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Thermal and electrical conductivity in white dwarfs and neutron stars

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Simple relativistic expressions are derived for the electron contribution to the thermal and electrical conductivities κ , σ in the degenerate core of a white dwarf and the degenerate layers of a neutron-star envelope (in which the density $\rho < 4 \times 10^{11}$ g/cm³). At temperatures T above the crystallization temperature T_M of the ions (the crystal melting point), the main factor governing κ and σ will be the scattering of electrons by ions; for $T < T_M$ the dominant process will be scattering by crystal lattice vibrations (phonons) or, if T is low enough, by impurities in the crystal structure. In the ionic-scattering case, κ and σ can be simply expressed in terms of the Coulomb logarithm, for which a convenient interpolation formula is obtained. For phonon scattering, unified analytic expressions are derived to describe the behavior of κ , σ at temperatures both above and below the Debye temperature Θ . This lattice-type scattering will be produced by acoustic phonons through *Umklapp* processes, and as a result when $T \ll \Theta$ the Wiedemann-Franz rule will qualitatively remain valid.

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1. INTRODUCTION

1. When studying many of the processes that take place in white dwarfs and neutron stars, such as the cooling of the star, the evolution of its magnetic field, or nuclear burning of accreted matter, it is important to have accurate values for the thermal conductivity as well as the electrical conductivity of the stellar material.

In this paper we shall calculate the electron thermal and electrical conductivities of the degenerate layers of neutron-star envelopes and the degenerate cores of white dwarfs. Matter in these regions consists chiefly of practically free degenerate electrons and nondegenerate, non-relativistic, fully ionized ions of charge $Z|e|$ and atomic number A . Heat and charge are transported there primarily by electrons. The density ρ and temperature T vary within the ranges

$$\begin{aligned} Z^2 \text{ g/cm}^3 < \rho < 4 \cdot 10^{11} \text{ g/cm}^3 \quad T < T_F, \\ T_F = \frac{\mu - mc^2}{k_B} = \frac{5.93 \cdot 10^9 (\rho_6/\mu_e)^{2/3}}{1 + [1 + (\rho_6/\mu_e)^{2/3}]^{1/2}} \text{ } ^\circ\text{K}. \end{aligned} \quad (1)$$

Here $\mu = m_* c^2 = (m^2 c^4 + c^2 p_F^2)^{1/2}$ represents the Fermi energy of electrons having a number density n_e , $p_F = \hbar (3\pi^2 n_e)^{1/3} \approx mc (\rho_6/\mu_e)^{1/3}$ denotes the Fermi momentum,

k_B is the Boltzmann constant, $\rho_6 \equiv \rho / (10^6 \text{ g/cm}^3)$, and $\mu_e = A/Z$. At densities $\rho \sim Z^2/\text{g/cm}^3$ the condition $Ze^2/a \ll k_B T_F$ for an ideal electron gas will be violated [$a = (3\pi n_i/4)^{1/3}$, with n_i the ion number density], while at densities $\rho > 4 \cdot 10^{11} \text{ g/cm}^3$ free neutrons¹ or perhaps pions (see, for example, Saakyan and Grigoryan²) will be formed in the electron gas.

The state of the ions is specified by the gas parameter $\Gamma = Z^2 e^2 / k_B T a$. If $\Gamma < 1$, the ions will constitute a Boltzmann gas; for Γ in the range $1 < \Gamma < \Gamma_M$, a liquid³; and if $\Gamma > \Gamma_M \approx 50$, they will form a crystal (Ref. 4). The crystallization temperature is given by

$$T_M = Z^2 e^2 / a k_B \Gamma_M = 1.51 \cdot 10^9 Z^{2/3} (\rho_6/\mu_e)^{1/3} (150/\Gamma_M) \text{ } ^\circ\text{K}, \quad (2)$$

while the Debye temperature of the crystal is⁵

$$\begin{aligned} \Theta &= 0.45 \hbar \omega_p k_B^{-1} = 2.4 \cdot 10^8 (\rho_6/\mu_e)^{1/3} (2/\mu_e)^{1/2} \text{ } ^\circ\text{K}, \\ \omega_p &= (4\pi Z^2 e^2 n_i / m_i)^{1/2}, \end{aligned} \quad (3)$$

where ω_p denotes the ion plasma frequency and m_i is the mass of the ions.

Figure 1 shows the temperatures defined by Eqs. (1)-(3) as functions of ρ for the case of an equilibrium nuclear composition¹ ($Z \gtrsim 26$). In this case the melting

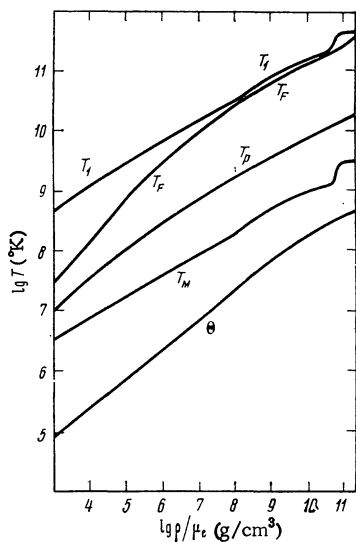


FIG. 1. The several characteristic temperatures of cool, dense matter with an equilibrium nuclear composition,¹ as functions of ρ/μ_e . T_F , Fermi temperature; T_M , crystallization temperature of ions; Θ , Debye temperature of crystal; T_i , temperature corresponding to a gas parameter $T_p = \hbar\Omega_p/k_B$ (Ω_p is the electron plasma frequency).

point $T_M > \Theta$, and for $T < T_F$ we will have $\Gamma < 1$. However, for matter with smaller values of Z (again for $T < T_F$) we can have $\Gamma < 1$. If Z is small and ρ is large, it is also possible (see, for example, D. Q. Lamb and Van Horn⁶) to have $\Theta > T_M$. The ions will then remain in liquid form even for $T < T_M$.

We would point out here that the influence of the magnetic field on the thermal and electrical conductivity is neglected in this paper. It is not difficult, however, to generalize the results to the case of a nonquantizing magnetic field, and we have done so in a separate paper.⁷

The thermal and electrical conductivity of metals on the earth have been studied very thoroughly (see, for example, Ziman's book⁸). But these results cannot be applied directly to astrophysical conditions because, in the first place, in astrophysics one must deal with relativistic electrons, and furthermore, the properties of astrophysical crystals are highly distinctive. In such crystals the electrons will be scattered by acoustic phonons through Umklapp processes – not at all what happens in ordinary terrestrial crystals (see Sec. 4).

The first authors to investigate the electron thermal conductivity κ and electrical conductivity σ under astrophysical conditions (for scattering of electrons by ions in the case $T > T_M$) were Marshak,⁹ in 1941, and Mestel¹⁰ and Lee,¹¹ in 1950. Lee obtained simple, relativistic equations expressing κ and σ in terms of the "Coulomb logarithm" (see Sec. 3), although the logarithm itself was not accurately evaluated.

Subsequent calculations^{12–17} of κ and σ for various values of Γ have on the whole been carried out numerically, and the results are presented in far more cumbersome form.¹ In Sec. 3 we shall demonstrate that actually these results can conveniently be expressed in terms of the Coulomb logarithm, and we shall obtain a simple interpolation formula for it.

Even less satisfactory are the calculations that have been made for the scattering of electrons from lattice vibrations (if $T < T_M$). Such calculations have been performed for $T < \Theta$ by several authors^{14,15,17–19} and for $T < \Theta$ by Flowers and Itoh¹⁷ and by Ewart et al.¹⁹ For $T < T_M$ detailed tables of κ and σ are available.¹⁷ In the case $T > \Theta$, however, the values of κ and σ given²) by Solinger¹⁸ and Ewart et al.¹⁹ are three times the values found by Hubbard and Lampe¹⁴ and by Kovetz and Shaviv,¹⁵ while they are four or five times Flowers and Itoh's values.¹⁷ Furthermore, in the case $T < \Theta$ the tables¹⁷ indicate that $\sigma \propto T^{-2}$, whereas Ewart et al.¹⁹ find that $\sigma \propto T^{-5}$. The various authors have not intercompared their results or analyzed the reasons for the discrepancies.

We shall show in Sec. 4 that the discrepancies are caused by comparatively inaccurate approximations having been used for the vibration spectrum of astrophysical crystals. Employing more exact data, we have obtained values of κ and σ which in the case $T > \Theta$ are about 13 times lower than those of Solinger¹⁸ and Ewart et al.,¹⁹ while in the case $T < \Theta$ they are about five times lower than found by Flowers and Itoh.¹⁷ In addition, the results of Ewart et al.¹⁹ for $T < \Theta$ are qualitatively incorrect. We would further point out that Flowers and Itoh¹⁷ have made certain mistakes in calculating the electrical conductivity due to the scattering of electrons by impurities (see Sec. 5) and in estimating the role of electron–electron scattering.²⁰ As a consequence, large parts of their extensive tables (including all of their Tables 2, 3, and 4, and the third column of their Table 1) are inaccurate.

To facilitate the calculations, we shall first express κ and σ in terms of the effective electron collision frequencies (Sec. 2). Simple equations will then be derived (Secs. 3–5), enabling these frequencies easily to be evaluated for various electron scattering mechanisms. We conclude (Sec. 6) by briefly suggesting some physical processes that it might be possible to investigate by using the values found for κ and σ .

2. EXPRESSIONS FOR CONDUCTIVITIES

2. In strongly degenerate electron gas ($T \ll T_F$), where heat and charge are transported chiefly by electrons whose energy ε is close to the quantity μ ($|\varepsilon - \mu| \ll k_B T$, $\varepsilon = (m^2 c^4 + c^2 p^2)^{1/2}$), it is convenient to express σ and κ in terms of the effective frequencies ν_σ , ν_κ of electron collision (see, for example, Flowers and Itoh¹⁷):

$$\sigma = n_e e^2 / m_e \nu_\sigma, \quad \kappa = \pi^2 k_B^2 T n_e / 3 m_e \nu_\kappa, \quad (4)$$

where $m_* = \mu/c^2$. The following expressions are useful for numerical calculations:

$$\sigma = 1.51 \cdot 10^{22} (\rho_0/\mu_e) [1 + (\rho_0/\mu_e)^{1/3}]^{-1/2} (10^{16} \text{ sec}^{-1} / \nu_\sigma) \text{ sec}^{-1}, \quad (5a)$$

$$\kappa = 4.11 \cdot 10^{15} (\rho_0/\mu_e) \times [1 + (\rho_0/\mu_e)^{1/3}]^{-1/2} T_0 (10^{16} \text{ sec}^{-1} / \nu_\kappa) \text{ erg} \cdot \text{cm}^{-1} \cdot \text{sec}^{-1} \cdot \text{deg}^{-1}, \quad (5b)$$

with $T_0 = T/(10^6 \text{ }^\circ\text{K})$.

To evaluate σ and κ we must determine the frequencies ν_σ and ν_κ which, generally speaking, are not equal to each other. If several independent electron scattering mechanisms are simultaneously operative and described by collision frequencies $\nu_\sigma^i, \nu_\kappa^i, \dots$, then the total fre-

quencies and the conductivities will be given (see, for example, Ziman⁸) by expressions of the form

$$v_{\sigma,\kappa} = v'_{\sigma,\kappa} + v''_{\sigma,\kappa} + \dots, \quad 1/\sigma = 1/\sigma' + 1/\sigma'' + \dots, \quad (6)$$

$$1/\kappa = 1/\kappa' + 1/\kappa'' + \dots$$

We turn now to the principal mechanisms for scattering electrons: against ions, with $T > T_M$ (Sec. 3), and against crystal lattice vibrations (Sec. 4) or impurities (Sec. 5), when $T < T_M$.

3. ION SCATTERING

Under the conditions (1) of interest to us, if $T > T_M$ the energy of an electron will change by an amount $\Delta\varepsilon \ll 2p_F^2 m_i^{-1} \ll k_B T$, when it is scattered by an ion; thus the scattering will be elastic. We can then readily calculate σ and κ (see, for example, Ziman⁸) in the approximation of the electron relaxation time $\tau(\varepsilon)$:

$$v_{\sigma} = v_{\kappa} = \tau^{-1}(\mu), \quad \tau^{-1}(\varepsilon) = n_i v \sigma_{tr}(\varepsilon), \quad (7)$$

$$\sigma_{tr}(\varepsilon) = \int d\sigma(\varepsilon, \vartheta) (1 - \cos \vartheta).$$

The two conductivities will satisfy the standard Wiedemann-Franz rule: $\kappa/\sigma = \pi^2 k_B^2 T / 3e^2$. Here n_i denotes the number density of the ions (which are stationary in comparison with the electrons); v is the velocity of an electron; $d\sigma(\varepsilon, \vartheta)$ represents the cross section for scattering into solid angle $d\vartheta$, averaged over the initial and summed over the final spin states of the electron; ϑ is the scattering angle; and σ_{tr} is the transport cross section. In Eq. (7) the integration extends over the full solid angle.

In the Born approximation (see, for example, Beresetskii et al.²¹)

$$d\sigma(\varepsilon, \vartheta) = (4\pi^2 \hbar^4 c^4)^{-1} |V_q|^2 e^2 \left(1 - \frac{v^2}{c^2} \sin^2 \frac{\vartheta}{2}\right) d\vartheta, \quad \hbar q = 2p \sin \frac{\vartheta}{2}. \quad (8)$$

where $\hbar q$ denotes the momentum transmitted in the scattering process, and V_q is the Fourier transform of the electron-ion interaction potential; we shall write it in the form $|V_q| = 4\pi Z e^2 \varphi_q$. As in Sec. 4, we here assume for simplicity that all the ions have the same charge; the results can readily be generalized to the case of a mixture of different ions, as Hubbard and Lampe^{12,14} have done.

Equations (7) and (8) now yield

$$\sigma_{tr}(\varepsilon) = 4\pi (Ze^2/pv)^2 \Lambda_{ei}(p), \quad \Lambda_{ei}(p) = \int_0^{2p/\hbar} q^2 \varphi_q^2 (1 - \hbar^2 c^2 q^2 / 4e^2) dq, \quad (9)$$

$$\tau(\varepsilon) = \frac{p^2 v}{4\pi Z^2 e^4 n_i \Lambda_{ei}(p)}, \quad (10)$$

$$v_{\sigma,\kappa}^{-1} = \frac{3\pi \hbar^3}{4m_e Z e^4 \Lambda_{ei}} = \frac{5.65 \cdot 10^{-17}}{Z \Lambda_{ei} [1 + (\rho_e/\mu_e)^{3/2}]^{1/2}} \text{ sec},$$

where $\Lambda_{ei} = \Lambda_{ei}(p_F)$ is the Coulomb logarithm. Although Eq. (10) has previously been obtained by Lee,¹¹ he determined Λ_{ei} inaccurately (see, for example, Lampe¹³). In subsequent calculations (such as those by Kovetz and Shaviv¹⁵ and Canuto¹⁶), Eq. (10) has generally been written in a far more cumbersome form, without separating out Λ_{ei} , even though there is a real advantage in doing so, because Λ_{ei} depends only weakly on T , ρ , and Z (see below). In order to evaluate Λ_{ei} we must know φ_q . As $q \rightarrow \infty$ (small impact parameters), $\varphi_q \approx q^{-2}$ will correspond to a

pure Coulomb potential. To avoid the logarithmic ("Coulomb") divergence of the integral in Eq. (9) at $q = 0$, the screening of the potential at large distances by the polarization of the plasma should be taken into account.

In a degenerate gas the controlling factor is ion screening, whose radius will be denoted by r_{\max} . Electron screening will be of little importance for $T_M < T \ll T_F$, as its radius $r_{TF} \gg r_{\max}$ (see below). Even with these simplifications it is a difficult matter to calculate Λ_{ei} accurately (see, for example, Lampe¹³), but we can obtain some simple approximate expressions. In particular, when integrating the second term in $\Lambda_{ei}(p)$, in which the Coulomb divergence does not develop, we may set $\varphi_q = q^{-2}$.

But the integral of the first term for $p = p_F$ is the Coulomb logarithm $\Lambda_{ei}^{(0)}$, which arises when one is considering a nonrelativistic gas (since the character of the ion screening is practically independent of the degree to which the electrons are relativistic). Then $\Lambda_{ei} = \Lambda_{ei}^{(0)} - v_F^2/2c^2$, where v_F is the Fermi velocity. For $\Gamma < 40$ the values of $\Lambda_{ei}^{(0)}$ can be found from Hubbard's results¹² [in his notation $\Lambda_{ei}^{(0)} = [2G_{\Gamma}(\kappa_F)]^{-1}$], and for $\Gamma \ll 1$, from Lampe's analysis.¹³ We would point out that with an appropriate choice for r_{\max} , these results may be written in the form $\Lambda_{ei}^{(0)} = \ln(r_{\max}/r_{\min})$. For $\Gamma \ll 1$, according to Hubbard¹² and Lampe,¹³ $r_{\max} \approx r_{Di}$, where $r_{Di} = v_{Ti} \cdot (\omega_p)^{-1}$ is the ion Debye radius. For $\Gamma \gg 1$, Hubbard's qualitative result¹² indicates that r_{\max} is slightly smaller than the average inter-ion distance ($r_{\max} \approx a/3$).

One can easily show that all of Hubbard's results¹² for $1 < \Gamma < 40$ correspond with good accuracy (at least as good as that of Hubbard's calculation itself) to $r_{\max} = (r_{Di}^2 + a^2/6)^{1/2}$ [In particular, for $\Gamma \gg 1$ the parameter $r_{\max} = a/\sqrt{6}$ fits better than $a/3$; and it is curious to see that the empirically selected parameter $a/\sqrt{6}$ practically coincides with q_D^{-1} , where $q_D = (6\pi^2 n_i)^{1/3}$ is the Debye wave number.] Since the above expression for r_{\max} also satisfies the required asymptotic relationship for $\Gamma \ll 1$, we propose that it be adopted as an interpolation formula for all Γ .

Then in view of the fact that $r_{\min} = \hbar/2p_F$, we may write

$$\Lambda_{ei} = \ln [(2\pi Z/3)^{1/2} (1.5 + 3/\Gamma)^{1/2}] - v_F^2/2c^2, \quad (11)$$

$$v_F^2/c^2 = (\rho_e/\mu_e)^{1/2} [1 + (\rho_e/\mu_e)^{3/2}]^{-1}.$$

Values of Λ_{ei} for $\rho_e > 1$ and $\Gamma > 1$ can also be obtained directly from Canuto's tabulations¹⁶ [in his notation $\Lambda_{ei} = [\rho_e/\mu_e](v_F/c)^3/4G$]; they also are in good agreement with the values given by the simple expression (11).

Finally, if $\Gamma \ll 1$ and $\rho_e > 1$, then Eqs. (10), (11) will yield Kovetz and Shaviv's results,¹⁵ apart from a number of small and inconsequential terms due to screening. It is also noteworthy that Eqs. (10), (11) can replace a whole series of awkward interpolation formulas proposed by Iben.²²

In the case $\Gamma > 1$, numerical calculations of κ and σ have been performed for three values of T by Flowers and Itoh.¹⁷ Their values of κ and σ are several times too high, compared with the values given by Eqs. (10), (11). Moreover, Flowers and Itoh do not give their final work-

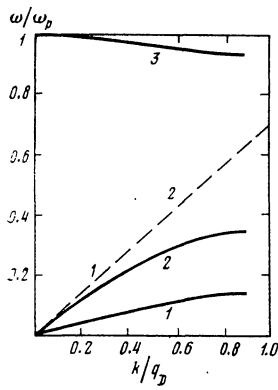


FIG. 2. Frequency ω as a function of wave number k for (1, 2) acoustic and (3) optical vibrations of a body-centered Coulomb crystal in the crystallographic orientation $[1, 1, 0]$, according to Clark.²⁴ The value $k = 0.905 q_D$ corresponds to the boundary of the Brillouin zone in this orientation. The dashed line represents Flowers and Itoh's approximation¹⁷ to the acoustic vibration spectrum.

ing formulas, which would have allowed their scheme of calculation to be checked. For $\rho_b < 1$ their results¹⁷ disagree with Hubbard's,¹² and for $\rho_b > 1$ they are inconsistent with Canuto's data,¹⁶ even though Flowers and Itoh's calculations in effect merely replicate the earlier ones.^{12,16}

We should also like to point out that some authors (Refs. 12, 15, 16) have eliminated the Coulomb divergence by taking ion-ion correlations into account, while Lampe¹³ has instead introduced into V_q the dielectric constant of the plasma. As a result, for $\Gamma \ll 1$ the values obtained by Lampe¹³ (the method he employed is more systematic) differ, although weakly, from Hubbard's values.¹² Furthermore, in the range $\Gamma > 10$ Hubbard¹² and Canuto¹⁶ obtained their results by modeling the ion distribution function [the weak oscillations in Λ_{ei} as a function of Γ found by Canuto for $\Gamma > 10$ are not described by Eq. (11); evidently they fall beyond the accuracy provided in Canuto's calculation]. A more systematic calculation of Λ_{ei} would therefore be desirable. Nevertheless, the simple expression (11) appears to afford enough precision for practical purposes.

One last point: for small Z and $T \ll T_p/3$ (Fig. 1), in addition to the scattering of electrons on ions, a definite contribution to the thermal conductivity might come from electron-electron scattering.^{13,20} This effect can easily be taken into account by accepting those results^{13,20} and applying Eq. (6).

4. SCATTERING FROM LATTICE VIBRATIONS

One of the principal mechanisms for scattering electrons at temperatures $T < T_M$ is scattering by thermal vibrations of the crystal lattice. Astrophysical, "Coulomb" crystals differ greatly from ordinary crystals on the earth. Under astrophysical conditions the kinetic energy of the electrons will far exceed the energy of electron-electron and electron-ion interaction and the energy of ion interaction. Hence the electrons will be practically free (except for the negligible fraction of electrons whose momenta are very close to the Brillouin zone boundaries; see, for example, Flowers and Itoh¹⁷), and the ions will crystallize against the background of a uniformly "smeared"

electronic charge.

The thermodynamic properties and vibrations of such crystals have been studied in some detail.^{4,5,23-25} Among the (cubic) Coulomb crystals, those most strongly bound are the body-centered crystals.^{25,4} Hence these are the ones that ought to form in white dwarfs and neutron stars.⁴ Of the three branches of lattice vibration $\omega_s(\mathbf{k})$ ($s = 1, 2, 3$; \mathbf{k} is the wave vector, confined to the first Brillouin zone) in such a crystal,^{24,5} two ($s = 1, 2$) are acoustic and one ($s = 3$), because of the long-range interaction between ions that is peculiar to Coulomb crystals, is optical (Fig. 2). As $k \rightarrow 0$, the acoustic modes will be transverse and the optical mode longitudinal. We would also mention²⁵

that $\sum_s \omega_s^2(\mathbf{k}) = \omega_p^2$ so that $\omega_s(\mathbf{k}) \leq \omega_p$.

Flowers and Itoh¹⁷ have obtained general expressions for the frequencies ν_σ, ν_κ when electrons are scattered from lattice vibrations. The variational solution they give for $T > \Theta$ [see Eq. (3)] agrees with the exact solution; for $T < \Theta$ it leads to insignificant deviations for ν_σ and deviations by no more than a few tens of percent for ν_κ (see, for example, Ziman⁸); this level of accuracy is fully acceptable for astrophysical applications.

In our notation, Flowers and Itoh's expressions³) may be written in the form

$$\nu_{\sigma,\kappa} = \frac{e^2 \omega_p^2}{v_F k_B T} \int_{(q < 2k_F)} \frac{dq}{4\pi k_F^2 q} \frac{1 - (\hbar c q / 2\mu)^2}{(q^2 + q_{TF}^2)^2} \times \sum_{s=1}^3 \frac{e^{z_s}}{(e^{z_s} - 1)^2} [\mathbf{q} \cdot \mathbf{e}_s(\mathbf{k})]^2 g^{(\sigma,\kappa)}(q, z_s). \quad (12)$$

$$g^{(\sigma)}(q, z) = q^2, \quad g^{(\kappa)}(q, z) = q^2 - q^2 z^2 / 2\pi^2 + 3k_F^2 z^2 / \pi^2, \quad k_F = p_F / \hbar, \quad z_s = \hbar \omega_s(\mathbf{k}) / k_B T. \quad (13)$$

Here $\hbar \mathbf{q}$ denotes the transferred momentum (its maximum possible value $q = 2k_F = r^{-1}$ restricts the region of integration); \mathbf{k} and $\mathbf{e}_s(\mathbf{k})$ represent the wave vector and the polarization unit vector of the phonon excited or absorbed by the electron ($\pm \mathbf{k} = \mathbf{q} - \mathbf{Q}$, where \mathbf{Q} is the reciprocal lattice vector for the Brillouin zone to which \mathbf{q} is confined). In Eq. (12) we have included only single-phonon transitions (only when $T \rightarrow T_M$ is this assumption not very accurate¹⁷) and have used static screening of electron-ion interactions by electrons. The screening radius (the Thomas-Fermi radius) is here given by $r_{\max} = q_{TF}^{-1} = v_F / \sqrt{3} \Omega_p$, where $\Omega_p = (4\pi n_e e^2 / m_*)^{1/2}$ is the electron plasma frequency.

The number of Brillouin zones coming within the region of integration in Eq. (12) is of order $(2k_F / q_D)^3 = 4Z$ [$q_D = (6\pi^2 n_i)^{1/3}$, and a single zone has a volume $V_B = 4\pi q_D^3 / 3$]. In astrophysics the case $4Z \gg 1$ is of special interest, and this is the case we shall consider. Umklapp processes will then make the main contribution to the scattering, with the vector \mathbf{q} falling in Brillouin zones distant from the first zone. This property is peculiar to astrophysical crystals and causes lattice scattering in them to be highly distinctive.

It happens that the quantity $(2k_F / q_D)^3$ introduced above is approximately equal to half the ratio of the volume n_i^{-1} allocable per ion to the de Broglie volume $4\pi / 3k_F^3$ of an electron. Hence the condition $4Z \gg 1$ actually means that

electrons will be scattered by individual ions that participate in the collective lattice oscillations. Thus when $4Z \gg 1$ the integral (12) need include only the contribution of distant Brillouin zones. In performing an integration within a single distant zone characterized by the vector Q , it suffices to set $q = Q$ in the integrand and to pass to an integration with respect to k within the first zone only. Since the condition (1) implies that $q_D^3/q_{TF}^3 \sim 2 \cdot 10^3 Z^{-1} (v_F/c)^3/2 \gg 1$, screening of the interaction need not be taken into account for the distant zones. The sum with respect to Q that is formed in Eq. (12) need merely be replaced by an integral over a spherical layer $Q_{\min} < Q < 2k_F$ ($Q_{\min} \sim q_D$) that contains the distant Brillouin zones in question.

The result of integrating the vector Q with respect to angles is expressed by the function

$$\Phi_{(x)}^{(n)} = \langle z_s^n e^{i\alpha} (e^{i\alpha} - 1)^{-2} \rangle, \quad \langle f_s(\mathbf{k}) \rangle = (3V_B)^{-1} \sum_{s=1}^3 \int_{(V_B)} dk f_s(\mathbf{k}), \quad (14)$$

$$x = \hbar \omega_p / k_B T,$$

where $n = 0, 2$, and the angle brackets signify an average over wave vectors and phonon polarizations. In the remaining integration over Q the contribution of small $Q \sim Q_{\min}$ may be neglected almost everywhere, and we may set $Q_{\min} = 0$. The only exception is in one term in v_M , where the expression in the integrand behaves like Q^{-1} for small Q ; we there take $Q_{\min} = q_D$. Strictly speaking, by virtue of its derivation the quantity $Q_{\min} \sim q_D$, but it is not accurately determined. This circumstance is unimportant, however, since the integral merely depends logarithmically on Q_{\min} . This last property does not imply that the contribution of the first Brillouin zone (that is, normal scattering processes) to the sum over Q is large. Straightforward estimates of this contribution indicate that for $4Z \gg 1$ it will be smaller than the contribution in the distant zones. [For purposes of estimates one should recognize that, according to the results of Cohen and Keffer²³ and Carr,⁵ as $k \rightarrow 0$ we will have $e_{1,2}(\mathbf{k})k/k \sim (k/q_D)^2$, and that when allowing for the inter-ion screening interaction⁴ for small $k \sim q_{TF} \ll q_D$, the optical branch of vibration will go over to the acoustic branch.]

Performing the integration, then, we obtain

$$v_x = (e^2/\hbar v_F) \omega_p x [(2 - v_F^2/c^2) \Phi^{(0)}(x) + (3\Lambda_{ph} - 1 + v_F^2/2c^2) \Phi^{(2)}(x)/\pi^2], \quad (15a)$$

$$v_o = (e^2/\hbar v_F) \omega_p x (2 - v_F^2/c^2) \Phi^{(0)}(x),$$

$$\Lambda_{ph} = \ln(2k_F/q_D) - v_F^2/2c^2 = 1/3 \ln(4Z) - v_F^2/2c^2. \quad (15b)$$

For $x \ll n + 2$, Eq. (14) yields

$$\Phi^{(n)}(x) = u_{n-2} x^{n-2} - 1/12 u_n x^{n+1} + 1/240 u_{n+2} x^{n+2} - 1/6048 u_{n+4} x^{n+4} + \dots, \quad u_i = \langle \omega_s^i(\mathbf{k}) \rangle / \omega_p^i. \quad (16)$$

The quantities u_i appearing in Eq. (16) (other than u_{-2}) have been evaluated by Coldwell-Horsfall and Maradudin (Ref. 25): $u_0 = 1$, $u_2 = 1/3$, $u_4 = 0.203$, $u_6 = 0.136$. The quantity u_{-2} has been calculated by Pollock and Hansen⁴: $u_{-2} = 13.0$. These authors further point out⁴ that the value quoted²⁵ for u_6 is 10% too low. However, there is no point in introducing such corrections, although small adjustments of this order might arise, for example, when allowance is made for the screening interaction between ions.⁴

In the case $x \gg n + 2$, it is adequate to employ in Eq. (25) the averaging formula²⁵

$$\langle f(\omega_s(\mathbf{k})) \rangle = \omega_p^{-3} \int_0^{\omega_p} d\omega \omega^2 f(\omega) [c_2 + c_4 (\omega/\omega_p)^2 + c_6 (\omega/\omega_p)^4 + \dots], \quad (17)$$

$$c_2 = 30.0, \quad c_4 = -86.3, \quad c_6 = 27.2, \dots$$

extending the upper limit of integration to infinity. We then find

$$\Phi^{(n)}(x) = c_2 x^{-3} (n+2)! \zeta(n+2) + c_4 x^{-5} (n+4)! \zeta(n+4) + c_6 x^{-7} (n+6)! \zeta(n+6) + \dots, \quad (18)$$

where $\zeta(n)$ denotes the Riemann zeta-function.

The asymptotic expressions (16), (18) suffice for constructing the relations $\Phi^{(0,2)}(x)$ over the entire range of x variation. The only function that is not very accurately determined is $\Phi^{(2)}(x)$ in the interval $6 < x < 10$ (the error is $\lesssim 20\%$), but $\Phi^{(2)}(x)$ is less important in this interval than $\Phi^{(0)}(x)$. A more accurate calculation would require an elaborate numerical integration in Eq. (14). The relationships we have obtained can be approximated, to an accuracy at least as good as that of the calculation itself, by the simple expressions

$$x^2 \Phi^{(0)}(x) = u_{-2} [1 + (3u_{-2} x / \pi^2 c_2)^2]^{-1/2} = 13 [1 + (\Theta/3.46T)^2]^{-1/2}, \quad (19a)$$

$$(x/\pi)^2 \Phi^{(2)}(x) = (x/\pi)^2 [1 + x^2 (15/4\pi^4 c_2)^{2/3}]^{-3/2} = 13 (\Theta/5.1T)^2 [1 + \Theta/4.17T]^2]^{-3/2}. \quad (19b)$$

To facilitate numerical estimates from Eqs. (15) and (19), we would point out that

$$x = \Theta/0.45T, \quad e^2 \omega_p u_{-2} / \hbar v_F x = 1.25 \cdot 10^{16} T_e c / v_F \text{ sec}^{-1} \quad (20)$$

and that, by Eq. (11), Λ_{ph} practically coincides with Λ_{ei} when $\Gamma \gg 1$.

By using Eqs. (3), (15), (19), and (20), we can now easily calculate the electrical and thermal conductivity due to lattice scattering. Figure 3 displays the results. It is remarkable that in these equations only Λ_{ph} depends explicitly on Z , and even that dependence is logarithmic. We shall consider two limiting cases in detail: A) $T \gg \Theta$; B) $T \ll \Theta$.

Case A. If $T \gg \Theta$, the change in the energy of an electron when it emits or absorbs a phonon will be $\Delta \varepsilon \sim \hbar \omega_s \ll k_B T$, so a relaxation time $\tau(\varepsilon)$ will exist which can readily be computed from Eq. (7). If the condition (1) holds, the time required by the electron to traverse an elementary cell will be much shorter than the characteristic periods of ion vibration, which are of order ω_p^{-1} . Furthermore, if $T \gg \Theta$ the vibrations of the ions may be treated classically. Then in the event that $4Z \gg 1$ (see above), each elementary event will involve elastic scattering of the electron in the potential (see, for example, Hubbard and Lampe¹⁴) $V(\mathbf{r}) = -Ze^2 (|\mathbf{r} - \xi|^{-1} - r^{-1}) \approx -Ze^2 (r\xi) r^{-3}$, produced by a single ion when it undergoes small displacements ξ relative to its equilibrium position.

The scattering cross section will therefore be given by Eq. (8), in which $|V_q|^2$ should be taken to be the squared absolute value of the Fourier transform of $V(\mathbf{r})$, averaged

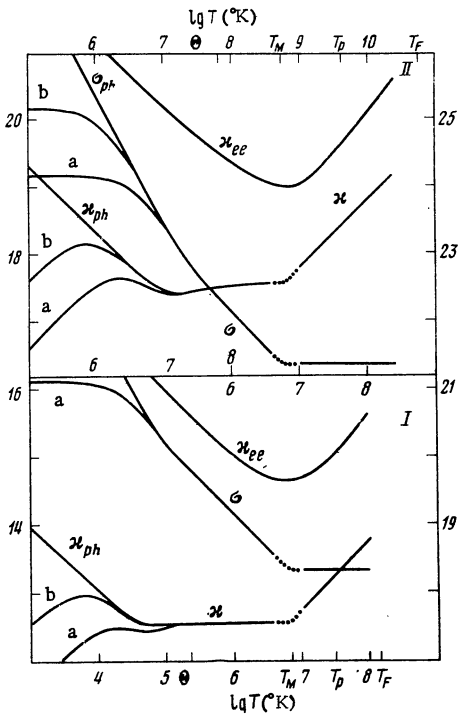


FIG. 3. Temperature dependence of the thermal conductivity κ and the electrical conductivity σ in matter having an equilibrium nuclear composition¹ (left-hand scale, $\log \kappa$; right-hand scale, $\log \sigma$), for: I) non-relativistic electrons ($\rho/\mu_e = 10^4$ g/cm³, $Z = 26$, $A = 56$); II) relativistic electrons ($\rho/\mu_e = 10^9$ g/cm³, $Z = 34$, $A = 84$). The curves for κ and σ presuppose that electrons are scattered by ions for $T > T_M$ (where T_M is the crystallization temperature), and by phonons and impurities for $T < T_M$. The pairs of curves a, b correspond to impurities having a number density and charge such that $x_{\text{imp}}(\Delta Z)^2 = 1, 0.1$, respectively; the curves κ_{ph}, σ_{ph} take only the phonons into account. In the dotted portions of the curves, at temperatures $T \approx T_M$, the behavior of κ and σ is not known in detail. In order graphically to demonstrate that the thermal conductivity κ_{ee} due to electron-electron collisions is unimportant for large Z , separate κ_{ee} curves have been plotted according to the results of Lampe¹³ and the authors.²⁰

over the ensemble of displacements ξ . Then

$$\sigma_{tr}(\epsilon) = (4\pi/3) (Ze^2/\hbar v)^2 (2-v^2/c^2) \bar{\xi}^2, \quad \bar{\xi}^2 = (3\hbar^2/2m_i k_B T) \langle z_s^{-1} (1-e^{-z_s})^{-1} \rangle. \quad (21)$$

Here ξ^2 represents the mean-square thermal displacement of the ion, neglecting quantum fluctuations, which are unimportant when $T \gg \Theta$ (see, for example, Coldwell-Horsfall and Maradudin²⁵). Inasmuch as $T \gg \Theta$, Eqs. (7) and (21) yield in the relaxation-time approximation

$$\tau^{-1}(\epsilon) = (e^2/\hbar v) (2-v^2/c^2) (k_B T/\hbar) u_{-2}, \quad \tau^{-1}(\mu) = \nu_\sigma = \nu_\kappa, \quad (22)$$

which naturally coincides with the leading terms in the expansion of the general expressions (15) for small x . Note that the quantities (21) do not depend on Z (see Solinger¹⁸).

The more limited case $\Theta < T < T_M$ has been treated in the literature.^{14,15,17-19} Some of these calculations^{14,15} are analogous to those we have performed in deriving Eq. (21), but the authors assume at the start of their derivation that $\omega_s(k) = \omega_p/\sqrt{3}$. Hence their result agrees with Eq. (21) if the quantity $u_{-2} = 3$, rather than $u_{-2} = 13.0$. Moreover, they maintain^{14,15} that their result holds true if Umklapp processes are unimportant; but actually it

holds only if these processes do play a decisive role. One can easily verify this statement by turning to Eq. (12): if the contribution of Umklapp processes were excluded (electrons then would be scattered not by the individual ions but by the lattice as a whole), one would obtain a result different from the expression (21).

Solinger¹⁸ and Ewart et al.¹⁹ have obtained expressions analogous to Eq. (21) by a longer route. But their expressions are much more cumbersome, since allowance is made for screening by electron-ion interaction, an unimportant effect in our case. Furthermore, these authors assume that the contribution of the acoustic phonons to u_{-2} is the same as the contribution of the optical phonon. Taking $\omega_3(k) = \omega_p$, they find that $u_{-2} = 1$. Actually, however, when $T > \Theta$ the optical phonon will contribute far less to the scattering than the acoustic phonons (since the latter have small phase velocities; see Fig. 2). When $T < \Theta$, the contribution of the optical phonon will of course be smaller still.

Finally, Flowers and Itoh¹⁷ replace the Brillouin zone (a rhombo-dodecahedron) by a sphere of radius q_D and consider that $\omega_{1,2}(k) = 0.7\omega_p k/q_D$, $\omega_s(k) \approx \omega_p$. As Fig. 2 indicates, by making this simplification Flowers and Itoh have raised the frequencies of the acoustic phonons, thereby diminishing u_{-2} . In fact, in their approximation $u_{-2} = 1/3 + 2 \cdot (0.7)^{-2} = 4.4$. Although Flowers and Itoh have not themselves evaluated σ and κ analytically, their numerical data agree with the values computed from Eq. (21) if this value is taken for u_{-2} . It is worth noting that even an estimate based on the Debye model yields $u_{-2} = 3\omega_p^2 \omega_D^{-2} \approx 15$, $\omega_D = k_D \Theta/\hbar$, an expression far closer to the correct value than the results we have discussed.^{14,15,17-19}

Case B. If $T < \Theta$, the relaxation-time approximation for describing lattice scattering will not work, in general, because the scattering becomes decidedly inelastic ($\Delta\epsilon \sim k_B T$). In ordinary metals on the earth (see, for example, Ziman⁸), normal scattering processes predominate, in which electrons are deflected by small angles when $T < \Theta$. One such scattering event will suffice to turn a "hot" electron into a "cool" one, but many events will be needed to lower the current velocity of the electron. Thus if $T \ll \Theta$, the collision frequencies $\nu_\kappa \gg \nu_\sigma$ in ordinary metals; the relaxation-time approximation and the Wiedemann-Franz rule will be strongly violated, and the conductivities $\kappa \propto T^{-2}$, $\sigma \propto T^{-5}$ [in order to obtain this result from Eq. (12), one should neglect the contribution of Umklapp processes and regard the longitudinal phonon as acoustic, with $q_{TF} \gg q_D$].

Matters are very different in astrophysical crystals where, according to Eq. (18), we will have for $T \ll \Theta$:

$$\nu_\sigma = \frac{\pi^2}{3} \frac{c_2 \omega_p}{x^2} \frac{e^2}{\hbar v_F} \left(2 - \frac{v_F^2}{c^2} \right), \quad (23)$$

$$\nu_\kappa = \frac{\pi^2}{5} \frac{c_2 \omega_p}{x^2} \frac{e^2}{\hbar v_F} \left(4\lambda_{ph} + 2 - \frac{v_F^2}{c^2} \right).$$

Accordingly, although $\nu_\kappa > \nu_\sigma$, in typical circumstances the frequencies differ by no more than a factor of 2-3, and $\sigma \propto T^{-2}$, $\kappa \propto T^{-1}$. The point here is that in Coulomb crystals the dominant role is played by Umklapp processes, accompanied by scattering at large angles. Such scattering will be equally effective for electrical and for thermal

resistance. Thus if $T < \Theta$, the relaxation-time approximation and the Wiedemann-Franz rule will be violated not qualitatively but merely numerically [one can easily show that for estimates when $T < \Theta$, an approximate relaxation time can be introduced according to the simple meaning of Eqs. (7) and (21)].

This important conclusion could have been drawn from Flowers and Itoh's calculations¹⁷ of σ and κ , but they make no mention of this circumstance. In fact, when $T < \Theta$ the results are much more sensitive to the details of the phonon spectrum than when $T > \Theta$. If we were to approximate the spectrum in the same manner as Flowers and Itoh, we would obtain $c_2 = 2 \cdot (0.7)^{-3} = 5.85$ in Eq. (23). Flowers and Itoh's values of σ and κ agree with those computed from Eq. (23) for $T < \Theta$ if c_2 has this value. If the expression (14) were averaged according to the Debye model, we would have $c_2 = 3(\omega_p/\omega_D)^3 \approx 27$ in Eq. (23), in reasonable accord with the exact value. Ewart et al.¹⁹ also calculate σ for $T < \Theta$, but they assume that only normal processes are important when $T < \Theta$, and they arrive at a qualitatively incorrect result: $\sigma \propto T^{-5}$.

The behavior of κ and σ for $T \sim \Theta$ is illustrated in Fig. 3. Notice that the transition from Eq. (22) to Eq. (23) actually occurs for $T \sim \Theta/4$. We would also point out that when $T \approx T_M$, Eqs. (11) and (21) may be somewhat inaccurate because of the complicated motion of the ions. It is remarkable that, despite this fact, an extrapolation of these expressions to $T = T_M$ gives, as Fig. 3 demonstrates, a practically continuous trend of the κ and σ curves through the melting point.

5. SCATTERING BY IMPURITIES

One would hardly expect astrophysical crystals to be perfect. Probably they will have a good many structural defects of various kinds - impurities, cracks, and so on, caused, for instance,¹⁷ by the crystallization process itself or the presence of irregularities in chemical composition. Scattering of electrons from these defects could become significant at low T .

Let us consider scattering by impurities (ions of charge $Z' \neq Z$ impregnated at the lattice points), the effect thought to be the most important.^{14,17,19} This scattering will take place elastically and will be similar to electron-ion scattering (Sec. 3), except that now the effective charge determined by the Coulomb potential of the impurity will be $\Delta Z = Z' - A$ (see, for example, Flowers and Itoh¹⁷), while the screening radius $r_{\max} = q_{TF}^{-1}$ [see Eq. (12)].

We can calculate the relaxation time from Eqs. (7) and (8). Following Flowers and Itoh, we take the Fourier transform of the potential in Eq. (8) to be $|V_a| = 4\pi|\Delta Z|e^2 \cdot (q^2 + q_{TF}^2)^{-1}$. If the conditions (1) hold, the ratio $r_{\max}/r_{\min} = 2k_F/q_{TF} \approx 20.6 (v_F/c)^{1/2} \gg 1$. Then making the substitution $n_i \rightarrow \sum_{Z'} n'$ in Eq. (7), where n' denotes the number density of impurities of charge Z' , we find that $\tau(\varepsilon)$ will be given by the first of Eqs. (10) if we make there the substitutions

$$Z^2 \rightarrow (\overline{\Delta Z})^2 x_{\text{imp}} = \sum_{Z'} n' (\Delta Z)^2 / n_i, \quad (24)$$

$$\Lambda_{ei}(p) \rightarrow \Lambda_{\text{imp}}(p) = \ln(2p/\hbar q_{TF})^{-1/2} - v^2/2c^2.$$

Here x_{imp} , $(\overline{\Delta Z})^2$ have been introduced as defined by Flowers and Itoh; they represent the relative abundance of impurities and the mean value of $(\Delta Z)^2$. As a result we obtain

$$\tau^{-1}(\mu) = 4m_e e^4 (\overline{\Delta Z})^2 x_{\text{imp}} \Lambda_{\text{imp}} / 3\pi \hbar^2 Z, \quad \Lambda_{\text{imp}} = \Lambda_{\text{imp}}(p_F), \quad (25)$$

which may be compared with the expression (10). Note that if $\rho_6 \gtrsim 0.1$ the Coulomb logarithm $\Lambda_{\text{imp}} \approx 2$. Furthermore, the sum with respect to Z' in Eq. (24) contains a term representing scattering by vacancies ($Z' = 0$). For vacancies of fixed number density, their contribution to the scattering will grow with Z .

Expressions analogous to Eq. (25) have been obtained by Flowers and Itoh¹⁷ and Ewart et al.¹⁹ Flowers and Itoh's equation is lacking a Lorentz factor m^*/m . And while their calculation, to judge from their tabulated data, was performed with the correct formula and κ has been properly computed, their values of σ are incorrect (they are too low by about one and one-half orders of magnitude). Moreover, the previous equations^{17,19} are much more cumbersome than our Eq. (25), as they retain several terms which are unimportant when $2k_F \gg q_{TF}$.

For $T < T_M$ one can easily allow for concurrent scattering by phonons and impurities by using Eqs. (6), as illustrated in Fig. 3. Comparison of Eqs. (22), (23) with Eq. (25) clearly shows that at high T , phonon scattering will predominate; at low T , scattering from impurities. The changeover temperature $T = T_{\text{imp}}$ depends on the quantity $x_{\text{imp}}(\overline{\Delta Z})^2$, whose value is unknown for the conditions in real stars. Figure 3 demonstrates that (for $Z \approx 30$), roughly speaking, we will have $T_{\text{imp}} > \Theta/4$ if $x_{\text{imp}}(\overline{\Delta Z})^2 > 1$, while $T_{\text{imp}} < \Theta/4$ in the opposite case (compare curves a and b).

6. CONCLUDING REMARKS

We see, then, that in many cases our values for the thermal and electrical conductivity differ significantly from those obtained in more simplified treatments by other authors. When calculations are made for specific processes that will occur in white dwarfs and neutron stars, such as the cooling of neutron stars,^{26,27} extrapolated nonrelativistic expressions for κ and σ are often applied to regions containing relativistic electron gas. According to Eqs. (10) and (15), the values of κ and σ obtained from such an extrapolation will be much too high [in particular, for $T > T_M$ and $\rho_6 \gg 1$ the values of κ are exaggerated by a factor $(\rho_6/\mu_e)^{2/3}$]. But neutron-star envelopes and the cores of massive white dwarfs consist chiefly of relativistic gas.

As a consequence, such results as the relation between the surface temperature and the internal temperature of a neutron star (which controls the cooling rate of the neutron star,^{26,27} the conditions whereby accreted matter undergoes thermonuclear burning,²⁸ and so on), when calculated on the basis of our improved values for κ , may turn out to differ substantially from earlier findings.^{26,27} We are devoting a separate paper²⁹ to such a calculation.

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- ¹)Canuto's equations¹⁶ contain a superfluous Lorentz factor m_*/m , and we find that the values of κ and σ given by Flowers and Itoh¹⁷ are several times too high (see Sec. 5).
- ²)We have in mind Solinger's final "literal" equation (28). In substituting numerical values of Solinger made several mistakes, so that his numerical result (34) is about 15 times smaller than his Eq. (28) actually implies.
- ³)The numerator of Flowers and Itoh's Eq. (52) contains an extraneous Lorentz factor m_*/m . However, to judge from their Tables 2 and 3, their numerical calculation of σ and κ has been made without this extra factor.
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