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A comparison study of the vertical and magnetic shear instabilities in accretion discs

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Abstract. We consider the stability properties of discs rotating with the angular velocity dependent on both the radial and vertical coordinates. A vertical dependence of Ω destabilizes the disc at any particular form of this dependence. The growth rate of the vertical shear instability is calculated and compared with that of the magnetic shear instability. We find that the vertical shear instability grows faster for a wide range of parameters.

Key words. accretion: accretion discs – hydrodynamics – instabilities – turbulence

1. Introduction

The standard alpha viscosity prescription in accretion discs requires a sufficiently strong turbulence that enhances the angular momentum transport. From the very beginning, differential rotation has been regarded as the most promising source of turbulence. In most accretion disc models, however, the radial dependence of the angular velocity satisfies the Rayleigh stability criterion

$$\frac{\partial(s^4\Omega^2)}{\partial s} > 0, \quad (1)$$

where s is the cylindric radius, and Ω is the angular velocity. The inequality (1) is usually regarded as a sufficient condition of linear stability of the hydrodynamic flow in accretion discs. Therefore, the origin of turbulence in discs is oftenly attributed to the well-known Velikhov-Chandrasekhar instability (Velikhov 1959; Chandrasekhar 1960) which can arise in magnetic differentially rotating fluid. This instability has been analysed in detail for stellar conditions (see Fricke 1969; Acheson 1978, 1979). The role of the magnetic shear instability in turbulization of astrophysical discs was first considered by Safronov (1969). Later on, Balbus & Hawley (1991) recognized that this instability can give rise to MHD turbulence in magnetic accretion discs since the necessary condition (a decrease of the angular velocity with the cylindrical radius) is fulfilled in thin discs. Note that the magnetic shear instability can arise only if the magnetic field is not very strong since a strong field stabilizes differential rotation (see, e.g., Urpin 1996). Numerical simulations done by a number of authors (Brandenburg et al. 1995; Hawley et al. 1995; Matsumoto & Tajima 1995; Torkelsson et al. 1995) indicate that this instability can provide an efficient mechanism of the angular momentum transport and the magnetic field amplification. Calculations show that the level of turbulence in magnetic accretion discs is sufficient to enhance the viscosity by orders of magnitude. For instance, estimates of the turbulent parameter α range from 0.001 to 0.7 depending on the magnetic field strength and the angular velocity. It is worth noting that a complete analytical and numerical treatment of MHD modes in stratified magnetic accretion discs demonstrates a much richer variety of instabilities than previously realized (Keppens et al. 2002), and the current view on the origin of MHD turbulence is likely highly simplified.

Recently, the question of a pure hydrodynamic origin of turbulence in astrophysical discs has been critically reexamined on the basis of a large body of experimental and numerical evidence (Richard & Zahn 1999; Longaretti 2002). This problem is of particular importance for protoplanetary discs because some of them (or their regions) are poorly conducting (Fleming et al. 2000; Sano et al. 2000), and non-MHD mechanisms should be operative to provide an efficient turbulent transport of the angular momentum. However, a non-MHD origin of turbulence cannot be excluded in accretion discs as well. For instance, analysing the behavior of various types of shear flows, Longaretti (2002) has argued that shearing sheet flows should be turbulent, and that the lack of turbulence in accretion disc simulations is most likely caused by a lack of resolution.

Likely, hydrodynamic processes in astrophysical discs are much more complex than in simple shear flows because discs are subject to both vertical and radial stratification. In such conditions, the Rayleigh criterion (1) is not true criterion and does not apply to accretion discs. Therefore, some linear instabilities can arise even if the stability condition (1) is fulfilled. For example, the vertical stratification, despite being usually stable, can provide a catalyzing effect which under certain conditions

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induces a linear non-axisymmetric instability of anticyclonically sheared flow (Molemaker et al. 2001). Recently, this instability was reexamine by making use of a different approach by Dubrulle et al. (2002) who enlarged the study towards conditions typical of astrophysical discs. They found that thin discs are potentially subject to this instability.

Apart from being vertically and radially stratified, discs rotate with the angular velocity, Ω, which depends on both the cylindrical radius, s, and the vertical coordinate, z (see, e.g., Kippenhahn & Thomas 1982; Urpin 1984; Kley & Lin 1992). It was first argued by Kippenhahn & Thomas (1982) that a slight baroclinity in the angular velocity distribution is necessary to fulfil hydrostatic and thermal equilibrium in accretion discs. The dependence of Ω on z is relatively weak but it can trigger a number of instabilities. For instance, Knobloch & Spruit (1986) have found that adiabatic baroclinic waves can be unstable in thin accretion discs if vertical and radial stratification is taken into account. The instability occurs only for non-axisymmetric perturbations and is analogous to baroclinic instability encountered in geophysics. Due to the dominant effect of the Kepler shear, instability is possible only if radial temperature gradients has a very short lengthscale (of the order of scaleheigh), or if the stratification is close to convective instability. Both these conditions are unlikely to be fulfilled and, probably, baroclinic waves play no important role in accretion discs.

The instability first considered by Goldreich & Schubert (1967) seems to be more suitable for the conditions of accretion discs. It does not require strong radial stratification but is caused by the vertical shear alone. This instability has been considered recently by Rüdiger et al. (2002) who examined the behaviour of linear and non-linear small-scale perturbations and concluded that the stabilizing effect of stratification is sufficient to stabilize even non-linear perturbations. Simulations have been done by making use of a version of the ZEUS-3D code which does not allow one to treat properly the effect of thermal conductivity and, therefore, the authors considered only adiabatic fluctuations. However, small-scale motions in discs are likely non-adiabatic. The exchange of heat between perturbations and the surrounding medium substantially reduces the influence of the buoyancy force and decreases the stabilizing effect of stratification (Urpin & Brandenburg 1998). As a result, the stability properties of thermally conducting discs can well be different from those obtained in the adiabatic limit.

In the present paper, we consider in detail how the vertical shear instability can operate in the conditions of accretion discs. The main difference to stellar conditions is caused by a high radiative thermal conductivity that makes the thermal relaxation time scale comparable to the hydrodynamic time scale. We compare the growth rates of the vertical and magnetic shear instabilities and find that the latter grows much more slowly within a wide range of wavelengths except perturbations with a very short wavelength. The turbulent viscosity caused by the vertical shear instability is estimated. We argue that the vertical gradient of Ω existing in accretion discs is sufficient to enhance substantially the angular momentum transport.

The paper is organized as follows. In Sect. 2, the main equations are presented governing the behaviour of perturbations in the Boussinesq approximation, and the dispersion equation for short wavelength perturbations is derived. The stability criterion is discussed in Sect. 3, and the growth time of instability is calculated in Sect. 4. In Sect. 5, we estimate the turbulent viscosity produced by the considered instability in accretion discs.

2. Dispersion equation for short-wavelength perturbations

Consider a non-magnetic axisymmetric disc of finite vertical extent, not necessarily thin. The unperturbed angular velocity depends on both s and z coordinates, $\Omega = \Omega(s, z)$; s, φ, z are cylindrical coordinates. We treat axisymmetric short-wavelength perturbations with the space-time dependence $\exp(\gamma t - \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r})$ where $\boldsymbol{k} = (k_s, 0, k_z)$ is the wave vector, $|\boldsymbol{k} \cdot \boldsymbol{r}| \gg 1$. Small perturbations will be marked by the subscript 1, whilst unperturbed quantities will have no subscript, except for indicating vector components when necessary. In the unperturbed state, the disc is assumed to be in hydrodynamic equilibrium,

$$\frac{\nabla p}{\rho} = \boldsymbol{G} = \boldsymbol{g} + \Omega^2 \boldsymbol{s}, \tag{2}$$

where $\boldsymbol{g}$ is the gravity of the central object. Solving Eq. (2) together with the thermal balance equation, one can obtain s- and z-dependences of ρ, p, and Ω. In a thin disc, these dependences can be represented in a simple analytical form (see Kley & Lin 1992).

We use the Boussinesq approximation since the growth time of instabilities associated with shear is typically much longer than the period of sound wave with the same wavelength. The linearized momentum, continuity and thermal balance equations read

$$\gamma \boldsymbol{V}_1 + 2\boldsymbol{\Omega} \times \boldsymbol{V}_1 + \boldsymbol{e}_\varphi s \boldsymbol{V}_1 \cdot \nabla\Omega = -\frac{\nabla p_1}{\rho} - \beta \boldsymbol{G} T_1 - \nu k^2 \boldsymbol{V}_1, \tag{3}$$

$$\boldsymbol{k} \cdot \boldsymbol{V}_1 = 0, \tag{4}$$

$$\gamma T_1 + \boldsymbol{V}_1 \cdot (\Delta\nabla T) = -\chi k^2 T_1, \tag{5}$$

where $\boldsymbol{V}_1$, p_1 and T_1 are perturbations of the hydrodynamic velocity, pressure and temperature, respectively; $\beta = -(\partial \ln\rho/\partial T)_{\rm p}$ is the thermal expansion coefficient, χ and ν are the thermal diffusivity and viscosity, respectively; $(\Delta\nabla T) = \nabla T - \nabla_{\rm ad} T$ is the difference between the actual and adiabatic temperature gradients; we denote by $\boldsymbol{e}_\varphi$ the unit vector in the azimuthal direction. In the momentum Eq. (3), it is assumed that the density perturbation in the buoyancy force is mainly determined by the temperature perturbation, thus $\rho_1 = -\rho\beta T_1$, in accordance with the idea of the Boussinesq approximation. As it was mentioned, perturbations are generally non-adiabatic in discs, and the effect of the radiative heat transfer has to be taken into account in Eq. (5). The disc is assumed to be optically thick, and the thermal diffusivity, χ, can be expressed in terms of the radiative thermal conductivity, κ, by $\chi = \kappa/\rho c_{\rm p}$ where $c_{\rm p}$ is the thermal capacity at constant pressure. Viscosity is taken into account in Eq. (3) because we consider short wavelength perturbations and ν enters this equation in a product with k^2.

However, we neglect the effect of viscous dissipation in Eq. (5) where its contribution is $\propto k$ and is negligible compared to advection.

In a stratified flow, the buoyancy force acts as a stabilizing factor if the temperature gradient is subadiabatic. However, stabilization due to buoyancy may be drastically reduced if the exchange of heat proceeds faster than the change of momentum caused by the buoyancy force (Townsend 1958). The effect of the thermal conductivity is of particular importance for short-wavelength perturbations which can exchange heat with the surrounding plasma on a very short time scale proportional to k^{-2}. From Eq. (5), one can express the temperature perturbation in terms of $\boldsymbol{V}_1$,

$$T_1 = -\frac{\boldsymbol{V}_1 \cdot (\Delta\nabla T)}{\gamma + \omega_\chi}, \tag{6}$$

where $\omega_\chi = \chi k^2$ is the inverse time scale of dissipation due to the thermal conductivity. For very short wavelengths ($k \to \infty$), one has $T_1 \to 0$ and, hence, the stabilizing effect of stratification in Eq. (3) tends to zero for such perturbations.

Equations (3)–(5) yield the following dispersion equation

$$\gamma^3 + a_2\gamma^2 + a_1\gamma + a_0 = 0 \tag{7}$$

where

$$\begin{aligned} a_2 &= \omega_\chi + 2\omega_\nu, \quad a_1 = Q^2 + \omega_g^2 + \omega_\nu(\omega_\nu + 2\omega_T), \\ a_0 &= \omega_\chi(Q^2 + \omega_\nu^2) + \omega_\nu\omega_g^2, \end{aligned} \tag{8}$$

and

$$Q^2 = 4\Omega^2\frac{k_z^2}{k^2} + 2\Omega s\frac{k_z}{k^2}\left(k_z\frac{\partial\Omega}{\partial s} - k_s\frac{\partial\Omega}{\partial z}\right),$$

$$\omega_g^2 = -\beta(\Delta\nabla T)\cdot\left[\boldsymbol{G} - \frac{\boldsymbol{k}}{k^2}(\boldsymbol{k}\cdot\boldsymbol{G})\right], \quad \omega_\nu = \nu k^2,$$

ω_g is the frequency of the buoyancy waves, and ω_ν is the inverse time scale of dissipation due to viscosity; Q^2 represents the effects associated with the angular velocity and its gradient. Equation (7) describes three low-frequency modes that can exist in incompressible non-magnetic fluids. The sound waves are excluded from consideration in the Boussinesq approximation.

In the case of vanishing viscosity, $\omega_\nu \to 0$, Eq. (7) can be rewritten as

$$\gamma^2 = -Q^2 - \frac{\gamma\,\omega_g^2}{\gamma + \omega_\chi}, \tag{9}$$

Equation (9) illustrates well the difference between the adiabatic and non-adiabatic calculations. If $\chi \neq 0$, Eq. (9) is cubic and has three different roots corresponding to three complex modes. If we assume adiabaticity of perturbations ($\omega_\chi = 0$) then the dispersion Eq. (9) degenerates to a quadratic equation,

$$\gamma^2 = -Q^2 - \omega_g^2, \tag{10}$$

which describes only two modes. Thus, one mode is lost in the adiabatic approximation and, occasionally, this lost mode is most unstable in the conditions of accretion discs.

3. The criteria of instability

The condition that at least one of the roots of Eq. (7) has a positive real part (unstable mode) is equivalent to one of the following inequalities

$$a_2 < 0\,, \quad a_1a_2 < a_0\,, \quad a_0 < 0 \tag{11}$$

being fulfilled (see, e.g., Aleksandrov et al. 1985; DiStefano III et al. 1994). Since ω_χ and ω_ν (and, hence, a_2) are positive defined quantities, the first condition, $a_2 < 0$, will never apply, and only the two other conditions determine the instability of accretion discs.

Substituting the values of a_0, a_1 and a_2, we can transform the second condition (11) to

$$(\omega_\chi + \omega_\nu)\omega_g^2 + 2\omega_\nu[Q^2 + (\omega_\nu + \omega_\chi)^2] < 0. \tag{12}$$

In accretion discs, the radiative viscosity dominates usually the molecular viscosity. Then, we have $\nu/\chi \sim (c_s/c)^2$ where c_s is the sound speed (Thomas 1930). Estimating the temperature of the disc as $T \sim 10^6$ K and $c_s \sim 10^7$cm/s, we obtain $\nu/\chi \sim 10^{-7}$. Since typically $\omega_g^2 \sim Q^2 \sim \Omega^2$, we can neglect in Eq. (12) all terms proportional to ω_ν if

$$\omega_g^2 \sim \Omega^2 \gg \omega_\nu\omega_\chi. \tag{13}$$

This inequality can be fulfilled for a wide range of k because ν is small, and we consider only perturbation satisfying Eq. (13). Then, the criterion (12) simplifies,

$$\omega_g^2 < 0. \tag{14}$$

This inequality describes the condition of convective instability in accretion discs. In the case $\Omega = 0$ and $\boldsymbol{g} \parallel \Delta\nabla T$, Eq. (14) reduces to the well-known Schwarzschild criterion of convection. Note that $\boldsymbol{g}$ and $\Delta\nabla T$ are generally not parallel in accretion discs, therefore a consideration of convection requires a detailed knowlegde of the 2D structure and is beyond the scope of the present paper.

The third criterion (11), $a_0 < 0$, yields

$$\omega_\chi(Q^2 + \omega_\nu^2) + \omega_\nu\omega_g^2 < 0. \tag{15}$$

Viscosity plays no role in this condition if

$$Q^2 \sim \Omega^2 > \frac{\nu}{\chi}\omega_\nu\omega_\chi. \tag{16}$$

This inequality is even less restricting than Eq. (13), and can be fulfilled for a wider range of k. For perturbations with such wavevectors, the inequality (15) is equivalent to $Q^2 < 0$, or

$$\frac{k_z^2}{k^2}\cdot\frac{\partial}{s^3\partial s}(s^4\Omega^2) - \frac{k_zk_s}{k^2}\cdot 2\Omega s\frac{\partial\Omega}{\partial z} < 0\,. \tag{17}$$

Note the difference between the criterion (17) and the condition of instability of adiabatic perturbations, $Q^2+\omega_g^2 < 0$, that can be obtained from Eq. (10). This difference is of crucial importance for convectively stable discs with $\omega_g^2 > 0$. For thin disks, the radial dependence of Ω is approximately given by the Keplerian law, $\Omega \propto s^{-3/2}$ and, hence, the first term on the l.h.s. of the inequality (17) is positive. The sign of the second term depends on the direction of a wave vector, and this term may cause a destabilizing effect. In thin disks with $z/s \ll 1$, the vertical dependence of Ω can be calculated analytically (see, e.g., Urpin 1984; Kley & Lin 1992), and we have

$$\partial\Omega/\partial z \approx q\Omega z/s^2 \tag{18}$$

where q is the parameter in the series expansion of $\Omega(s,z)$ around the equator. This parameter can be calculated using Eq. (A31) of the paper by Kley & Lin (1992) and is of the order of 0.05−0.13 depending on the zone of a disc. Then, the term associated with the vertical shear dominates the r.h.s. of the inequality (17) if

$$k_s > |\,(k_z/2q)(s/z)\,|\,. \tag{19}$$

The instability arises for perturbations with the wavelength much shorter in the radial direction than vertically. Evidently, the necessary condition of instability can not be fulfilled at the central plane of the disk where $\partial\Omega/\partial z = 0$.

Note that for any dependence of Ω on z, there exists a region in the plane (k_s, k_z) where the condition of instability, $Q^2 < 0$, is satisfied. In the general case, the instability arises if the components of a wave vector satisfy the inequality

$$\left|\frac{k_s}{k_z}\right| > \frac{|\partial(s^4\Omega^2)/\partial s|}{|2\Omega s^4 \partial\Omega/\partial z|}, \tag{20}$$

and k_s and k_s are of the same/opposite sign if the quantity $[\partial(s^4\Omega^2)/\partial s]/[\partial\Omega/\partial z]$ is positive/negative.

4. The growth rate of instability

Since the coefficients of Eq. (7) are real there exist three real roots or one real and two complex conjugate roots. The number of roots with a positive real part is determined by Routh criterium (DiStefano III et al. 1994), which states that the number of unstable modes of a cubic Eq. (7) is given by the number of changes of sign in the sequence

$$\left\{1,\ a_2,\ \frac{a_2 a_1 - a_0}{a_2},\ a_0\right\}. \tag{21}$$

If we assume that conditions (13) and (16) are satisfied and viscosity is negligible then this sequence reads

$$\{1\ ,\ \ \omega_\chi\ ,\ \ \omega_g^2\ ,\ \ \omega_\chi Q^2\}. \tag{22}$$

Assuming the disc to be convectively stable ($\omega_g^2 > 0$), we obtain that under the condition (17) only one real mode is unstable. If the disc is convectively unstable ($\omega_g^2 < 0$) and the condition (17) holds then again there should be only one unstable mode. However, if disc is convectively unstable but the condition (17) is not fulfilled then two modes are unstable.

The roots γ_i ($i = 1, 2, 3$) of the cubic Eq. (7) can be represented as $\gamma_i = x_i - a_2/3$. The expressions for x_i are

$$x_1 = u + v\ ,\quad x_2 = \varepsilon_1 u + \varepsilon_2 v\ ,\quad x_3 = \varepsilon_2 u + \varepsilon_1 v, \tag{23}$$

where

$$\varepsilon_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2},\quad (u,v) = \left(-q \pm \sqrt{q^2 + p^3}\right)^{1/3}, \tag{24}$$

and

$$q = \frac{\omega_\chi}{6}\left(\frac{2}{9}\omega_\chi^2 + 2Q^2 - \omega_g^2\right),\quad p = \frac{1}{9}(Q^2 + \omega_g^2 - \omega_\chi^2) \tag{25}$$

(see, e.g., Bronstein & Semendyaev 1957). These expressions are rather cumbersome but they can be simplified very much in some limiting cases. We consider the most important cases of a low thermal frequency,

$$\omega_\chi < |Q^2 + \omega_g^2|^{1/2}, \tag{26}$$

and a high thermal frequency,

$$\omega_\chi > |Q^2 + \omega_g^2|^{1/2} \tag{27}$$

(but still assuming that Eqs. (13) and (16) hold). The condition (26) corresponds to perturbations with a relatively large lengthscale, the condition (27) is adopted to relatively small scale perturbations. In the Keplerian disc, we have

$$\omega_\chi = \chi k^2 \sim \Omega\left(\frac{\chi}{c_s H}\right)(kH)^2, \tag{28}$$

where H is the half-thickness of the disc; we took into account that $c_s/H \sim \Omega$. Then, $\omega_\chi \sim \Omega$ if

$$k \sim k_{\rm cr} = \left(\frac{\Omega}{\chi}\right)^{1/2} \sim \frac{1}{H}\left(\frac{c_s H}{\chi}\right)^{1/2}. \tag{29}$$

The radiative thermal diffusivity, χ, is large in accretion disc models, and usually $\chi/c_s H \geq 1$. Therefore, the thermal frequency is typically larger then the Keplerian frequency for short wavelength perturbations with $kH \gg 1$. Since $|\omega_g| \sim \Omega$, the conditions (26) and (27) are equivalent to $k < k_{\rm cr}$ (relatively large scales) and $k > k_{\rm cr}$ (relatively small scales), respectively.

Under the condition (26) (or $k < k_{\rm cr}$), we have from the expressions (23)–(25) with the accuracy in term $\propto\omega_\chi$,

$$\gamma_1 \approx -\frac{\omega_\chi Q^2}{Q^2 + \omega_g^2},\quad \gamma_{2,3} \approx \pm i\sqrt{Q^2 + \omega_g^2} - \frac{\omega_\chi \omega_g^2}{2(Q^2 + \omega_g^2)}. \tag{30}$$

In convectively stable Keplerian discs ($\omega_g^2 > 0$), we have $Q^2 + \omega_g^2 > 0$ (Urpin & Brandenburg 1998; Rüdiger et al. 2002) and, hence, only the first mode can be unstable if $Q^2 < 0$. The growth time of instability, τ_L, is given by

$$\tau_L \approx \frac{1}{\omega_\chi}\left|\frac{Q^2 + \omega_g^2}{Q^2}\right|. \tag{31}$$

If the thermal conductivity is small or the wavelength of perturbations is relatively long, then the growth time becomes large, and the instability is inefficient.

If the condition (27) is fulfilled then, with the accuracy in terms $\propto\omega_\chi^{-1}$, the roots are

$$\gamma_1 \approx -\omega_\chi\ ,\quad \gamma_{2,3} \approx \pm i\sqrt{Q^2} - \frac{\omega_g^2}{2\omega_\chi}. \tag{32}$$

In convectively stable discs, one of the mode is unstable if $Q^2 < 0$. In this case, the growth time of unstable mode, τ_S, is given by

$$\frac{1}{\tau_S} \sim |Q|. \tag{33}$$

Since the condition (17) can be satisfied only for perturbations with $|k_s/k_z| \gg 1$, the growth rate depends on the ratio k_z/k_s rather than on the wavelength of perturbations. The maximum growth rate is reached at

$$\frac{k_z}{k_s} \approx \frac{s^4 \Omega(\partial\Omega/\partial z)}{\partial(s^4\Omega^2)/\partial s}. \quad (34)$$

In the Keplerian disc, this ratio is

$$\frac{k_z}{k_s} \approx q \cdot \frac{z}{s}, \quad (35)$$

and the corresponding maximum growth rate is given by

$$\frac{1}{\tau_{S\,\max}} \approx \Omega \left| \frac{qz}{s} \right|. \quad (36)$$

Thus, the growth time of instability is short and is comparable to the time scale of vertical shear.

The mechanism of instability is qualitatively simple, particularly in the limit of high ω_χ. Let us assume that the initial laminar rotation is slightly perturbed and a perturbation of the azimuthal velocity, $V_{1\varphi}$, depends both on s- and z-coordinates. This additional rotation produces a perturbation of the pressure, p_1 which depends on s and z as well. If we neglect stratification, the magnitude of p_1 can be estimated from the "geostrophic equlibrium" which implies a balance of the pressure and Coriolis forces in s-direction. If $k_s \gg k_z$, one has for the pressure perturbation

$$p_1/p \sim (2i\Omega/c_s^2 k_s) V_{1\varphi}. \quad (37)$$

The buoyancy force is strongly suppressed for short-wavelength perturbations, therefore the vertical component of the pressure force cannot be compensated by other forces. This uncompensated component generates a flow with the vertical acceleration

$$\dot{V}_{1z} \approx i k_z p_1/\rho. \quad (38)$$

Since Ω depends on z, the vertical flow redistributes the angular momentum. Advection of the angular momentum by the vertical flow alters $V_{1\varphi}$ with the rate

$$\frac{\Delta V_{1\varphi}}{\Delta t} \approx -s \cdot \frac{\partial \Omega}{\partial z} V_{1z}. \quad (39)$$

Depending on the sign of $(\partial\Omega/\partial z)(k_z/k_s)$, this additional rotation can increase or decrease the initial perturbation of the azimuthal velocity causing instability or stability of the differential rotation.

The origin of turbulence in accretion discs is oftenly attributed to the magnetic shear instability (Balbus & Hawley 1991). It is interesting, therefore, to compare the properties of these instabilities. In the presence of a poloidal field, both instabilities can arise in a differentially rotating fluid only if the field strength is weaker than some critical value which depends on the type of instability (Urpin & Brandenburg 1998). In the Keplerian disc, the stabilizing field that suppresses the vertical shear instability, $B_{\rm VS}$, is given by

$$B_{\rm VS} \sim \sqrt{8\pi\rho}\; q\, \frac{\Omega}{k} \left|\frac{z}{s}\right|. \quad (40)$$

The magnetic shear instability represents instability of Alfvén waves which are better adopted to the magnetic field and, therefore, requires generally a stronger field to be suppressed. In this case, the stabilizing field is

$$B_{\rm MS} \sim \sqrt{12\pi\rho}\; \frac{\Omega}{k}. \quad (41)$$

This field is approximately (s/H) times greater than $B_{\rm VS}$. Therefere, if the magnetic field satisfies the condition $B_{\rm MS} > B > B_{\rm VS}$ then only the magnetic shear instability can exist in the disc. In a weaker field, $B < B_{\rm VS}$, both instabilities arise. To have impression about the stabilizing fields, we can estimate $B_{\rm MS}$ and $B_{\rm VS}$ in the accretion disc around the neutron star with more or less standard mass ($\sim$1.4 $M_\odot$) and accretion rate ($\sim 10^{-8}$ $M_\odot$/yr). Using the disc model calculated, for example, by Urpin (1983) we obtain $\Omega \sim 1$ s^{-1}, $\rho \sim 10^{-3}$ g/cm^3, $H \sim 10^8$ cm at $s = 10^9$ cm. Then, for perturbations with the wavelength $\sim 0.1\,H \sim 10^7$ cm, we have $B_{\rm VS} \sim 10^6$ G and $B_{\rm MS} \sim 10^7$ G. Note that both values are rather strong compared, for example, to the dipole magnetic field of the "standard" neutron star which is $\sim 10^3 - 10^4$ G at this distance. Therefore, it is plausible that $B < B_{\rm VS}$ in a significant fraction of the disc volume, and both instabilities can generally exist.

Compare the growth rates of the instabilities. The growth rate of the magnetic shear instability under the condition (27) and for $B \ll B_{\rm MS}$ is given by

$$\gamma_{\rm MS}^2 \approx -2s\Omega\, \frac{k_z}{k^2}\, \frac{\omega_{\rm A}^2}{Q^2} \left(k_z \frac{\partial\Omega}{\partial s} - k_s \frac{\partial \Omega}{\partial z} \right), \quad (42)$$

where $\omega_{\rm A} = (\boldsymbol{k}\cdot\boldsymbol{B})/\sqrt{4\pi\rho}$ is the Alfvén frequency (see Urpin & Brandenburg 1998). On contrast to the vertical shear instability, $\gamma_{\rm MS}$ is maximal for $k_z \gg k_s$. For such perturbations, we obtain from Eq. (42) the following expression for the growth time in the case of Keplerian rotation

$$\frac{1}{\tau_{\rm MS}} \approx \sqrt{3}\omega_{\rm A}. \quad (43)$$

It is easy to check that everywhere within the range $B < B_{\rm VS}$ where both instability can exist the growth rate of the vertical shear instability is greater.

Since both $B_{\rm MS}$ and $B_{\rm VS}$ depends on the wavevector of perturbations, we can describe the above regimes in terms of the wavelengths as well. In a given magnetic field, B, the condition $B < B_{\rm VS}$ is equivalent to

$$\lambda > \frac{\sqrt{2}\pi}{q} \cdot \frac{c_{\rm A}}{\Omega} \cdot \frac{s}{H}, \quad (44)$$

where $\lambda = 2\pi/k$ is the wavelength and $c_{\rm A} = B/\sqrt{4\pi\rho}$ is the Alfvén velocity. Therefore, the vertical shear instability grows faster for perturbations with the wavelength satisfying the inequality (44). The condition $B_{\rm MS} > B > B_{\rm VS}$ can be rewritten as

$$\frac{\sqrt{2}\pi}{q} \cdot \frac{c_{\rm A}}{\Omega} \cdot \frac{s}{H} > \lambda > \frac{2\pi}{\sqrt{3}} \cdot \frac{c_{\rm A}}{\Omega}. \quad (45)$$

If the wavelength satisfies this condition then only the magnetic shear instability arises but the vertical shear instability is suppressed. If

$$\lambda < \frac{2\pi}{\sqrt{3}} \cdot \frac{c_{\rm A}}{\Omega} \tag{46}$$

then the both instabilities do not exist.

5. Discussion

We have shown that accretion discs always are subject to instability for short wavelength perturbations. This instability is associated with the heat transport and the vertical shear, and can arise at any dependence of the angular velocity on the vertical coordinate, z. The most rapidly growing perturbations in a thin accretion discs have the growth rate of the order of $\Omega|qz/s|$. For these perturbations, the ratio of the vertical and radial components of the wavevector is small, $k_z/k_s \sim z/s$, or, in other words, the radial wavelength has to be approximately a factor s/z shorter than the vertical one. Perturbations arise faster near the disc surface than near the central plane. The growth time of the most unstable perturbations is relatively short and, likely, the considered instability may be a candidate for the origin of turbulence in discs.

We can estimate the turbulent viscosity associated with the vertical shear instability. The turbulent viscosity concept is of limited validity in complex situations and has been rightly criticized (see, e.g., Tennekes & Lumley 1972). In simple shear flows, however, this concept provides more or less accurate scalings of mean flow properties, and it has been widely used to describe turbulent transport in accretion discs. A calculation of the coefficient of viscosity requires generally a numerical solution of hydrodynamic equations with non-linear terms. However, the order of magnitude estimate of turbulence produced by the vertical shear can be obtained by making use the qualitative approach developed by a number of authors for laboratory experiments with shear flows (see, for a detail, Townsend 1958; Zahn 1983).

Consider a stratified inviscid plane Couette flow with the velocity $W(z)$ directed along x-axis and the gravity g against z-axis. As mentioned above, the buoyancy force acts as a stabilizing factor if $\omega_g^2 > 0$. The influence of this force can be characterized by the dimensionless number $R = \omega_g^2/({\rm d}W/{\rm d}z)^2$. The stabilizing influence of buoyancy is strong at $R \geq R_{\rm cr} \sim 1$ and the flow is stable. On the contrary, the stabilization due to buoyancy is negligible at small $R \leq R_{\rm cr} \sim 1$. However, this inference is invalid for dissipative flows where the stabilizing effect of buoyancy can be suppressed by an efficient heat transport. This effect was first estimated by Townsend (1958) who showed that the critical value, $R_{\rm cr}$, is increased by the factor $\omega_\chi/({\rm d}W/{\rm d}z)$ for perturbations with $\omega_\chi > ({\rm d}W/{\rm d}z)$. Due to this, the perturbations with the wavelength satisfying the inequality

$$\frac{\omega_g^2}{({\rm d}W/{\rm d}z)^2} \cdot \frac{({\rm d}W/{\rm d}z)}{\omega_\chi} \leq 1, \tag{47}$$

may be unstable. The turbulent viscosity, $\nu_{\rm T}$, is determined by the largest among the unstable scales satisfying the condition (47), L_m, and can be estimated as (see, e.g., Zahn 1983)

$$\nu_{\rm T} \sim \frac{1}{3} L_m^2 \frac{{\rm d}W}{{\rm d}z}. \tag{48}$$

We will use this estimate of $\nu_{\rm T}$ for turbulence associated with the vertical shear in discs.

Since only perturbations with $k_z/k_s \leq qz/s$ are unstable in accretion discs, the vertical shear instability likely generates a highly anisotropic turbulence. For such turbulence, the eddy viscosity is also anisotropic and, hence, should be represented by a tensor. The angular momentum transfer in the radial direction is determined by the $s\varphi$-component of the viscosity tensor, therefore we will estimate this component alone. Since the growth rate of unstable perturbations with $k < k_{\rm cr}$ decreases rapidly with an increase of the wavelength, the main contribution to the eddy viscosity is likely provided by small scale turbulence with $k > k_{\rm cr}$. By analogy with a plane-parallel flow, the radial component of viscosity can be represented in the form (48) but, in our case, ${\rm d}W/{\rm d}z$ should be replaced by the vertical shear, $s\partial\Omega/\partial z \sim qz\Omega/s^2$, and the lengthscale L_m should be taken equal to the maximal radial wavelength of unstable small scale perturbations (see Eq. (29))

$$L_m \sim \frac{2\pi}{k_{\rm cr}} \sim 2\pi \left(\frac{\chi}{\Omega}\right)^{1/2}. \tag{49}$$

Then, the order of magnitude estimate of $\nu_{\rm T}$ is

$$\nu_{\rm T} \sim \frac{4}{3}\pi^2 q\chi \left(\frac{z}{s}\right). \tag{50}$$

This value of turbulent viscosity is sufficient to provide an effective angular momentum transport in accretion discs. The viscosity depends on both the radial and vertical coordinates and reaches its maximum near the disc surface. Equation (50) is not probably applicable near the equatorial plane where $\partial\Omega/\partial z \approx 0$. At small z, the viscosity can be enhanced due to overshooting by turbulent eddies generated above and below the central plane.

Note that hydrodynamic motions generated by instability will try to redistribute the angular momentum in the disc and smooth the vertical gradient of Ω. The characteristic time scale of this process, $\tau_{\rm inst}$, should obviously be longer than the growth time of instability. For typical conditions of accretion discs, however, $\tau_{\rm inst} > \tau_S$ is longer than both the time scale required to reach hydrostatic equilibrium, $\tau_{\rm h} \sim H/c_{\rm s} \sim \Omega^{-1}$, and the thermal time scale, $\tau_{\rm th} \sim \omega_\chi^{-1}$. Therefore, a z-dependence of all quantities (including Ω) in z-unstable discs is primarily determined by hydrostatic equilibrium and thermal balance but instability leads only to small departures from the equilibrium state. This justifies the choice of a z-dependence in Eq. (17) and argues that instability will lead to disc turbulence rather than self-stabilize the flow by redistributing the angular momentum to the state with $\Omega = \Omega(s)$.

In the present paper, we have addressed the behaviour only of axisymmetric perturbations. Most likely, however, that the obtained results can apply to non-axisymmetric perturbations as well, if the azimuthal wavelength is much longer than the vertical one, $k_z \gg k_\varphi$. Note that, generally, the time behaviour of non-axisymmetric perturbations can be rather complicated. Korycansky (1992) studied non-axisymmetric perturbations in a rotating flow with the radial shear and vertical stratification, taking into account dissipative effects but neglecting the vertical shear. He found that if heat diffusion and viscosity are included, then non-axisymmetric perturbations with some particular dependence on s decay asymptotically after the initial transient growth (even if the disc is convectivelly unstable or the angular momentum gradient is negative but not very large). This conclusion has been obtained by making use of the "shearing sheet" approximation that is quite different from the "eigenfunction" approach used in this paper. The eigenfunctions can grow or decay only exponentially independently whether local or global instabilities are considered. Of course, a superposition of eigenfunctions can exhibit a much more complex behaviour like that treated by Korycansky (1992). As it was correctly recognized by the auther, the ultimate dissipation of any particular sheared perturbations does not imply that an entire wave packet has to decay.

Likely, turbulence produced by the instability considered should be substantially anisotropic since the radial turbulent lengthscale is about $|s/z|$ times shorter than the vertical one for unstable perturbations. In spite of a relatively short radial coherence length, the generated turbulence may be efficient in the radial transport of angular momentum. Our estimates show that in thin accretion discs the turbulent viscosity can be comparable to the radiative thermal diffusivity. Direct numerical simulations of the vertical shear instability in accretion disc and calculations of α and other turbulent parameters will be addressed in a forthcoming paper.

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