Comment on "Equation of state of a dense and magnetized fermion system"

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Contrary to what is claimed by Ferrer *et al.* [Phys. Rev. C **82**, 065802 (2010)], the magnetic field of a neutron star cannot exceed 10^{19} G and the thermodynamic pressure of dense magnetized fermion gas is isotropic.

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The authors of a recent paper [1] construct thermodynamics of charged fermions in a strong magnetic field $\boldsymbol{B}$ where the Landau quantization of orbital motion is important and thermodynamic quantities depend on $\boldsymbol{B}$. The subject attracts considerable attention, with the most important applications to neutron stars possessing strong magnetic fields. The authors conclude that (i) the magnetic field in a neutron star can exceed 10^{19} G and (ii) the gas of particles in a quantizing magnetic field has anisotropic pressure. We point out that both statements are inaccurate.

Maximum field strength. The well-known estimate based on the virial theorem [2] gives the maximum magnetic field in a neutron star $B_{\max} \sim 10^{18}$ G [3]. The authors of Ref. [1] claim that this estimate can be relaxed. As an alternative, they propose arbitrary simplistic parametrizations of mass density ρ and field strength B as functions of the radial coordinate r within the star, treating the parameters of these functions as "totally arbitrary." For certain values of these parameters they obtain $B_{\max} > 10^{19}$ G.

However, the density and field distributions are not arbitrary, but must satisfy stability equations for a magnetized star with a realistic equation of state. Detailed self-consistent numerical simulations (for example, [4,5]) show that, depending on the adopted equation of state in the stellar core, $B_{\max}$ takes values of (0.3–3.0) $\times 10^{18}$ G, in disagreement with Ref. [1] but in good agreement with Ref. [3]. A large variety of equations of state were explored in numerical simulations [4]. The obtained ρ and B distributions are different from artificial distributions of Ref. [1], leading to different values of $B_{\max}$.

Isotropy of pressure. The consideration of the pressure in Ref. [1] is based on the articles by Canuto and Chiu [6] who showed that kinetic pressures $p_\parallel^{\rm kin}$ and $p_\perp^{\rm kin}$ of charged particles along and across $\boldsymbol{B}$, calculated as ensemble averages of respective currents of kinetic momenta, are different. The authors of Ref. [1] repeat the consideration [6] using a more general formalism and arrive at the same conclusions. According to Refs. [1,6], the total anisotropic pressure is the sum of the magnetic pressure related to the Maxwell stress tensor, and the kinetic pressure. The longitudinal and transverse kinetic pressures are $p_\perp^{\rm kin} = -\Omega - MB$ and $p_\parallel^{\rm kin} = -\Omega$, where Ω is the grand-canonical potential per unit volume and $\boldsymbol{M}$ is the magnetization (directed along $\boldsymbol{B}$ in the quasistationary approximation adopted in these studies).

However, the deficiency of the approach of Ref. [6] was pointed out long ago by Blandford and Hernquist [7]. It is well known that the total microscopic electric current density $\boldsymbol{j}$ is composed of the free (or conduction) current term $\boldsymbol{j}_f$ and bound current term $\boldsymbol{j}_b$ due to magnetization (the dynamical polarization contribution to $\boldsymbol{j}_b$ in the quasistationary approximation is negligible). The magnetization current density equals (in Gaussian units) $\boldsymbol{j}_b = c\,\nabla \times \boldsymbol{M}$; in case of boundaries, this volume current should be supplemented by the surface current $c\boldsymbol{M} \times \boldsymbol{B}/B$ (see, e.g., Ref. [8]). The total *thermodynamic* pressure P in a magnetized plasma is the sum of the kinetic pressure and an additional contribution due to the Lorentz force density related to the magnetization currents. If we compress a plasma across $\boldsymbol{B}$, then the magnetization current density induces an additional contribution MB to the force density. As a result, the transverse component of the total (thermodynamic) plasma pressure equals $p_\perp^{\rm kin} + MB = p_\parallel^{\rm kin}$, so that the total plasma pressure $P = -\Omega$ is isotropic.

In spite of the simplicity of the above arguments, they are sometimes ignored in the literature, like in Ref. [1]. Therefore, in order to make them still more transparent, let us illustrate the pressure isotropy with two graphic examples.

As the simplest example, consider a plasma contained in a finite cylinder in vacuum with a uniform external $\boldsymbol{B}$ field along the cylinder axis. At equilibrium in the absence of external forces, the sum of the force densities exerted on the side wall of the cylinder by the transfer of kinetic momenta of plasma particles and by the surface magnetization current equals $p_\perp^{\rm kin} + MB = -\Omega$. It is the same as the force density $p_\parallel^{\rm kin} = -\Omega$ exerted on the head wall. Hence the plasma pressure, which can be determined in this experiment by measuring forces on the cylinder walls, is isotropic.

As another example, more relevant to astrophysics, consider a volume element in a magnetized star. Let the element be sufficiently small and distributions of $\boldsymbol{B}$, temperature T, and gravitational acceleration $\boldsymbol{g}$ be sufficiently smooth, so that we can assume constant $\boldsymbol{B}$, T, and $\boldsymbol{g}$ within this volume. Let the z axis be directed along $\boldsymbol{g}$. Then ρ and $\Omega(\rho, B, T)$ depend on z, resulting in z-dependent magnetization $\boldsymbol{M} = -\partial\Omega(\rho, \boldsymbol{B}, T)/\partial\boldsymbol{B}$. Hydrostatic balance implies that the density of gravitational force, $\rho\boldsymbol{g}$, be balanced by the density of forces created by plasma particles (the gradient of kinetic pressure and Lorentz force due to plasma magnetization).

Now let us compare two limiting cases. If $\boldsymbol{B}$ is parallel to $\boldsymbol{g}$, the z component of the Lorentz force is absent, and we get the standard equation of hydrostatic equilibrium $\rho g = dp_{\parallel}^{\rm kin}/dz = dP/dz = -d\Omega/dz$.

If $\boldsymbol{B}$ is perpendicular to $\boldsymbol{g}$, then the kinetic pressure gradient $dp_{\perp}^{\rm kin}/dz$ acts in parallel with the Lorentz force density $B dM/dz$. Note that in our case $dM/dz \neq 0$, simply because $d\rho/dz \neq 0$ (ρ depends on z) in the gravity field. Since B and T are constant,

$$\frac{dM}{dz} = \frac{\partial M(\rho, T, B)}{\partial \rho}\frac{d\rho}{dz} = -\frac{\partial^2 \Omega(\rho, T, B)}{\partial\rho\,\partial B}\frac{d\rho}{dz}. \tag{1}$$

Then the equilibrium condition takes the same standard form

$$\rho g = \frac{dp_{\perp}^{\rm kin}}{dz} + B\frac{dM}{dz} = \frac{d}{dz}(-\Omega - MB) + B\frac{dM}{dz} = -\frac{d\Omega}{dz}.$$

Thus, the gradient $d\rho/dz = -(\partial\Omega/\partial\rho)^{-1}\rho g$ does not depend on $\boldsymbol{B}$-field direction, which means that the hydrostatic equilibrium is determined by the isotropic thermodynamic pressure P, in accordance with the results of Ref. [7].

Since the forces created by bound currents are small in the majority of applications, the equations of magnetohydrodynamics (MHD) are commonly derived neglecting the magnetization. However, the magnetization term is easily recovered by substituting the general expression $\boldsymbol{j} = \boldsymbol{j}_f + \boldsymbol{j}_b$ into the microscopic Lorentz force density $\boldsymbol{j} \times \boldsymbol{B}/c$ that is included in the derivation of MHD equations from the first principles (e.g., [9], Chap. VIII). Moreover, thermodynamics of magnetized media is well studied in the theory of magnetics (e.g., [9], Chap. IV). Of course, everyone is free to use anisotropic kinetic pressure in MHD equations and add the magnetization force density explicitly. However, it seems more natural to follow the traditional approach and use the isotropic thermodynamic pressure that automatically includes the contribution of the magnetization.

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[1] E. J. Ferrer, V. de la Incera, J. P. Keith, I. Portillo, and P. L. Springsteen, Phys. Rev. C **82**, 065802 (2010).
[2] S. Chandrasekhar and E. Fermi, Astrophys. J. **118**, 116 (1953); **122**, 208(E) (1955).
[3] D. Lai and S. L. Shapiro, Astrophys. J. **383**, 745 (1991).
[4] M. Bocquet, S. Bonazzola, E. Gourgoulhon, and J. Novak, Astron. Astrophys. **301**, 757 (1995).
[5] K. Kiuchi and K. Kotake, Mon. Not. R. Astron. Soc. **385**, 1327 (2008).
[6] V. Canuto and H.-Y. Chiu, Phys. Rev. **173**, 1210 (1968); **173**, 1220 (1968).
[7] R. D. Blandford and L. Hernquist, J. Phys. C **15**, 6233 (1982).
[8] D. J. Griffith, *Introduction to Electrodynamics*, 3rd ed. (Prentice-Hall, London, 1999), Chap. 6.
[9] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskiĭ, *Electrodynamics of Continuous Media*, 2nd ed. (Butterworth-Heinemann, Oxford, 1984).