorption feature in

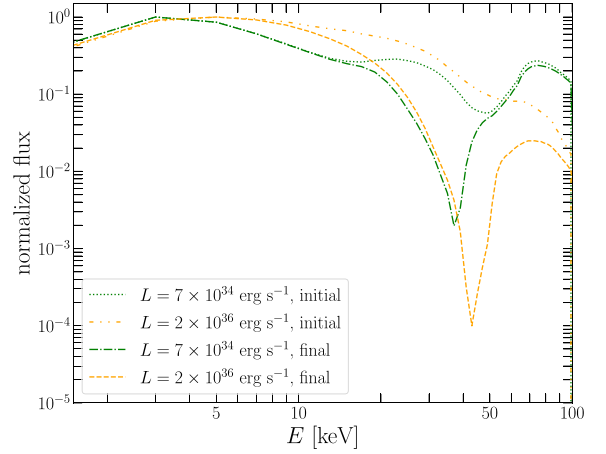


Figure 11. Radiative flux into the upper hemisphere ($\vartheta \leq 90^\circ$) as a function of photon energy for the same model spectra of radiation from the NS surface as in Fig. 7 for the states with the lowest and highest luminosities: $L_X = L_1$ (dot-dashed curve) and $L_X = L_3$ (dashed curve). The initial model spectra from the NS surface are also shown (dotted (L_1) and double-dot-dashed (L_3) curves, respectively). The parameters R_c and d_w are chosen consistently with the accretion luminosity according to equations (60) and (61).

the polarization-averaged outgoing spectrum in the case of the plasma approximation, but there appear strong cyclotron features when vacuum polarization effects are included. It is because only in the latter case both cross-sections for the X- and O-modes display a strong cyclotron resonance (see Fig. 2).

Besides, it should be noted that the cyclotron line position and shape can be also affected by some other factors, which are not currently included in our model, such as variations of temperature and magnetic field over the height above the NS surface (see e.g. O. Nishimura 2008, for a discussion).

4.4 Linear polarization

One of the key features of our model is its ability to provide not only the energy spectrum, but also the polarization by self-consistent radiation hydrodynamical simulation. The degree of linear polarization with respect to the magnetic field direction is

$$P_L = \frac{I_O - I_X}{I_O + I_X} P_0, \quad (63)$$

where I_O is the O-mode intensity, I_X is the X-mode intensity and

$$P_0 = \frac{|K_O|^2 - 1}{|K_O|^2 + 1} = \frac{1 - |K_X|^2}{1 + |K_X|^2} = -P_X \quad (64)$$

is the linear polarization of the O-mode, $K_O = K_2$ and $K_X = K_1$ being the O- and X-mode polarization parameters given by equation (18).

Fig. 12 presents P_L as a function of photon energy E in different model approximations. The simulations were performed for the ring geometry with $R_c = 0.5 \text{ km}$, $d_w = 15 \text{ m}$, and accretion rate $\dot{M} = 2.5 \times 10^{15} \text{ g s}^{-1}$. Interpolation tables were calculated with $\rho = 4 \times 10^{-4} \text{ g cm}^{-3}$ and $k_B T = 5 \text{ keV}$.

First, we compare the results obtained in the approximation for the polarization parameter given by equation (17), which does not take into account polarization of vacuum (double-dot-dashed line), with more accurate results (solid line) obtained with taking the vacuum polarization into account according to equations (18)

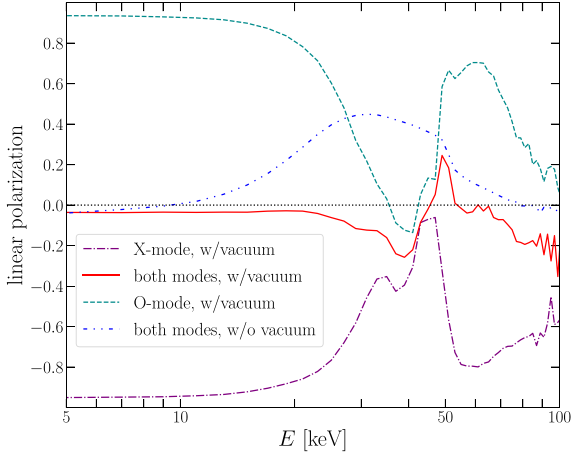


Figure 12. Linear polarization degree P_L as a function of photon energy, averaged for radiation emitted into the upper hemisphere ($\vartheta \leq 90^\circ$) (with weights proportional to the total flux in each direction ϑ) for the ring geometry with $R_c = 500$ m, $d_w = 15$ m, and accretion rate $\dot{M} = 2.5 \times 10^{15}$ g s $^{-1}$. The calculation of P_L including vacuum polarization effects (solid line) is compared with the cold collisionless plasma approximation neglecting the vacuum polarization from the NS surface. The dot-dashed (X-mode) and dashed (O-mode) curves show the results of analogous calculations in the cases of purely elliptical hotspot surface polarization, where the seed photons are injected in the channel in only one of the NMs, with the vacuum polarization effects taken into account.

and (19). These two sets of simulations were performed assuming that the non-polarized blackbody radiation is emitted from the NS surface. Despite the surface radiation has been assumed unpolarized, we observe significant linear polarization P_L of the outgoing radiation at sufficiently high energies, $E \gtrsim 20$ keV. It means that the accretion channel substantially contributes into the polarization at such photon energies even in the subcritical XRPs. We also observe an important effect of the vacuum polarization. In the approximation where the vacuum polarization is neglected, P_L is positive and rather large (reaching ~ 40 per cent) in a wide energy band close to E_B . If the vacuum polarization effects are taken into account, then P_L is also non-zero at $E \sim E_B$, but in a narrower band; in this case it is smaller in magnitude and sign-alternating around E_B due to the strong redistribution of photons between NMs near the resonant energy. At $E \ll E_B$, P_L is negative and moderate (several percent) in both cases with and without the vacuum polarization effects. Note that only the latter (lower) energy range is accessible with the current polarization measurements by IXPE (2–8 keV).

Secondly, we carried out calculations for two extreme cases, when only one of the two NMs is emitted from the hotspot surface (the dashed and dot-dashed lines in Fig. 12). It is clearly seen that at $E \ll E_B$ (in the IXPE band) the degree of linear polarization strongly correlates with that of the seed radiation. This is expected, because the accretion channel of a subcritical pulsar has a small optical thickness at such low photon energies. However, in a band around $E \sim 40$ keV, which corresponds to the cyclotron energy red-shifted due to the plasma motion, P_L is strongly reduced in magnitude. This means that the scattering in the accretion channel strongly mixes the two NMs in this energy band and to lesser extent redistributes the photons between them at other energies.

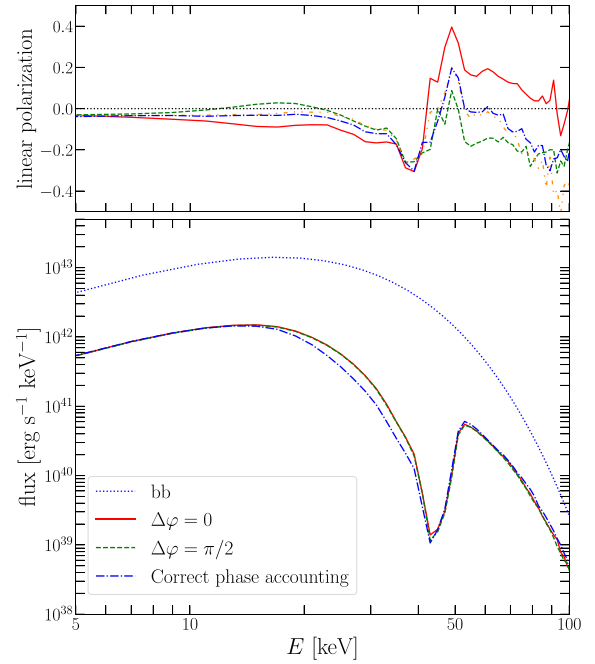


Figure 13. Average linear polarization degree (upper panel) and flux (lower panel) of radiation emitted into the upper hemisphere in the ring geometry of the accretion channel with $R_c = 500$ m and $d_w = 15$ m as a function of photon energy for mass accretion rate $\dot{M} = 2.5 \times 10^{15}$ g s $^{-1}$, calculated using the cross-sections with fixed azimuthal scattering angle $\Delta\varphi = 0$ (solid line), $\Delta\varphi = \pi/2$ (dashed line), $\Delta\varphi = \pi$ (double-dot-dashed line), and with the accurate $\Delta\varphi$ dependence according to equation (56) (dot-dashed line). The dotted line in the lower panel represents the spectrum of the blackbody radiation with $k_B T = 6$ keV, which was used as the hotspot boundary condition; the vacuum polarization effects are included.

To summarize, the results shown in Fig. 12 indicate that an accurate model of the hotspot surface radiation is needed to calculate the properties of the outgoing radiation at $E \ll E_B$. On the other hand, these results demonstrate that the reprocessing of the NMs polarization by the accretion channel not only plays a key role in formation of the polarization of outgoing radiation near the resonance, but also may contribute appreciably to P_L at lower and higher energies.

We should note that most of the previous simulations of XRP radiation, including our Paper I, neglected the azimuthal dependence of the scattering cross-sections. Here, we find that the accurate φ -dependence of the cross-sections according to equation (56) is not negligible. To illustrate it, in Fig. 13 the results of an accurate calculation of the polarization degree and the spectral flux are compared with the results obtained using the scattering cross-sections at fixed $\Delta\varphi = 0, \pi/2$, or π . The spectrum in the lower panel is insensitive to the choices with fixed or variable $\Delta\varphi$. However, for polarization (the upper panel) we see a considerable difference between the results obtained with fixed $\Delta\varphi$ and with using the cumulative distribution function (58).

The upper panel of Fig. 14 presents the degree of linear polarization as a function of photon energy for accretion rates $\dot{M} = 10^{15}$, 2.5×10^{15} , and 5×10^{15} g s $^{-1}$ in the ring geometry, taking into account the vacuum polarization effects. The lower panel of Fig. 14 presents the respective radiation energy fluxes emitted into the upper hemisphere. The unpolarized blackbody radiation was assumed as the boundary condition at the NS surface, with

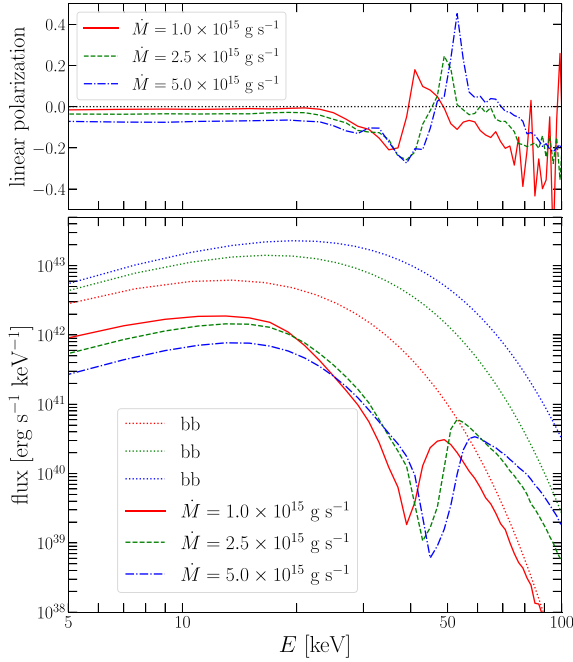


Figure 14. Same as in Fig. 13 for mass accretion rates $\dot{M} = 10^{15} \text{ g s}^{-1}$ (thick solid line), $\dot{M} = 2.5 \times 10^{15} \text{ g s}^{-1}$ (dashed line), and $\dot{M} = 5.0 \times 10^{15} \text{ g s}^{-1}$ (dash-dotted line). The dotted lines in the lower panel represent the spectra of the blackbody radiation with (from bottom to top) $k_B T = 4.6, 6.0$, and 7.1 keV , which were used as the hotspot boundary conditions.

temperatures determined by the energy balance, as described above: $k_B T = 4.6, 6.0$, and 7.1 keV for the lowest, middle, and highest accretion rate, respectively (the corresponding spectra are shown in the lower panel by dotted lines). The underlying interpolation tables have been calculated at $E_B = 50 \text{ keV}$, $\rho = 4 \times 10^{-4} \text{ g cm}^{-3}$, and $k_B T = 5 \text{ keV}$.

As in Fig. 13, the degree of linear polarization is substantial at photon energies around the cyclotron resonance, so we see the same changes of polarization signs near this energy. The change of sign of P_L in the upper panels of Figs 13 and 14 occurs near the energy where the spectral flux in the lower panels has a minimum. The non-monotonic behaviour of the polarization degree around this resonance in the upper panel of Fig. 14 is an artefact related to our use of the tabulated cross-sections to speed-up calculations: we have checked that this non-monotonic structure changes with a change of the grid of energies and angles for the tabulation. However, the average trend of $P_L(E)$ is rather robust, so that its maxima and minima around the cyclotron resonance remain nearly the same within this wavy structure. Since the seed surface radiation is set unpolarized, the observed polarization of the outgoing radiation is produced by the radiative transfer in the accretion channel. The cyclotron feature in the spectral flux, seen in the lower panel, shifts to higher energies with increasing accretion rate, in accordance with observations of subcritical XRPs (A. Mushtukov & S. Tsygankov 2024). The growth of P_L near the cyclotron resonance in the upper panel is also produced by scatterings in the accretion channel.

5 SUMMARY AND DISCUSSION

We have presented the results of self-consistent numerical simulations of the structure of an XRP accretion channel and radia-

tion, emitted from a polar hotspot and reprocessed by Compton scattering in the strongly magnetized plasma, which falls vertically along the magnetic field lines, assuming that the electrons reside on the ground Landau level. The finite-volume method of solution of the system of equations of radiation hydrodynamics was coupled to the Monte Carlo simulation of two-mode radiative transfer in an accretion channel, which has the shape of either filled (circle geometry) or hollow (ring geometry) cylinder. We have jointly accounted for the effects of the strongly magnetized plasma and the QED vacuum in the accretion channel.

We have shown that in an accretion channel of a subcritical XRP the polarization of the NMs is mostly determined by the vacuum polarization effects. As a consequence, the polarization of both NMs is almost linear and their scattering cross-sections have comparable cyclotron resonances. It slightly increases the plasma deceleration in comparison to the pure plasma approximation. The vacuum polarization changes the shape of the spectra formed in the accretion channel around the electron cyclotron energy. The cyclotron absorption feature becomes quite pronounced in the outgoing spectra. Without allowance for the vacuum polarization, this absorption is smeared away because of the mostly non-resonant character of the O-mode scattering (see Fig. 10; the absence of this smearing in P. Fotiadis et al. 2026 is caused by averaging cross-sections over polarizations, thus neglecting the difference between the X- and O-modes). The vacuum polarization makes both NMs almost equally resonant (Fig. 2), thus enhancing the photon depletion in the core of the line.

However, the quantitative results of our simulations are only preliminary, because of simplifications inherent in our model. In particular, taking the vacuum polarization into account, we obtain a more pronounced cyclotron depression than is observed in real XRP spectra. A possible cause of this disagreement may be our neglect of population and radiative decay of higher Landau levels, which may cause the ‘spawning’ of the cyclotron photons (R. A. Araya & A. K. Harding 1999).

An important part of our model is the seed radiation that is emitted from the hotspot on the NS surface and scattered by the plasma in the channel. The blackbody seed radiation does not contain the cyclotron absorption feature, which is expected in the radiation of underlying atmosphere (see e.g. A. A. Mushtukov et al. 2021; E. Sokolova-Lapa et al. 2021). To test the influence of such a feature, we performed a set of simulations in which the blackbody hotspot spectrum was replaced by the spectrum of the subcritical pulsar 1A 0535+262 in different luminosity states. We found that this modification of the initial spectrum noticeably reduces plasma deceleration by radiation, because the plasma interacts mostly with the resonant photons and remains transparent for photons below the cyclotron resonance.

To test an effect of a possible polarization of the seed radiation, we performed the simulations assuming that the hotspot radiation consists entirely of one of the two NMs. As noted in Paper I, at $E \ll E_B$ the polarization of the outgoing radiation is almost completely determined by that of the seed radiation because the channel is transparent to radiation at these energies. Near the resonance the opposite situation occurs and the channel is highly opaque. We demonstrate that the initial polarization state is completely erased at $E \sim E_B$ because of the multiple scattering in the accretion channel. It means that at such photon energies it is necessary to treat accurately the radiation transfer in the falling plasma even at low accretion rates.

A number of subcritical XRPCs demonstrate a positive correlation between the energy of the cyclotron absorption feature in the spectrum and luminosity. One of various hypotheses to explain this property of the resonance absorption features proposes the Doppler shift of Comptonized photons in accretion channels. In order to check it, we carried out simulations with different accretion rates and found that our radiation hydrodynamical model agrees with the proposed mechanism: the cyclotron feature energy increases with increasing accretion rate although this effect is not very strong. In addition, we have found a positive correlation between the luminosity and the degree of linear polarization at the energies near and above the cyclotron resonance. Such energies are not observed by the IXPE telescope, so this result cannot yet be verified experimentally.

The angular dependence of the outgoing flux (Fig. 9) controls the formation of the beaming patterns and light curves at different photon energies. The polarization degree also shows such angular dependence, which results in formation of the *X-ray polarization light curves* of the XRPCs. These beaming patterns and light curves are studied in more detail in another paper (I. D. Markozov et al. 2026).

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DATA AVAILABILITY

The calculations presented in this paper were performed using a private code developed and owned by the corresponding author. All the data appearing in the figures are available upon request.

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APPENDIX A: SCATTERING OF THE NORMAL MODES

Equation (38) explicitly reads

$$a_{j'j} = \sum_{l'=1,2} \sum_{l=1,2} M'_{j'l'} a_{l'l}^{\ell} M_{jl}^*, \quad (\text{A1})$$

where the matrix elements M_{jl} are given by any of equations (29) or (30), the amplitudes for linear polarizations $a_{l'l}^{\ell}$ are given by equations (34)–(36) and the primed quantities are taken for the final state with $E' = E_f(p)$ according to equation (42). As mentioned above, we fix the phases in equations (29) and (30) as $\phi_1 = 0$ and $\phi_2 = \pi$. The result can be written in the form

$$a_{j'j} = C_{j'j}^{(0)} + \frac{C_{j'j}^{(1)} E e^{i(\varphi' - \varphi)}}{E - E_B + i\hbar v_{\text{eff},e}}, \quad (\text{A2})$$

where

$$C_{11}^{(0)} = (1 + K_1^2)^{-1/2} (1 + K_1'^2)^{-1/2} [2K_1' K_1 \sin \theta' \sin \theta + A_+ (1 + K_1' K_1 \cos \theta' \cos \theta + K_1' \cos \theta' + K_1 \cos \theta)], \quad (\text{A3})$$

$$C_{11}^{(1)} = (1 + K_1^2)^{-1/2} (1 + K_1'^2)^{-1/2} [1 + K_1' K_1 \cos \theta' \cos \theta - K_1' \cos \theta' - K_1 \cos \theta], \quad (\text{A4})$$

$$C_{22}^{(0)} = (1 + K_1^2)^{-1/2} (1 + K_1'^2)^{-1/2} \text{sign}(K_1') \text{sign}(K_1) \times [2 \sin \theta' \sin \theta + A_+ (K_1' K_1 + \cos \theta' \cos \theta - K_1 \cos \theta' - K_1' \cos \theta)], \quad (\text{A5})$$

$$C_{22}^{(1)} = (1 + K_1^2)^{-1/2} (1 + K_1'^2)^{-1/2} \text{sign}(K_1') \text{sign}(K_1) \times [K_1' K_1 + \cos \theta' \cos \theta + K_1 \cos \theta' + K_1' \cos \theta], \quad (\text{A6})$$

$$C_{12}^{(0)} = (1 + K_1^2)^{-1/2} (1 + K_1'^2)^{-1/2} \text{sign}(K_1) [2K_1' \sin \theta' \sin \theta + A_+ (K_1' \cos \theta' \cos \theta - K_1 - K_1' K_1 \cos \theta' + \cos \theta)], \quad (\text{A7})$$

$$C_{12}^{(1)} = (1 + K_1^2)^{-1/2} (1 + K_1'^2)^{-1/2} \text{sign}(K_1) \times [K_1' \cos \theta' \cos \theta - K_1 + K_1' K_1 \cos \theta' - \cos \theta], \quad (\text{A8})$$

$$C_{21}^{(0)} = (1 + K_1^2)^{-1/2} (1 + K_1'^2)^{-1/2} \text{sign}(K_1') \times [2K_1 \sin \theta' \sin \theta + A_+ (K_1 \cos \theta' \cos \theta - K_1' - K_1' K_1 \cos \theta + \cos \theta')], \quad (\text{A9})$$

$$C_{21}^{(1)} = (1 + K_1^2)^{-1/2} (1 + K_1'^2)^{-1/2} \text{sign}(K_1') \times [K_1 \cos \theta' \cos \theta - K_1' + K_1' K_1 \cos \theta - \cos \theta'], \quad (\text{A10})$$

and A_+ is defined by equation (37). Here, we have separated resonant contributions with coefficients $C_{j'j}^{(1)}$ from non-resonant terms with coefficients $C_{j'j}^{(0)}$, which is convenient for numerical implementation – for example, to optimize the integration in equation (46).

Making use of equations (33) and (A2), and taking into account that $C_{j'j}^{(0)}$ and $C_{j'j}^{(1)}$ in equations (A3)–(A10) are real, we can write the differential cross-sections in the right-hand side of equation (39) as

$$\frac{d\sigma_{j'j}^0}{d\Omega_f^0} = \frac{3\sigma_T}{32\pi} \left[|C_{j'j}^{(0)}|^2 + \frac{|C_{j'j}^{(1)}|^2 + 2C_{j'j}^{(0)} C_{j'j}^{(1)} (1 - u_e)}{(1 - u_e)^2 + (\hbar v_{\text{eff},e}/E)^2} \right], \quad (\text{A11})$$

where $u_e = (E_B/E)^2$ as defined by equation (15). We see, in particular, that the factors $\text{sign}(K_1)$ and $\text{sign}(K_1')$ that are present in equations (A5)–(A10) are squared in equation (A11). Therefore, the change of these signs at $\theta_B = \pi/2$ or $\theta_B' = \pi/2$ may only lead to a change of sign of the amplitudes $a_{j'j}$, but $d\sigma_{j'j}^0/d\Omega_f^0$ remains continuous when crossing these angles.

We have checked that equations (A1)–(A11) can also be derived from equations (1), (10), and (11) of J. Ventura (1979) using an appropriate coordinate transformation.

APPENDIX B: CODE VERIFICATION

In this appendix, we describe some of the tests of the code used for its verification.

B1 Magnetized atmosphere with coherent scattering

We have tested our Monte Carlo radiation transfer code on several radiative transfer problems with known analytical solutions. The most complicated of these tests involves almost all parts of our Monte Carlo code: simulation of the radiation emitted by a semi-infinite uniform atmosphere with a strong magnetic field, assuming a coherent resonant scattering. This problem has an analytical solution in the diffusion approximation for two NMs (A. D. Kaminker et al. 1982). To test our Monte Carlo code, we compute intensities of the two NMs emitted by the semi-infinite magnetized atmosphere in the coherent scattering approximation and compare the results with equation (19) of A. D. Kaminker et al. (1982).

Fig. B1 shows the angle dependence of the X- and O-mode intensities, calculated numerically and analytically for a monochromatic radiation with photon energy $E = 70$ keV, which experiences coherent scattering in a semi-infinite atmosphere with the cyclotron energy $E_B = 51.1$ keV and vacuum parameter $W = 29.3$. These values correspond to the case (Bb) (very large vacuum polarization, near the cyclotron resonance) of A. D. Kaminker et al. (1982). The number of seed photons was $N_{ph} = 10^7$. The numerical noise seen in the lower panel of Fig. B1 is higher than in the upper one, because most of the photons come out of the atmosphere in the X-mode. We see that our numerical solution satisfactorily reproduces the analytical one within the numerical noise. Thus, we can conclude that our Monte Carlo code is free of a systematic bias (within the model assumptions).

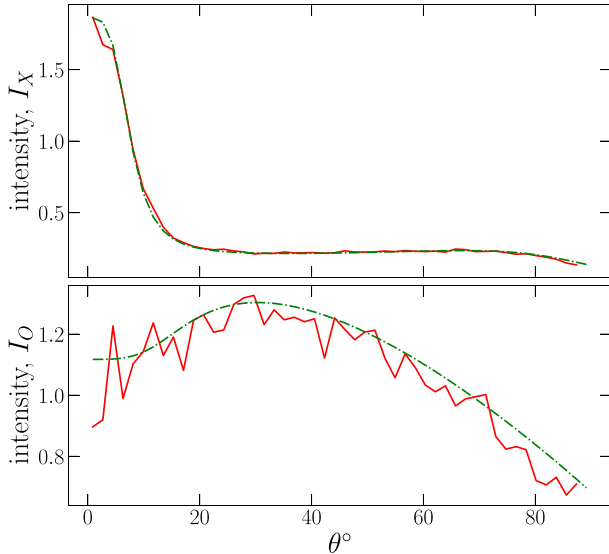


Figure B1. Integrated intensity in X-mode (the upper panel) and O-mode (the lower panel) in relative units (normalized to the average) as a function of the angle ϑ to the outer normal to the plane-parallel semi-infinite magnetized NS atmosphere. At both panels, the dash-dotted line corresponds to the analytical formula and the solid one corresponds to our programme results. Computations were performed for the photon energy $E = 70$ keV, the cyclotron energy $E_B = 51.1$ keV and the vacuum parameter $W = 29.3$.

B2 Radiation hydrodynamics: forward-backward scattering

The second part of our tests deals with the coupling of radiative transfer with hydrodynamics. For this purpose, we consider a tube filled with a plasma, interacting with radiation, which is injected from one side of the tube (the bottom) and emitted from the other side (the top of the tube). To obtain an analytical solution, we consider one-dimensional propagation of photons along the tube and neglect the gravity.

First we consider the case where the radiation interacts with the matter only via coherent forward-backward scattering. For simplicity, we assume that the probabilities of photon scattering in the forward or backward directions are equal in the reference frame of the tube. A constant radiative flux from the bottom of the tube is treated as a parameter of the simulation.

This model setup is described by equations

$$\frac{d\rho v}{dx} = 0, \quad \frac{d}{dx}(\rho v^2) + \frac{dP}{dx} = \rho a_r, \quad \frac{d}{dx} \left(\frac{Pv\gamma}{\gamma - 1} + \frac{\rho v^3}{2} \right) = 0, \quad (\text{B1})$$

where γ is the adiabatic index of the plasma, $a_r = \sigma_T F_0 / m_i c$ is the acceleration by the radiation force, m_i is a proton mass, and F_0 is a flux from the bottom boundary, which can be expressed through the tube cross-section S_b and bottom luminosity L_b as $F_0 = L_b / S_b$. The zero right-hand side of equation (B1) is due to the absence of energy exchange at the coherent scattering.

This system of equations has an analytical solution:

$$\rho v = \rho_0 v_0, \quad P(v) = \frac{c_1}{v} - \rho_0 v_0 v^{\frac{\gamma-1}{2\gamma}}, \quad (\text{B2})$$

$$\frac{(v^2 - v_0^2)(\gamma+1)}{4\gamma} - \frac{c_1}{\rho_0 v_0} \ln \left(\frac{v}{v_0} \right) = a_r (x - x_0). \quad (\text{B3})$$

Here, ρ_0 and v_0 are the inflow mass density and velocity at the top of the tube, placed at the coordinate $x = x_0$,

$$c_1 = P_0 v_0 + \rho_0 v_0^3 \frac{\gamma - 1}{2\gamma} \quad (\text{B4})$$

is the integration constant and P_0 is the top-side pressure. We use the following method to verify our numerical model. For certain values of F_0 , P_0 , v_0 , and ρ_0 we compute hydrodynamical profile of the tube with a fixed length H according to equation (B3). After that, we perform a simulation with the same boundary conditions and constant values of $P = P_0$, $v = v_0$, and $\rho = \rho_0$ over the tube as the initial condition. The system relaxes to a steady-state solution, which is then compared with the analytical solution.

The upper panel of Fig. B2 demonstrates the results of comparison of the numerical solution with the semi-analytical one. We see that these solutions almost coincide, which means that the test has been passed successfully.

B3 Radiation hydrodynamics: pure absorption

Now let us consider the same model tube as in the previous section, but with pure absorption of radiation by the plasma instead of scattering, with the absorption cross-section σ set equal to the Thomson cross-section σ_T . In addition, we assume that photons are emitted from the lower boundary only in the direction strictly normal to the boundary surface. This test is useful for examining the explicit Euler method implemented in our code, because absorption produces a steeper solution profile than scattering. The system of radiation hydrodynamical equations in this case has the form:

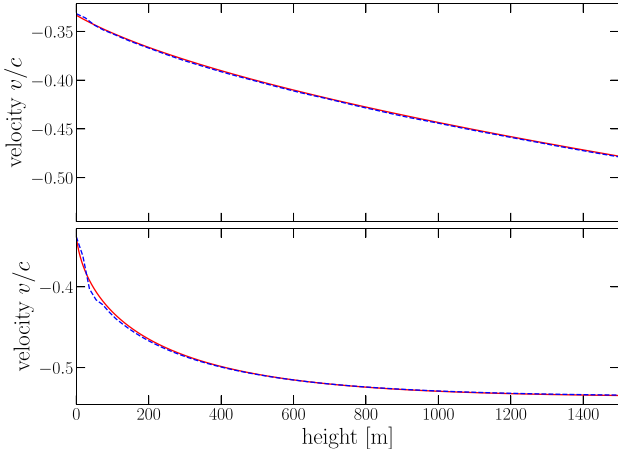


Figure B2. Velocity profiles in the one-dimensional tube for the forward-backward radiation scattering (upper panel, see Section B2; scattering cross-section $\sigma = \sigma_T = 6.65 \times 10^{-25} \text{ cm}^2$, radiation acceleration at the lower boundary $a_r = 1.43 \times 10^{14} \text{ cm s}^{-2}$) and pure absorption (lower panel, see Section B3; absorption cross-section $\sigma = \sigma_T$, $a_r = 5.1 \times 10^{14} \text{ cm s}^{-2}$). The matter is treated as an ideal gas with an adiabatic index $\gamma = 5/3$. In both panels, conditions at the upper boundary of the simulation domain are $v_0 = 0.54c$, $P_0 = 4 \times 10^{14} \text{ dyn cm}^{-2}$, and $\rho_0 = 7 \times 10^{-5} \text{ g cm}^{-3}$. At the lower boundary, we set zero gradient for hydrodynamical variables and a constant flux of the radiation propagating along the normal to the surface.

$$\frac{d\rho v}{dx} = 0, \quad (\text{B5})$$

$$\frac{d}{d\tau}(\rho v^2) + \frac{dP}{d\tau} = -\frac{\sigma F_0}{cm_i} e^{-\tau(x)}, \quad (\text{B6})$$

$$\frac{d}{d\tau} \left(\frac{Pv\gamma}{\gamma-1} + \frac{\rho v^3}{2} \right) = \frac{\sigma F_0}{cm_i} v e^{-\tau(x)}, \quad (\text{B7})$$

where $\tau = (\rho/m_i)\sigma x$ is the optical depth. Unlike the pure scattering case, in the present case the radiative flux is not conserved in the computational volume, but exponentially decreases.

We obtained the profiles of hydrodynamical quantities as functions of τ by solving the system of equations (B5)–(B7) and restore the $x(\tau)$ profile as

$$x(\tau) = \int_0^\tau \frac{d\tau'}{\rho(\tau')}. \quad (\text{B8})$$

To compare the results of our numerical model with a steady-state solution, we proceed as follows. First, the flux F_0 and hydrodynamical quantities P_0 , v_0 and ρ_0 at the bottom boundary are specified. After that, the system (B5)–(B7) is being solved until the optical depth τ reaches a certain value τ_s , which is the additional parameter of the calculation. After that the total length H of the computational domain that corresponds to the optical depth τ_s is obtained from equation (B8). This length H as well as the flux from the bottom boundary F_0 and values of pressure p_1 , density ρ_1 , and velocity v_1 at the optical depth τ_s are supplied to our radiation hydrodynamics code as boundary conditions. The simulation with these conditions starts with p_1 , ρ_1 , and v_1 constant over all tube. Then, it evolves to the steady-state distribution which can be compared with the solution of the system (B5)–(B7).

The results of the comparison of our code calculations for this model with the semi-analytical solution are presented in the lower panel of Fig. B2. Numerical and semi-analytical velocity profiles are in good agreement, which means that our code works accurately for this model.

This paper has been typeset from a $\text{\LaTeX}$ file prepared by the author.