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On geodesic motion of particles with zero energy in Kerr and rotating dirty black holes

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Abstract We investigate radial geodesic motion of particles with zero and negative energy in Kerr and rotating dirty black hole spacetimes. The first part of the paper uses a reduced one-dimensional radial equation and decomposes it into formal, coordinate-dependent radial contributions. These terms are useful for comparing the role of the conserved energy in Schwarzschild, Kerr, and stationary, axisymmetric geometries, but they should not be interpreted as covariant forces or as locally measured three-accelerations. The second part switches to an observer-based inertial force formalism inside the ergoregion and analyzes the locally measured gravitational, centrifugal, and Coriolis-type contributions for a chosen family of stationary observers. We show that the gravitational contribution is insensitive to the energy, whereas the rotation-related terms depend on the energy and on the angular momentum. We also prove that circular orbits of zero-energy particles are absent in the considered class of rotating dirty black holes.

1 Introduction

Black holes play an important role in modern theoretical physics and astrophysics. Their vicinity can serve as a natural supercollider for particles [1]. Recent observations by the Event Horizon Telescope Collaboration revealed images of black holes at the centers of the M87 [2] and Milky Way [3] galaxies, and the sizes of the observed dark spots have already been used to constrain the spins of these objects [4]. Analyses of black hole shadows provide information about compact-object geometry and spin; see, for example, the review [5]. The Kerr metric is one of the simplest exact vacuum solutions of the Einstein equations describing a rotating black hole (see [6] for a comprehensive review). Black holes whose exterior geometry is affected by surrounding matter are usually called *dirty* black holes [7]. In the dirty black hole context, one usually focuses on how the surrounding matter modifies the exterior geometry relevant for particle motion and observations. Such effects have been investigated for spherically symmetric [8] and stationary, axisymmetric [9] configurations, for black holes surrounded by cosmological fields [10–12], and for radiating, dynamically evolving spacetimes [13, 14].

In [15], it was shown that the center-of-mass energy of two colliding particles may become arbitrarily large near an extremal Kerr black hole. This effect requires fine-tuning of the particle parameters and has been studied in a series of papers for rotating [16–23], charged [24–27], and magnetized [28] black holes. Although the Banados–Silk–West effect does not require circular orbits, near-horizon turning points and fine-tuned orbital configurations are often useful for realizing large center-of-mass energies. In Schwarzschild spacetime this effect is absent; however, related scenarios have been discussed for Vaidya [29] and Schwarzschild [30] black holes.

In Schwarzschild spacetime circular geodesics of massive particles exist only for $r > 3M$, the limiting case being the photon sphere at $r = 3M$. This is too far from the horizon to support a near-horizon mechanism of the above type. A radial turning point corresponds to $\dot{r} = 0$, that is, to a zero of the radial function introduced below. As the simplest illustration, consider static spherically symmetric metrics described by a single function $f(r)$,

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2. \tag{1.1}$$

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This is not the most general static spherically symmetric line element, but it is enough to make the point. For metrics of the form (1.1) one can write

$$V_{\rm rad}(r) = f(r)\left(\frac{L^2}{r^2}+1\right) - E^2, \tag{1.2}$$

where $E = -u_\mu \xi^\mu_{(t)}$ and $L = u_\mu \xi^\mu_{(\varphi)}$ are the conserved energy and angular momentum per unit mass of a particle with four-velocity u^μ. These quantities are generated by the stationary and the axial Killing vector fields

$$\xi^\mu_{(t)} = (\partial_t)^\mu = \delta^\mu_0, \qquad \xi^\mu_{(\varphi)} = (\partial_\varphi)^\mu = \delta^\mu_3, \tag{1.3}$$

where the coordinates are ordered as $x^\mu = (t, r, \theta, \varphi)$. The notation (1.3) is used throughout the paper; every other Killing vector that appears below is a linear combination of $\xi_{(t)}$ and $\xi_{(\varphi)}$. In this convention the turning point satisfies $V_{\rm rad}(r_{tp}) = 0$. At a horizon with $f(r_h) = 0$, the condition $r_{tp} = r_h$ requires $E = 0$. Thus, in this class of spherically symmetric black holes, a massive particle with zero energy cannot exist outside the event horizon.

The situation changes for rotating black holes. In Kerr spacetime the surface $g_{tt} = 0$ is the static limit surface, or ergosurface, and the ergoregion is the domain between it and the event horizon; the latter is the Killing horizon generated by $\chi^\mu = \xi^\mu_{(t)} + \Omega_H \xi^\mu_{(\varphi)}$, where Ω_H is the angular velocity of the horizon. Inside the ergoregion the stationary Killing vector $\xi_{(t)}$ becomes spacelike. Consequently, owing to frame dragging, there are no timelike worldlines with $\varphi = $ const in this region: every causal trajectory must rotate in the direction of the black hole rotation, $\dot\varphi > 0$ for $a > 0$. This property enables the Penrose process [31]. According to this mechanism, particles with negative energy, defined with respect to infinity, may exist inside the ergoregion as a result of decay or collision processes. The Penrose process has been investigated extensively for rotating [32–41], charged [42–45], and dynamical [29, 46] black holes. It was later shown that particles with zero energy but nonzero momentum may also exist in the ergoregion of a rotating black hole [47, 48].

A word on terminology is in order here. Sections 2, 3, and 5 rewrite the reduced radial equation in a chosen radial coordinate and split it into separate radial terms, while Sect. 4 uses the same equation to rule out circular orbits. This splitting is a coordinate-dependent diagnostic of the radial dynamics. It is not a covariant definition of force and must not be confused with a locally measured three-acceleration. The local, observer-dependent inertial force interpretation is introduced separately in Sect. 6, where a family of stationary observers is chosen explicitly. This distinction follows the standard observer-based treatment of inertial forces in stationary spacetimes [49–56].

In this paper, we investigate timelike geodesics in Kerr and rotating dirty black hole spacetimes in order to determine how the reduced radial equation and the local observer-based inertial force profiles depend on the sign of the energy. We also address the existence of circular orbits for zero-energy particles in the ergoregion of a rotating dirty black hole.

This work is organized as follows. In Sect. 2, we briefly review the reduced radial equation for timelike geodesic motion in Schwarzschild spacetime. In Sect. 3, we analyze the corresponding formal radial terms for equatorial Kerr geodesics. In Sect. 4, we prove the absence of circular orbits for zero-energy particles in a class of stationary, axisymmetric dirty black holes. In Sect. 5, we extend the formal radial decomposition to this class of metrics. In Sect. 6, we switch to the observer-based inertial force formalism in the Kerr ergoregion. Concluding remarks are presented in Sect. 7.

Throughout the paper, we use geometrized units $G = c = 1$, and the metric signature is chosen as $(-, +, +, +)$.

2 Timelike geodesics in Schwarzschild spacetime

The Kerr black hole can be regarded as a rotating generalization of the static, spherically symmetric Schwarzschild black hole. For this reason, before addressing the rotating case, it is useful to recall the reduced radial equation in Schwarzschild spacetime. This provides a reference point for separating terms already present in the non-rotating geometry from terms induced by rotation.

In Schwarzschild coordinates the line element is

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{2.1}$$

where M is the black hole mass and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric on the unit two-sphere. The spacetime is static and spherically symmetric, and its Killing fields (1.3) generate conserved quantities along geodesics.

Spherical symmetry allows us to choose the orbital plane as $\theta = \pi/2$. In that plane, the conserved energy per unit mass and the azimuthal angular momentum per unit mass are

$$E = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}, \qquad L = r^2\frac{d\varphi}{d\tau}. \tag{2.2}$$

Here τ is the proper time of the particle. The sign convention follows from $E = -u_\mu \xi^\mu_{(t)}$ and gives $E > 0$ for future-directed particles outside the horizon.

For timelike motion, $g_{\mu\nu}u^\mu u^\nu = -1$. Combining this normalization with Eq. (2.2) gives the reduced radial equation

$$\left(\frac{dr}{d\tau}\right)^2 + V_{\rm rad} = 0, \qquad V_{\rm rad} = -\frac{2M}{r} + \frac{L^2}{r^2} - \frac{2ML^2}{r^3} + 1 - E^2. \tag{2.3}$$

By construction, $V_{\rm rad}$ contains the constant $1 - E^2$, so that turning points are given by $V_{\rm rad} = 0$, while circular orbits require in addition $dV_{\rm rad}/dr = 0$.

The following decomposition is only a formal rewriting of the radial equation in the chosen coordinate r:

$$a_r^{\rm form} \equiv -\frac{1}{2}\frac{dV_{\rm rad}}{dr} = \frac{d^2r}{d\tau^2}. \tag{2.4}$$

It should not be interpreted as a covariant local force measured by an observer. With this convention the three radial contributions are

$$a_N^{\rm form} = -\frac{1}{2}\frac{d}{dr}\left(-\frac{2M}{r}\right) = -\frac{M}{r^2}, \tag{2.5}$$

$$a_C^{\rm form} = -\frac{1}{2}\frac{d}{dr}\left(\frac{L^2}{r^2}\right) = \frac{L^2}{r^3}, \tag{2.6}$$

$$a_{\rm gr}^{\rm form} = -\frac{1}{2}\frac{d}{dr}\left(-\frac{2ML^2}{r^3}\right) = -\frac{3ML^2}{r^4}. \tag{2.7}$$

The first term reproduces the inverse-square attraction in the weak field limit, the second is the angular momentum barrier in the chosen orbital plane, and the third is the strong field relativistic correction. The subscripts used here and in the rest of the paper indicate the origin of each term: N for the Newtonian-like attraction, C for the centrifugal (angular momentum) barrier, and gr for the general relativistic strong field correction; in the rotating case we also use rot and fd for the contributions generated by rotation and by frame dragging. The superscript form marks all of them as formal, coordinate-dependent contributions rather than locally measured forces.

In the Schwarzschild radial equation, the energy enters only as an integration constant, or equivalently as a constant shift of $V_{\rm rad}$ in Eq. (2.3). Therefore, it does not contribute to $dV_{\rm rad}/dr$. This is the limiting case against which the energy dependence of rotating spacetimes will be compared.

3 Formal radial terms for particles with zero energy in Kerr spacetime

The gravitational field of a rotating black hole is described by the Kerr metric. In Boyer–Lindquist coordinates, which are regular in the stationary region outside the event horizon considered here, the line element takes the form

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mra\sin^2\theta}{\rho^2}dtd\varphi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \\ &+ \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\varphi^2, \end{aligned} \tag{3.1}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2. \tag{3.2}$$

The Kerr metric is stationary and axisymmetric; hence, it admits the Killing fields $\xi^\mu_{(t)}$ and $\xi^\mu_{(\varphi)}$ introduced in Eq. (1.3). It is also asymptotically flat. The Komar integrals confirm that M coincides with the ADM mass, while $J = aM$ represents the total angular momentum.

The static limit surface, or ergosurface, is given by $g_{tt} = 0$, i.e., by the condition that the Killing field $\xi_{(t)}$ becomes null:

$$r_e = M + \sqrt{M^2 - a^2\cos^2\theta}. \tag{3.3}$$

The event and the Cauchy horizons are determined by $\Delta = 0$,

$$r_+ = M + \sqrt{M^2 - a^2}, \qquad r_- = M - \sqrt{M^2 - a^2}, \tag{3.4}$$

where the subscripts + and − label the outer (event) and the inner (Cauchy) horizon, respectively. Throughout the paper we assume $a < M$, so that both horizons exist and are distinct; the extremal case $a = M$ and the over-extremal case $a > M$, which describes a naked singularity, are not considered. The region between r_+ and the static limit surface is the ergoregion. It closes up on the symmetry axis, where $r_e(\theta = 0) = r_+$.

Inside the ergoregion the Killing vector $\xi_{(t)}$ becomes spacelike. Due to frame dragging, no timelike worldline with $\varphi = $ const exists there; in particular, there are no static observers, and every causal trajectory co-rotates with the black hole. This restricts the

admissible worldlines, but it does not make the region inaccessible: timelike trajectories exist at any $r_+ < r < r_e$ as long as they carry angular motion. Negative and zero values of $E = -u_\mu \xi^\mu_{(t)}$ are energies defined with respect to infinity; any admissible local observer measures a positive local energy.

The conserved quantities follow from the Killing symmetries. They are equivalently obtained from the geodesic Lagrangian:

$$E = -u_t = -(g_{00}\dot{t} + g_{03}\dot{\varphi}), \tag{3.5}$$

$$L = u_\varphi = g_{33}\dot{\varphi} + g_{03}\dot{t}. \tag{3.6}$$

Here the dot denotes differentiation with respect to proper time.

Besides E and L, geodesic motion in Kerr spacetime admits a fourth integral, which is generated by the irreducible Killing tensor of Carter [57],

$$K_{\mu\nu} = 2\rho^2 l_{(\mu} n_{\nu)} + r^2 g_{\mu\nu}, \qquad \nabla_{(\lambda} K_{\mu\nu)} = 0, \tag{3.7}$$

where l^μ and n^μ are the two principal null directions of the Kerr spacetime, normalized by $l_\mu n^\mu = -1$. The corresponding conserved quantity is $\mathcal{K} = K_{\mu\nu} u^\mu u^\nu$, and the Carter constant is defined as

$$Q = \mathcal{K} - (L - aE)^2 = u_\theta^2 + \cos^2\theta \left[a^2\left(1 - E^2\right) + \frac{L^2}{\sin^2\theta} \right]. \tag{3.8}$$

We restrict attention to equatorial geodesics, $\theta = \pi/2$ and $u_\theta = 0$, which are sufficient for the present analysis; by Eq. (3.8) this sector corresponds exactly to $Q = 0$, so that $\mathcal{K} = (aE - L)^2$ there. The combination $(aE - L)^2$ that appears below is therefore the equatorial value of the Killing tensor constant, and not a new independent constant.

Solving Eqs. (3.5) and (3.6) gives

$$\dot{t} = \frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2a^2 M}{r} \right) E - \frac{2MaL}{r} \right], \tag{3.9}$$

$$\dot{\varphi} = \frac{1}{\Delta}\left(\frac{2MaE}{r} + L\left(1 - \frac{2M}{r}\right) \right). \tag{3.10}$$

The normalization $g_{\mu\nu} u^\mu u^\nu = -1$ then yields the radial equation

$$\dot{r}^2 = E^2 + \frac{2M}{r^3}(aE - L)^2 + \frac{a^2 E^2}{r^2} - \frac{L^2}{r^2} - \frac{\Delta}{r^2}. \tag{3.11}$$

Introducing $2E_{\rm eff} = E^2 - 1$, this equation can be written as

$$\frac{\dot{r}^2}{2} + V_{\rm rad} = E_{\rm eff}, \qquad V_{\rm rad} = \frac{L^2}{2r^2} - \frac{M}{r} - \frac{M}{r^3}(aE - L)^2 - \frac{a^2}{r^2} E_{\rm eff}. \tag{3.12}$$

With the normalization of Eq. (3.12), differentiation along the trajectory gives $\ddot{r} = -dV_{\rm rad}/dr$, which is the analogue of Eq. (2.4). Again, this is a formal radial decomposition in the Boyer–Lindquist coordinate r, and not a locally measured force.

The first two formal terms reproduce the Schwarzschild limit:

$$a_N^{\rm form} = -\frac{M}{r^2}, \qquad a_C^{\rm form} = \frac{L^2}{r^3}. \tag{3.13}$$

The strong field correction is

$$a_{\rm gr}^{\rm form} = -\frac{3M}{r^4}(aE - L)^2. \tag{3.14}$$

In the static limit $a = 0$, it reduces to the Schwarzschild expression (2.7). The same functional form is recovered for $E = 0$, simply because a and E enter only through the product aE. The whole energy dependence of this term therefore comes from the frame-dragging combination $aE - L$.

The remaining term is rotation induced and disappears when $a = 0$:

$$a_{\rm rot}^{\rm form} = -\frac{2a^2 E_{\rm eff}}{r^3}. \tag{3.15}$$

It should not be called a locally measured Coriolis force: it is a formal radial contribution induced by frame dragging in the reduced equation. It vanishes for $E = \pm 1$; for $|E| < 1$ one has $E_{\rm eff} < 0$, so its sign is opposite to the case $|E| > 1$.

Thus, in Kerr spacetime the reduced radial equation contains terms depending explicitly on E and L. This feature becomes essential in the ergoregion, where particles with zero and negative energies are allowed.

4 Geodesics for particles with zero energy in stationary, axisymmetric spacetime

As shown in Sect. 1, a particle with $E = 0$ cannot be located outside the horizon of a static spherically symmetric black hole, whereas inside the ergoregion of a rotating black hole particles with $E \leq 0$ are allowed. For Kerr spacetime, it is known that such particles cannot move along circular orbits inside the ergoregion [37, 47, 48]. The purpose of this section is to show that the same conclusion holds for a considerably broader class of rotating dirty black holes.

Let the metric be written as

$$ds^2 = -N^2 dt^2 + R^2(d\varphi - \omega dt)^2 + A^{-1}dr^2. \tag{4.1}$$

We work in the equatorial plane, so that the metric functions N, R, ω and A depend on r only. The spacetime is stationary and axisymmetric, and the coordinates in Eq. (4.1) are adapted to the Killing fields (1.3): t and φ are the parameters along their orbits, so that the metric functions do not depend on them and E and L are conserved. We consider the class of black holes in which the lapse N and the radial function A vanish simultaneously at the event horizon, with A/N^2 finite and nonzero there [37]; the outer horizon is then located at the largest relevant root of $N = 0$.

For equatorial Kerr, the metric functions obey

$$R^2\omega^2 - N^2 = -1 + \frac{2M}{r}, \qquad \omega = \frac{2Mar}{(r^2+a^2)^2 - a^2\Delta}, \qquad R^2 = r^2 + a^2 + \frac{2Ma^2}{r}. \tag{4.2}$$

The ergoregion is characterized by $g_{tt} = R^2\omega^2 - N^2 > 0$, or equivalently $N^2 < \omega^2 R^2$. These functions also illustrate the near-horizon behavior assumed above: Eqs. (4.2) give $N^2 = \Delta/R^2$, while the radial function is $A = \Delta/r^2$, so that both vanish exactly at $\Delta = 0$, that is, at $r = r_+$, whereas $A/N^2 = R^2/r^2$ stays finite and nonzero.

Using the notation of [37], the first integrals are

$$\dot{t} = \frac{X}{N^2}, \qquad \dot{\varphi} = \frac{L}{R^2} + \frac{\omega X}{N^2}, \qquad X = E - \omega L. \tag{4.3}$$

The normalization condition gives

$$\frac{\dot{r}^2}{2} + V_{\rm rad} = 0, \qquad V_{\rm rad} = -\frac{A}{2N^2}Z^2, \qquad Z^2 = X^2 - N^2 - \frac{N^2}{R^2}L^2. \tag{4.4}$$

A circular orbit requires

$$Z^2 = 0, \qquad (Z^2)' = 0, \tag{4.5}$$

where a prime denotes d/dr and the prefactor A/N^2 in Eq. (4.4) is assumed to be regular and nonzero. For $E = 0$ and the forward-in-time condition $\dot{t} > 0$, one obtains

$$X = -\omega L, \qquad L \leq 0. \tag{4.6}$$

The value $L = 0$ is excluded as well, since it gives $X = 0$ and $Z^2 = -N^2 < 0$, which is incompatible with $\dot{r}^2 \geq 0$. A zero-energy particle thus has strictly negative angular momentum, while frame dragging still makes it move in the direction of the black hole rotation. The circular orbit conditions then lead to

$$L_c = -\frac{N}{\sqrt{\omega^2 - N^2/R^2}}, \tag{4.7}$$

$$(Z^2)' = \frac{N^2}{\omega^2 - N^2/R^2}\left(\omega^2 - \frac{N^2}{R^2}\right)' - 2NN'. \tag{4.8}$$

Introducing $\gamma = \omega^2 R^2 - N^2$, one may write

$$(Z^2)' = N^2\left[\ln\left(\frac{\gamma}{N^2R^2}\right)\right]'. \tag{4.9}$$

For $E = 0$ one has $\omega R > N$ and hence $\gamma > 0$. Under the monotonicity assumptions used in [37], in particular $N' \geq 0$, $R' \geq 0$, and the decreasing behavior of ωR in the relevant region, one has $\gamma' < 0$. Hence $[\ln(\gamma/(N^2R^2))]' < 0$, and therefore $(Z^2)'$ cannot vanish at the candidate point. Thus, circular orbits for zero-energy particles are absent in this class of rotating dirty black hole spacetimes.

This extends the corresponding statement for negative energy particles [37]: under the stated assumptions, a particle with $E \leq 0$ can stay in the ergoregion only on a non-circular trajectory with a turning point.

5 Formal radial terms in stationary, axisymmetric spacetime

We now extend the coordinate-dependent radial decomposition to the metric (4.1). This section still concerns the reduced radial equation in the chosen coordinate r; it does not define locally measured forces. The local observer-based interpretation is postponed to Sect. 6.

Near the horizon, we assume the regular behavior $A \sim N^2$. It is convenient to fix the radial coordinate by requiring $A = N^2$ exactly; outside the horizon this can always be done, since the reparametrization $d\tilde{r} = (N/\sqrt{A})\,dr$ is a simple quadrature. With this choice Eq. (4.4) becomes

$$\frac{\dot{r}^2}{2} + \mathcal{V}_{\rm rad} = 0, \qquad \mathcal{V}_{\rm rad} = -\frac{1}{2}\left(X^2 - N^2 - \frac{N^2L^2}{R^2}\right). \tag{5.1}$$

Differentiating this equation along the radial motion gives the formal radial acceleration

$$\ddot{r} = -\mathcal{V}'_{\rm rad} = -X\omega' L - NN' + L^2\left(\frac{N^2R' - NN'R}{R^3}\right), \tag{5.2}$$

where we used $X' = -\omega' L$, which follows from the constancy of E and L.

For the class of metrics parametrized by

$$N^2 = 1 - \frac{2M(r)}{R}, \tag{5.3}$$

we have

$$NN' = -\frac{M'R - MR'}{R^2}. \tag{5.4}$$

Substitution into Eq. (5.2) yields

$$\ddot{r} = \underbrace{-X\omega' L}_{\rm I} + \underbrace{\frac{M'R - MR'}{R^2}}_{\rm II} + \underbrace{\frac{L^2R'}{R^3}}_{\rm III} - \underbrace{\frac{L^2(3MR' - M'R)}{R^4}}_{\rm IV}. \tag{5.5}$$

The factor L^2 in the last term is essential. In the Schwarzschild limit $\omega = 0$, $R = r$, $M' = 0$, Eq. (5.5) reduces to

$$\ddot{r} = -\frac{M}{r^2} + \frac{L^2}{r^3} - \frac{3ML^2}{r^4}, \tag{5.6}$$

which is exactly the result obtained in Sect. 2.

The terms in Eq. (5.5) can therefore be classified formally as follows:

$$a_{\rm fd}^{\rm form} = -X\omega' L, \tag{5.7}$$

$$a_N^{\rm form} = \frac{M'R - MR'}{R^2}, \tag{5.8}$$

$$a_C^{\rm form} = \frac{L^2R'}{R^3}, \tag{5.9}$$

$$a_{\rm gr}^{\rm form} = -\frac{L^2(3MR' - M'R)}{R^4}. \tag{5.10}$$

The subscripts carry the same meaning as in Sect. 2. The first term is generated by frame dragging and vanishes whenever $\omega = \text{const}$, in particular in the static limit; the second generalizes the Newtonian attraction; the third is the centrifugal angular momentum barrier; and the fourth is the general relativistic strong field correction. As before, these names describe the role each term plays in the reduced radial equation, and not invariant local forces.

6 Observer based inertial forces in the ergoregion

We now switch from the formal radial decomposition of Sects. 2–5 to a local, observer-based inertial force formalism. In this section, the three-dimensional quantities are defined only after a stationary observer congruence has been chosen; they are therefore physically different from the coordinate-dependent radial terms used above.

The introduction of three-dimensional kinematical quantities in general relativity is essentially observer dependent. In particular, the notions of three-velocity and three-acceleration acquire a precise physical meaning only after a specific family of observers has been chosen. In this work, we are interested in the properties of particles with zero and negative energy with respect to infinity, $E \leq 0$, which play a fundamental role in rotational energy extraction mechanisms such as the Penrose process. A consistent description

of the dynamics of such particles therefore requires a reference frame that remains well defined inside the ergoregion of a Kerr black hole. In the Kerr spacetime with signature $(-+++)$, the metric component g_{tt} becomes positive inside the ergoregion, and the stationary Killing vector $\xi_{(t)}$ ceases to be timelike. As a consequence, observers with static worldlines ($d\varphi = 0$) do not exist in this region, and a direct construction of three-dimensional quantities with respect to static observers is impossible. Nevertheless, this does not preclude the introduction of physically meaningful three-dimensional quantities within the ergoregion. One may instead consider families of admissible stationary observers whose worldlines remain timelike, such as zero angular momentum observers (ZAMO) [58] or stationary observers rotating with a constant angular velocity Ω. The choice of Ω observers is particularly convenient, since it preserves stationarity while explicitly incorporating the effects of frame dragging, which are essential for understanding the dynamics of particles with $E \leq 0$. We consider a family of stationary observers associated with the Killing combination

$$\zeta^{\mu} = \xi^{\mu}_{(t)} + \Omega\, \xi^{\mu}_{(\varphi)}, \qquad \Omega = \text{const}, \tag{6.1}$$

where $\xi_{(t)}$ and $\xi_{(\varphi)}$ are the Killing fields (1.3) and Ω is a free parameter; the horizon generator χ of Sect. 1 is the particular member of this family with $\Omega = \Omega_H$. Equivalently, one can introduce rotating coordinates $\tilde{\varphi} = \varphi - \Omega t$, so that

$$\tilde{g}_{tt} = g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi} \equiv -N_{\Omega}, \qquad \tilde{g}_{t\tilde{\varphi}} = g_{t\varphi} + \Omega g_{\varphi\varphi} \equiv \mathcal{A}, \qquad \tilde{g}_{\tilde{\varphi}\tilde{\varphi}} = g_{\varphi\varphi}. \tag{6.2}$$

Here $\sqrt{N_{\Omega}}$ is the lapse of the Ω observers, so that $N_{\Omega} > 0$ exactly where their worldlines are timelike, and $\mathcal{A}$ is the off-diagonal (shift) component of the metric in the rotating frame. The admissible range of Ω for fixed $(r,\, \theta)$ is determined by the light surfaces $N_{\Omega} = 0$:

$$\Omega_{-} < \Omega < \Omega_{+}, \qquad \Omega_{\pm} = \frac{-g_{t\varphi} \pm \sqrt{\mathcal{D}}}{g_{\varphi\varphi}}, \qquad \mathcal{D} \equiv g^2_{t\varphi} - g_{tt} g_{\varphi\varphi}. \tag{6.3}$$

The integrals of motion of a particle with respect to infinity are defined by the Killing vectors, $E = -u_t$ and $L = u_{\varphi}$, i.e., by the same per unit mass quantities as in Sects. 2–5. When passing to t, $\tilde{\varphi}$ coordinates, the covariant components of the four-velocity transform as $u_{\tilde{t}} = u_t + \Omega u_{\varphi}$, $u_{\tilde{\varphi}} = u_{\varphi}$, so that the energy with respect to the N_{Ω}-time is

$$E_{\Omega} \equiv -u \cdot \zeta = E - \Omega L. \tag{6.4}$$

At the same time, in the Penrose process the relevant quantity is E, which in the ergoregion may satisfy $E \leq 0$. The four-velocity of an Ω observer is

$$U^{\mu} = \frac{1}{\sqrt{N_{\Omega}}}(1, 0, 0, \Omega). \tag{6.5}$$

The energy of a particle of mass m measured locally by this observer is then

$$E_{\text{loc}} = -m\, u \cdot U = m\gamma, \qquad \gamma \equiv -u \cdot U = \frac{E - \Omega L}{\sqrt{N_{\Omega}}} > 0. \tag{6.6}$$

To introduce observable three-dimensional quantities, we use the orthonormal basis (tetrad) of the observer. The radial axis is $e^{\mu}_{(\hat{r})} = (0,\, 1/\sqrt{g_{rr}},\, 0,\, 0)$. The azimuthal axis is constructed by projecting $\xi^{\mu}_{(\varphi)}$ into the local three-space of the observer: $w^{\mu} = \xi^{\mu}_{(\varphi)} + (U \cdot \xi_{(\varphi)})U^{\mu}$, $e^{\mu}_{(\hat{\varphi})} = w^{\mu}/\sqrt{w \cdot w}$. For $\mathcal{A} = g_{t\varphi} + \Omega g_{\varphi\varphi}$ one finds

$$e^{t}_{(\hat{\varphi})} = \frac{\mathcal{A}}{\sqrt{N_{\Omega}(N_{\Omega} g_{\varphi\varphi} + \mathcal{A}^2)}}, \qquad e^{\varphi}_{(\hat{\varphi})} = \frac{N_{\Omega} + \Omega\mathcal{A}}{\sqrt{N_{\Omega}(N_{\Omega} g_{\varphi\varphi} + \mathcal{A}^2)}}. \tag{6.7}$$

The local components of the particle four-velocity are $u^{(\hat{0})} = \gamma$, $u^{(\hat{i})} = u_{\mu} e^{\mu}_{(\hat{i})}$, and the measured three-velocity is

$$v^{(\hat{i})} = \frac{u^{(\hat{i})}}{u^{(\hat{0})}} = \frac{u^{(\hat{i})}}{\gamma}. \tag{6.8}$$

The decomposition of the particle acceleration into gravitational, Coriolis, and centrifugal contributions is performed within the standard inertial force formalism for stationary spacetimes [49–53]:

$$a^{\hat{r}}_{\text{sum}} = a^{\hat{r}}_{\text{grav}} + a^{\hat{r}}_{\text{cor}} + a^{\hat{r}}_{\text{cent}}. \tag{6.9}$$

With our notation, it is convenient to use the derivative along the physical radial direction $\partial_{\hat{r}} = (1/\sqrt{g_{rr}})\, d/dr$. The gravitational contribution is

$$a^{\hat{r}}_{\text{grav}} = -\partial_{\hat{r}} \ln\sqrt{N_{\Omega}} = -\frac{1}{\sqrt{g_{rr}}} \frac{d}{dr} \ln\sqrt{N_{\Omega}}. \tag{6.10}$$

The centrifugal contribution is

$$a^{\hat{r}}_{\text{cent}} = \kappa^{\hat{r}}\, (v^{(\hat{\varphi})})^2, \qquad \kappa^{\hat{r}} = \frac{1}{2}\partial_{\hat{r}} \ln \rho^2_{\Omega}, \qquad \rho^2_{\Omega} = g_{\varphi\varphi} + \frac{\mathcal{A}^2}{N_{\Omega}}, \tag{6.11}$$

where ρ_Ω is the gyration radius of the orbit: by construction $\rho_\Omega^2 = w \cdot w$, so that $2\pi\rho_\Omega$ is the proper circumference of the circle r, $\theta = \text{const}$ measured in the local rest space of the observer, while $\kappa^{\hat{r}}$ is the curvature that enters the centrifugal term [49, 53]. On the equatorial plane, where $\mathcal{D} = g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi} = \Delta$, one has $\rho_\Omega^2 = \Delta/N_\Omega$. The Coriolis contribution (in our conventions) reads

$$a^{\hat{r}}_{\rm cor} = v^{(\hat{\varphi})}\, H_{\hat{\theta}}, \qquad H_{\hat{\theta}} = \frac{\sqrt{N_\Omega}}{r}\frac{d}{dr}\left(\frac{\mathcal{A}}{N_\Omega}\right). \tag{6.12}$$

The precise numerical coefficients in the Coriolis term depend on the adopted convention for defining the gravitomagnetic field H; all numerical plots presented below were produced using exactly the form given in Eqs. (6.10)–(6.12). Because of the rather cumbersome analytic expressions, it is convenient to analyze the behavior of inertial forces by constructing numerical profiles of the radial acceleration components $a^{\hat{r}}$ as functions of the radial coordinate r.

An observer rotating with a fixed dimensionless angular velocity $\Omega^* = \Omega M$ remains timelike between the inner and the outer light surfaces, on which $N_\Omega = 0$. Such an observer penetrates the ergoregion down to the inner light surface $r^{\rm in}_{\rm LS}(\Omega)$, so that the radial interval accessible inside the ergoregion is $r^{\rm in}_{\rm LS}(\Omega) < r < 2M$. To enable a meaningful comparison of accelerations inside the ergoregion, the particle parameters are chosen such that the trajectories pass through the same radial region and simultaneously satisfy the condition of radial motion $(u^r)^2 \geq 0$. For this purpose, we fix the Lorentz factor at a reference point r_* located inside the ergoregion, $r^{\rm in}_{\rm LS}(\Omega) < r_* < 2M$, and impose the condition $\gamma(r_*) = \gamma_*$. This determines the corresponding angular momentum as a function of the particle energy,

$$L(E,\Omega) = \frac{1}{\Omega}\Big[E - \gamma_*\sqrt{N_\Omega(r_*)}\Big], \tag{6.13}$$

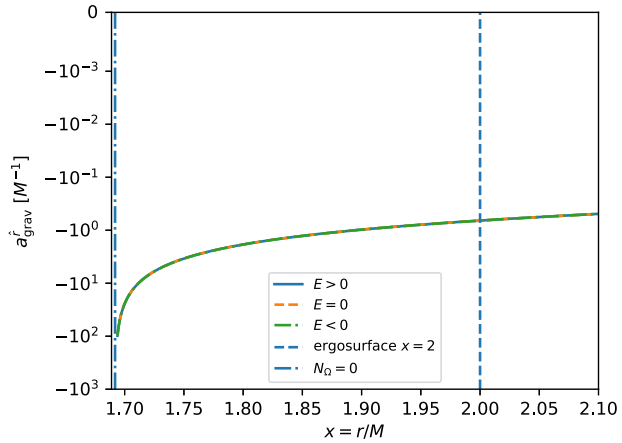
so that particles with $E > 0$, $E = 0$, and $E < 0$ possess the same degree of local relativistic motion at the chosen reference radius. The comparison of acceleration profiles reveals several characteristic features. First, the gravitational (gravitoelectric) contribution $a^{\hat{r}}_{\rm grav}$ (Fig. 1a) is completely insensitive to the particle energy, which is consistent with the formal radial limit of Sects. 2–5, now expressed in local observer-based quantities. In contrast, the rotation-related terms exhibit a pronounced dependence on the particle energy and angular momentum. They do so, however, in different ways. The sign of the centrifugal term $a^{\hat{r}}_{\rm cent} = \kappa^{\hat{r}}(v^{(\hat{\varphi})})^2$ is fixed by $\kappa^{\hat{r}}$ alone and is therefore the same for all three trajectories, the particle entering only through the magnitude of $v^{(\hat{\varphi})}$. Throughout the region considered here $\kappa^{\hat{r}} < 0$, so that this term is directed toward the black hole and adds to the gravitational attraction instead of providing an outward repulsive effect. It does reverse, at the minimum of the gyration radius ρ_Ω [59], but for the parameters of Fig. 1 that radius lies at $x \simeq 2.11$, outside the ergoregion and hence outside the domain of interest. The Coriolis term $a^{\hat{r}}_{\rm cor} = v^{(\hat{\varphi})}H_{\hat{\theta}}$ is linear in $v^{(\hat{\varphi})}$ and behaves differently: its sign follows the sense in which the particle moves relative to the observer, and it is here that particles with $E > 0$ and $E \leq 0$ differ qualitatively. For particles with positive energy $E > 0$, the radial acceleration varies smoothly with radius and becomes ill defined only in the immediate vicinity of the surface $N_\Omega = 0$, where the observer congruence itself becomes lightlike. For particles with $E \leq 0$, on the other hand, the azimuthal velocity measured by the Ω observers changes sign inside the ergoregion, at $x \simeq 1.80$ for $E = 0$ and at $x \simeq 1.75$ for $E < 0$. The Coriolis term reverses there, whereas $a^{\hat{r}}_{\rm cent}$, being quadratic in $v^{(\hat{\varphi})}$, only touches zero and stays directed inwards; the narrow spikes in Fig. 1b are these zeros. The reason is that such particles have $L < 0$: frame dragging still makes them co-rotate with the black hole, $\dot{\varphi} > 0$, but relative to observers rotating with constant Ω their motion is retrograde [32]. We stress that the radial domain shown in Fig. 1 is bounded from below by the inner light surface of the chosen observer family, and not by the event horizon. The profiles therefore describe the local kinematics in this domain only. Whether a given trajectory eventually crosses the horizon cannot be decided in this frame; it would require an observer family that stays regular down to the horizon, such as ZAMOs, or a direct integration of the geodesic equations.

It is instructive to compare these locally measured quantities with the formal radial terms of Sect. 2 in a case where both can be written in closed form. For $a = 0$ and $\Omega = 0$ the Ω observers reduce to the static observers of Schwarzschild spacetime, for which $N_\Omega = f \equiv 1 - 2M/r$, $g_{rr} = f^{-1}$, $\mathcal{A} = 0$ and $\rho_\Omega^2 = r^2$. In this limit Eqs. (6.10)–(6.12) give

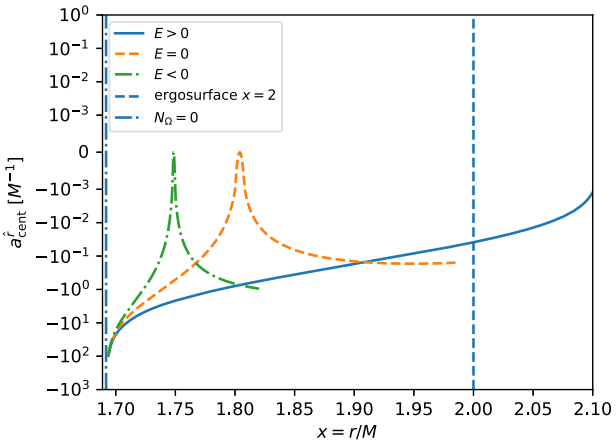
$$a^{\hat{r}}_{\rm grav} = -\frac{M}{r^2\sqrt{f}}, \qquad a^{\hat{r}}_{\rm cent} = \frac{\sqrt{f}}{r}\Big(v^{(\hat{\varphi})}\Big)^2 = \frac{L^2}{r^3}\frac{f^{3/2}}{E^2}, \qquad a^{\hat{r}}_{\rm cor} = 0, \tag{6.14}$$

where $v^{(\hat{\varphi})} = L\sqrt{f}/rE$ is the orbital velocity measured by the static observer and the Coriolis term vanishes because $\mathcal{A} = 0$ in a static spacetime. The comparison with Eqs. (2.5)–(2.7) is now explicit. The two descriptions agree only in the weak field, slow motion limit $f \to 1$, $E \to 1$, where $a^{\hat{r}}_{\rm grav} \to a_N^{\rm form} = -M/r^2$ and $a^{\hat{r}}_{\rm cent} \to a_C^{\rm form} = L^2/r^3$. Elsewhere, they differ in an essential way: $a^{\hat{r}}_{\rm grav}$ diverges as $r \to 2M$, because the static observers themselves cease to exist there, whereas $a_N^{\rm form}$ remains finite; the locally measured centrifugal term carries the energy dependence contained in $\gamma = E/\sqrt{f}$, whereas $a_C^{\rm form}$ does not; and the strong field term $a_{\rm gr}^{\rm form} = -3ML^2/r^4$ has no separate counterpart in Eq. (6.14) at all, since the observer-based decomposition of the radial acceleration contains only two terms in this limit.

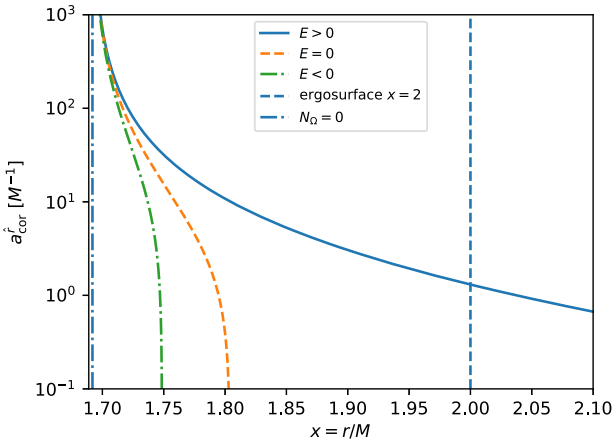
The two descriptions nevertheless agree on physical results, as they must. For a circular geodesic the contributions of the radial motion drop out, and the balance $a^{\hat{r}}_{\rm grav} + a^{\hat{r}}_{\rm cent} = 0$ gives $v^2 = M/(r-2M)$ for the locally measured orbital speed, while $a_N^{\rm form} + a_C^{\rm form} + a_{\rm gr}^{\rm form} = 0$ gives $L^2 = Mr^2/(r-3M)$. Together with $E^2 = (r-2M)^2/(r^2-3Mr)$ and $v^{(\hat{\varphi})} = L\sqrt{f}/rE$, the two conditions are identical, and both reproduce the photon sphere as the limiting radius $r \to 3M$, where $v \to 1$ and $L \to \infty$. This is



(a) Gravitational acceleration $a^{\hat{r}}_{\rm grav}$



(b) Centrifugal acceleration $a^{\hat{r}}_{\rm cent}$



(c) Coriolis acceleration $a^{\hat{r}}_{\rm cor}$

Fig. 1 Radial components of the inertial force decomposition inside the equatorial ergoregion of a Kerr black hole: $a^{\hat{r}}_{\rm grav}$, $a^{\hat{r}}_{\rm cent}$, and $a^{\hat{r}}_{\rm cor}$, evaluated with respect to the family of Ω observers. The Killing energies per unit mass are $E = 1.2,\ 0,\ -1.2$, with the corresponding angular momenta $L(E)$ chosen such that the Lorentz factor at $x_* = r_*/M$ satisfies $\gamma(x_*) = \gamma_*$. Parameters: $\Omega M = 0.1$, $a/M = 0.98$, $x_* = 1.85$, and $\gamma_* = 4$. The plotted radial interval is restricted by the inner light surface $x^{\rm in}_{\rm LS}$, where $N_\Omega = 0$, and by the equatorial static limit $x_{\rm ergo} = 2$, with $x^{\rm in}_{\rm LS} \simeq 1.692$

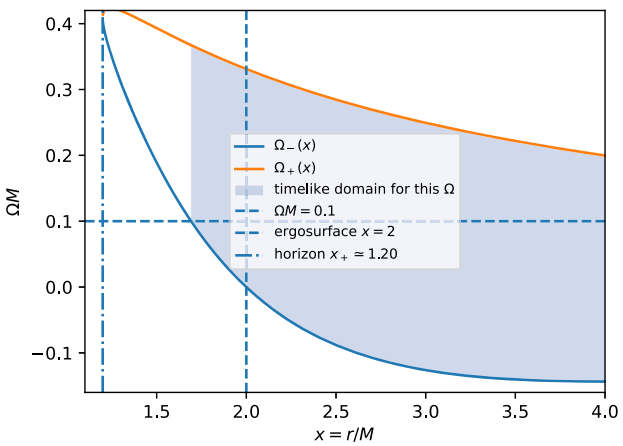


Fig. 2 Allowed angular velocities Ω of stationary observers in the Kerr spacetime as functions of the radial coordinate $x = r/M$ on the equatorial plane. The curves $\Omega_\pm(x)$ correspond to the light surfaces defined by $\zeta^\mu \zeta_\mu = 0$, where ζ is the Killing combination (6.1). For a fixed value of Ω (horizontal dashed line), the observer congruence is timelike only in the region $\Omega_-(x) \le \Omega \le \Omega_+(x)$. The vertical dashed lines indicate the event horizon x_+ and the equatorial static limit $x = 2$. The shaded region shows the domain where Ω observers can exist

precisely the status we ascribe to the terms of Sects. 2, 3, and 5: they are a convenient bookkeeping of the reduced radial equation in the coordinate r, useful for tracing where the Killing energy enters, whereas only the quantities (6.10)–(6.12), defined with respect to an explicitly chosen observer congruence, are locally measurable (Fig. 2).

7 Conclusion

In this work, we have analyzed geodesic motion of particles with zero energy in Kerr and rotating dirty black hole spacetimes. The formulation separates two levels of description. Sections 2, 3, and 5 split a reduced one-dimensional radial equation into formal, coordinate-dependent radial terms, while Sect. 4 uses the same equation to exclude circular orbits. Section 6 uses an observer-based inertial force formalism, in which local three-dimensional quantities are defined only with respect to a chosen stationary observer congruence.

For the Schwarzschild limit, the derivative of the radial function contains the Newtonian attraction, the angular momentum barrier, and the strong field relativistic correction. The conserved energy enters only as a constant in the radial equation and therefore does not affect these formal radial terms. In Kerr spacetime, the reduced radial equation contains additional rotation-dependent combinations such as $(aE - L)^2$ and $a^2 E_{\text{eff}}$, so the sign and the magnitude of the energy become dynamically relevant in the formal radial decomposition.

For the class of stationary, axisymmetric dirty black holes considered here, the radial decomposition obtained above reduces to the Schwarzschild result in the static limit. In particular, the strong field term necessarily carries the factor L^2, and the centrifugal denominator is R^3, as required by the Schwarzschild limit.

A robust qualitative result is the absence of circular orbits for zero-energy particles in the ergoregion under the stated monotonicity assumptions. Together with previous results for negative energy particles, this shows that a particle with zero or negative Killing energy cannot stay on a circular orbit in the considered rotating dirty black hole geometries.

The observer-based analysis in Sect. 6 clarifies which features survive as locally measured inertial force profiles. The locally measured gravitational contribution is insensitive to the particle Killing energy for the selected observer family, whereas the rotation-related Coriolis and centrifugal contributions depend on the energy and on the angular momentum. This distinction is essential: the paper does not claim that the formal radial terms of Sects. 2–5 are covariant physical forces.

These results are relevant for the Penrose process, near-horizon collisions, and possible observational modeling of particle dynamics in the ergoregion. A quantitative connection to black hole shadow observables would require a separate ray tracing or kinetic calculation and is not established by the present radial analysis alone.

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Data availability No Data associated in the manuscript.

Declarations

Conflict of interest The authors declare that there are no conflict of interest or conflict of interest.

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