DESCRIPTION OF A RELATIVISTIC ELECTRON IN A QUANTIZING MAGNETIC FIELD. TRANSVERSE TRANSPORT COEFFICIENTS OF AN ELECTRON GAS

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Many relations needed for calculating elementary processes with the participation of relativistic electrons in a quantizing magnetic field in the Landau gauge for the vector potential are obtained (properties of the wave functions, matrix elements, and quadratic combinations of them). Simple expressions are found for the transverse transport coefficients in a relativistic electron gas with quantizing magnetic field that can be used in solid-state physics and astrophysics.

1. Introduction

In solid-state physics and astrophysics much interest has recently been shown in the investigation of elementary processes with the participation of electrons, which may be relativistic, in quantizing magnetic fields. Magnetic fields which are quantizing (10^5-10^6 G) for a solid-state plasma were achieved comparatively long ago in the laboratory. Various processes in them (kinetics and thermodynamics of the carriers, propagation of electromagnetic and acoustic waves, etc.) have been studied intensively both experimentally and theoretically (see, for example, [1]). And because of the great variety of structures of the energy bands, it is not only a nonrelativistic electron gas (quadratic dispersion law) but also a relativistic gas that is of interest for the solid-state plasma. Indeed, it is well known that in a number of semiconductors and semimetals the behavior of the electrons is described by a relativistic Dirac equation in which the velocity of light is replaced by the corresponding effective "limiting" velocity of the carriers (see, for example, [2]).

The interest in such a theme in astrophysics is due to the discovery of superstrong magnetic fields (up to 10^{13} G, see, for example, [3]), which quantize the electron component of a plasma, in the surface layers of neutron stars (radio and x-ray pulsars, x-ray bursters, gamma sources, etc.). Both in neutron stars and near their surfaces, various processes with the participation of electrons (cyclotron and synchrotron radiation, interaction of photons with matter, thermodynamics and kinetics of the electron gas, etc; see, for example, [4, 5]) play an extremely important part.

In the study of these processes one uses an elegant but rather laborious mathematical formalism. An exposition of the formalism when the cylindrical gauge of the vector potential of the magnetic field is used is given in the monograph of Sokolov and Ternov [6]. However, for the solution of many problems it is more convenient to use the Landau gauge of the vector potential.

In the present paper, we give a systematic exposition of the mathematical formalism needed for calculations in the Landau gauge. Most of the expressions given here are original. Some formulas are common to the cylindrical gauge and the Landau gauge. For generality, we take them from [6].

In Sec. 2 of this paper, we describe the properties of the wave functions of relativistic electrons in a constant homogeneous field in the Landau gauge (Subsection A), the properties of the matrix elements, which can be expressed in terms of normalized associated Laguerre functions (Subsection B), the general properties of these functions (Subsection C), their quasiclassical representations (Subsection D), and various sums containing quadratic combinations of them (Subsection E). In Sec. 3, these formulas are used to obtain simple expressions for the transverse transport coefficients – the transverse conductivity, the thermal conductivity, and the specific thermoelectric force – in a relativistic electron gas. These expressions apply both in solid-state physics as well as in astrophysics.

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2. General Properties of Matrix Elements

A. Wave Functions. In the Landau gauge of the vector potential $\mathbf{A} = (-By, 0, 0)$ of the magnetic field $B \equiv B_z$, the states of a free relativistic electron are conveniently specified [7] by the quantum numbers ε, p_z, p_x, n, s. Here, ε is the electron energy, p_z is the momentum component along the magnetic field, $n = 0, 1, 2, \ldots$ is the number of the Landau level, p_x characterizes the y coordinate of the center of the Larmor gyration of the electron: $y_B = p_x/m\omega_B$, $s = \pm 1$ is the sign of the projection of the momentum $\mathbf{p} + e\mathbf{A}/c$ onto the spin, $\omega_B = eB/mc$ is the cyclotron frequency, and m and $-e$ are the mass and charge of the electron. The energy is [7]

$$\varepsilon = \eta (m^2c^4 + c^2p_z^2 + 2mc^2 n\hbar\omega_B)^{1/2}, \tag{1}$$

where $\eta = \pm 1$ is the sign of the energy.

Solving the Dirac equation in the standard representation, we obtain the following expression for the wave function:

$$\psi(\mathbf{r}) = \frac{\exp\{i(p_x x + p_z z)/\hbar\}}{\sqrt{L_x L_z}} \begin{Bmatrix} \alpha A \mathcal{H}_{n-1}(\xi) \\ -s\alpha B \mathcal{H}_n(\xi) \\ s\eta\beta A \mathcal{H}_{n-1}(\xi) \\ -\eta\beta B \mathcal{H}_n(\xi) \end{Bmatrix}, \tag{2}$$

where

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \left[\frac{1}{2}\left(1 \pm \frac{mc^2}{\varepsilon}\right)\right]^{1/2}, \quad \begin{Bmatrix} A \\ B \end{Bmatrix} = \left[\frac{1}{2}\left(1 \pm \frac{sp_z c}{\sqrt{\varepsilon^2 - m^2c^4}}\right)\right]^{1/2}, \tag{3}$$

$$\xi = \sqrt{\frac{m\omega_B}{\hbar}}(y - y_B), \quad \mathcal{H}_n(\xi) = \left(\frac{m\omega_B}{\pi\hbar}\right)^{1/4} \exp(-\xi^2/2)\,(2^n n!)^{-1/2} H_n(\xi), \tag{4}$$

$\mathcal{H}_n(\xi)$ are the normalized harmonic oscillator functions, $H_n(\xi)$ are Hermite polynomials; L_x and L_z are the normalization lengths. The levels with $n \neq 0$ are nondegenerate with respect to s. To the Landau ground level ($n = 0$) there corresponds only a single value of s, which corresponds to orientation of the spin in the opposite direction to the magnetic field ($s = -\text{sign}\, p_z$ for $p_z \neq 0$, which gives $A = 0$; for $p_z = 0$, the value of s is not determined, and in this case one must additionally set $B = 1$, and then the dependence of $\psi(\mathbf{r})$ on s disappears). The positron wave function in the standard representation in the state p_z, p_x, n, s, η is obtained from the electron function (2) for the state $-p_z$, $-p_x$, n, s, $-\eta$ by the operation of charge conjugation. This positron wave function is the same as the function found by Klepikov [7]. However, Klepikov erroneously associated this function with an electron in the state p_z, p_x, n, s, η. This did not influence the correctness of the probabilities of the various processes calculated in [7], since the errors in the probabilities disappear after averaging over the orientations of the particle spins.

In calculations using the Landau gauge, one frequently employs a different set of wave functions (see, for example, the monograph of Akhiezer and Berestetskii [8]). These functions are linear combinations of the functions (2) with $s = \pm 1$ chosen to make one of the bispinor components vanish. For this reason, they have a less symmetric form and describe quantum states in which the projection of the spin onto the momentum does not have a definite value.

B. Matrix Elements. Using the functions (2), we find matrix elements of the operators of the form $\hat{f}e^{i\mathbf{qr}}$, where $\hat{f} = \hat{1}, \hat{\beta}, \hat{\alpha}_i$ $(i = x, y, z)$, and $\hat{\beta}$ and $\hat{\alpha}$ are the Dirac matrices; such matrix elements are frequently encountered in applications. We obtain

$$(\psi' | \hat{f} \exp(i\mathbf{qr}) | \psi) = \delta_{p_z', p_z + \hbar q_z} \delta_{p_x', p_x + \hbar q_x} \bar{f}; \tag{5}$$

$$\begin{Bmatrix} \hat{1} \\ \hat{\beta} \end{Bmatrix} = (s's\alpha'\alpha \pm \eta'\eta\beta'\beta)\,(B'BI_{n',n} + s's A'A I_{n'-1,n-1}), \quad \bar{\alpha}_i = (\alpha' s\beta\eta + s'\beta'\eta'\alpha)\,a_i, \quad a_x = -A'BsI_{n'-1,n} - s'B'AI_{n',n-1},$$
$$a_y = iA'BsI_{n'-1,n} - is'B'AI_{n',n-1}, \quad a_z = A'AI_{n'-1,n-1} - s'sB'BI_{n',n}, \tag{6}$$

where, following [7], we have introduced the functions

$$I_{n',n} = \int_{-\infty}^{+\infty} dy\, e^{iq_y y} \mathcal{H}_{n'}(\xi') \mathcal{H}_n(\xi) = F_{n',n}(u) \exp\{i(n - n')\varphi + iq_y(y_B + y_B')/2\}, \tag{7}$$

$$F_{n',n}(u) = (-)^{n'-n} F_{n,n'}(u) = \sqrt{\frac{n'!}{n!}}\, u^{(n-n')/2} e^{-u/2} L_{n'}^{n-n'}(u), \tag{8}$$

$$u=\hbar q_\perp^2/2m\omega_B, \quad q_\perp^2=q_x^2+q_y^2, \quad \varphi=\operatorname{arctg} q_y/q_x, \tag{9}$$

$L_n^s(u)$ is a Laguerre polynomial, and $F_{n',n}(u)$ is a normalized Laguerre function. If $n < 0$, and (or) $n' < 0$, the functions (7) and (8) are equal to zero. For the reasons given above, the expressions (6) differ from the analogous expressions in [7].

To calculate the probabilities of various processes, we require quadratic combinations of the matrix elements (5) summed over the spin variables. In this connection, we calculate the quantities

$$Q=\sum_{ss'p_z'p_x'}|(\psi'|e^{iqr}|\psi)|^2, \quad Q_0=\sum_{ss'p_z'p_x'}|(\psi'|\hat{\beta}e^{iqr}|\psi)|^2, \quad Q_{ij}=\sum_{ss'p_z'p_x'}(\psi|e^{-iqr}\hat{\alpha}_i|\psi')(\psi'|\hat{\alpha}_j e^{iqr}|\psi). \tag{10}$$

Rotating for simplicity the coordinate system to ensure $q_y = 0$ (and $I_{n',n}=F_{n',n}(u)$), and setting $\hbar = c = 1$, we obtain

$$\left\{\begin{matrix}Q\\Q_0\end{matrix}\right\}=\frac{1}{2}\left(1+\frac{m^2\pm p_zp_z'}{\varepsilon\varepsilon'}\right)(F^2_{n'-1,n-1}+F^2_{n',n})\pm 2\frac{m\omega_B}{\varepsilon\varepsilon'}\sqrt{nn'}F_{n'-1,n-1}F_{n',n},$$

$$\left\{\begin{matrix}Q_{xx}\\Q_{yy}\end{matrix}\right\}=\frac{1}{2}\left(1-\frac{m^2+p_zp_z'}{\varepsilon\varepsilon'}\right)(F^2_{n',n-1}+F^2_{n'-1,n})\pm 2\frac{m\omega_B}{\varepsilon\varepsilon'}\sqrt{nn'}F_{n',n-1}F_{n'-1,n},$$

$$Q_{xy}=-Q_{yx}=\frac{i}{2}\left(1-\frac{m^2+p_zp_z'}{\varepsilon\varepsilon'}\right)(F^2_{n',n-1}-F^2_{n'-1,n}), \tag{11}$$

$$\left\{\begin{matrix}Q_{xz}\\-iQ_{yz}\end{matrix}\right\}=\left\{\begin{matrix}Q_{zx}\\iQ_{zy}\end{matrix}\right\}=-\frac{1}{\varepsilon\varepsilon'}\left(\frac{m\omega_B}{2}\right)^{1/2}[\sqrt{n}p_z'(F_{n'-1,n-1}F_{n'-1,n}\pm F_{n',n}F_{n',n-1})+\sqrt{n'}p_z(F_{n',n}F_{n'-1,n}\pm F_{n'-1,n-1}F_{n',n-1})],$$

$$Q_{zz}=\frac{1}{2}\left(1-\frac{m^2-p_zp_z'}{\varepsilon\varepsilon'}\right)(F^2_{n'-1,n-1}+F^2_{n',n})-2\frac{m\omega_B}{\varepsilon\varepsilon'}\sqrt{nn'}F_{n'-1,n-1}F_{n',n}.$$

Formulas (11) for Q_{ij} are identical with the ones obtained in [7, 9]. Note that the formulas in [7] are given for an arbitrarily oriented coordinate system ($q_y \neq 0$) and can be obtained from (11) by rotation of the axes through arbitrary angle φ. Note that Q and Q_{ij} are components of a 4-tensor of second rank, and Q_0 is a 4-scalar.

C. Properties of the Functions. The normalized associated Laguerre functions $F_{n',n}(u)$ $(n, n'\geqslant 0)$, determined in (8), satisfy the differential equation (see, for example, [10])

$$u\frac{d^2}{du^2}F_{n',n}(u)+\frac{d}{du}F_{n',n}(u)+\left[\frac{n+n'+1}{2}-\frac{u}{4}-\frac{(n-n')^2}{4u}\right]F_{n',n}(u)=0 \tag{12}$$

and the orthogonality condition (see, for example, [10])

$$\int_0^\infty du\,F_{n',n}(u)F_{n'',n}(u)=\int_0^\infty du\,F_{n,n'}(u)F_{n,n''}(u)=\delta_{n',n''}. \tag{13}$$

From (4) and (7) there follow the helpful integral relations [12]

$$F_{n',n}(u)=e^{-u/2}(\pi 2^{n+n'}n!n'!)^{-1/2}\int_{-\infty}^{+\infty}dt\,e^{-t^2}H_n\left(t+\sqrt{\frac{u}{2}}\right)H_{n'}\left(t-\sqrt{\frac{u}{2}}\right), \tag{14}$$

$$F_{n',n}(u)F_{m',m}(u)=\int_0^\infty dt\,F_{n',m'}(t)F_{n,m}(t)J_{n'-n-m'+m}(2\sqrt{ut}), \tag{15}$$

where $J_l(x)$ is a Bessel function.

In the special cases $n' = 0, 1, 2$, we have

$$F_{0,n}=(n!)^{-1/2}u^{n/2}e^{-u/2}, \quad F_{1,n}=(n!)^{-1/2}u^{(n-1)/2}e^{-u/2}(n-u), \quad F_{2,n}=(2n!)^{-1/2}u^{(n-2)/2}e^{-u/2}(u^2-2nu+n^2-n). \tag{16}$$

From the recursion relations for the Laguerre polynomials [11], we can obtain various recursion relations for $F_{n',n}(u)$. We give some of them:

$$\sqrt{(n+1)(n'+1)}F_{n'+1,n+1}+\sqrt{nn'}F_{n'-1,n-1}=(n+n'+1-u)F_{n',n}; \tag{17}$$

$$\sqrt{n}F_{n'-1,n}+\sqrt{n'}F_{n',n-1}=-lF_{n'-1,n-1}/\sqrt{u}, \quad \sqrt{n'}F_{n'-1,n}+\sqrt{n}F_{n',n-1}=-lF_{n',n}/\sqrt{u}, \quad \sqrt{n}F_{n'-1,n-1}-\sqrt{n'}F_{n',n}=\sqrt{u}F_{n'-1,n}; \tag{18}$$

$$\sqrt{n'}F_{n'-1,n}-\sqrt{n}F_{n',n-1}=-\sqrt{u}\left(F_{n',n}+2\frac{d}{du}F_{n',n}\right),\quad \sqrt{n}F_{n'-1,n}-\sqrt{n'}F_{n',n-1}=\sqrt{u}\left(F_{n'-1,n-1}-2\frac{d}{du}F_{n'-1,n-1}\right); \tag{1}$$

$$2\sqrt{n}(F_{n'-1,n-1}F_{n'-1,n}+F_{n',n}F_{n',n-1})=\sqrt{u}(F^2_{n',n-1}+F^2_{n'-1,n})-l(F^2_{n',n}+F^2_{n'-1,n-1})/\sqrt{u}; \tag{2}$$

$$2\sqrt{n/u}(F_{n'-1,n-1}F_{n'-1,n}-F_{n',n}F_{n',n-1})=F^2_{n'-1,n-1}-F^2_{n',n}+F^2_{n'-1,n}-F^2_{n',n-1}-\frac{d}{du}(F^2_{n',n}+F^2_{n'-1,n-1}); \tag{2}$$

$$\begin{aligned}4\sqrt{nn'}F_{n'-1,n}F_{n',n-1}&=\frac{l^2}{u}(F^2_{n',n}+F^2_{n'-1,n-1})-(n+n')(F^2_{n'-1,n}+F^2_{n',n-1}),\\ 4\sqrt{nn'}F_{n'-1,n-1}F_{n',n}&=(n+n')(F^2_{n'-1,n-1}+F^2_{n',n})-u(F^2_{n'-1,n}+F^2_{n',n-1}).\end{aligned} \tag{2}$$

Here and below, $l=n'-n$.

The behavior of $F_{n',n}(u)$ at small and large u follows from the asymptotic behavior [11] for the Laguerre polynomials.

For $u\ll(1+|l|)/\min(n, n')$ the principal term in the Laguerre polynomial is the one containing the lowest power of u. Then

$$F_{n',n}(u)=\frac{(-)^{(l+|l|)/2}}{|l|!}Cu^{|l|/2}u^{-u/2},\quad C=\sqrt{\frac{[\max(n,n')]!}{[\min(n,n')]!}}. \tag{23}$$

In the interval $u\ll(1+|l|)/\nu$ in (23) (here and below $\nu\equiv n+n'+1$) we can successively set $e^{-u}=1$, and we then obtain the well-known formula of [6]. If $n\ll n'$ or $n'\ll n$, the given interval is much narrower than the interval of applicability of (23); if $|l|\sim n\sim n'$, the two intervals coincide.

For $u\ll|l|+1$, the substitution $u=t^2/2\nu$ reduces Eq. (12) to the Bessel equation. Using (23) to determine the normalization factor, we obtain

$$F_{n',n}(u)=C(2/\nu)^{|l|/2}J_l(-\sqrt{2\nu u}). \tag{24a}$$

In particular, for $|l|\ll\nu$ we have the well-known [6] result

$$F_{n'n}(u)=J_l(-\sqrt{2\nu u}). \tag{24b}$$

For $u\gg(\sqrt{n}+\sqrt{n'})^2$, we obtain

$$F_{n',n}(u)=(-)^{n'}(n!n'!)^{-1/2}u^{(n+n')/2}e^{-u/2}. \tag{25}$$

For $n=0$ or (and) $n'=0$, formulas (23) and (25) are identical and are exact (see (16)) in the entire range of u.

D. Quasiclassical Expressions for the Functions $F_{n',n}(u)$. Equation (12) is equivalent to the equation [6]

$$\frac{d^2}{du^2}(\sqrt{u}F_{n',n}(u))-f(u)\sqrt{u}F_{n',n}(u)=0, \tag{26}$$

where

$$f(u)=\frac{l^2-1}{4u^2}-\frac{\nu}{2u}+\frac{1}{4}=\frac{1}{4u^2}(u-u_1)(u-u_2),$$

and $u_{1,2}=\nu\mp(\nu^2-l^2+1)^{1/2}$ are the roots of the equation $f(u)=0$.

Since $n, n'\geq 0$, it follows that $u_2>0$; $u_1>0$ for $|l|>1$, $u_1=0$ for $|l|=1$ and $u_1<0$ for $l=0$. For problems in which values $n, n'\gg 1$ are important, it is convenient to solve Eq. (26) in the quasiclassical approximation. The quasiclassical turning points are the points $u=u_{1,2}\approx(\sqrt{n}\mp\sqrt{n'})^2$.

In the classically forbidden regions $0<u<u_1$ and $u>u_2$ and not too near the turning points (for $u_1-u\gg(u_1^2/\Delta)^{1/3}$ and $u-u_2\gg(u_2^2/\Delta)^{1/3}$, respectively, where $\Delta=u_2-u_1$), the standard procedure for finding quasiclassical solutions using (23) and (25) to determine the normalization constants leads to the expressions

$$F_{n',n}(u<u_1)=(-)^{(l+|l|)/2}(4\pi|p|u)^{-1/2}e^{|p|u-\Phi}, \tag{27a}$$

$$F_{n',n}(u>u_2)=(-)^{n'}(4\pi|p|u)^{-1/2}e^{-|p|u-\Phi}, \tag{27b}$$

$$\Phi=\frac{|l|}{2}\ln\frac{(\sqrt{u_1|u_2-u|}+\sqrt{u_2|u_1-u|})^2}{u\Delta}+\frac{\nu}{2}\ln\frac{(\sqrt{|u_2-u|}-\sqrt{|u_1-u|})^2}{\Delta}, \tag{28}$$

where $p=p(u)=\sqrt{-f(u)}$.

In the neighborhoods of the turning points for $|u - u_1| \ll u_1, \Delta$ and for $|u - u_2| \ll \Delta$, introducing the variables $\xi_{1,2} = \mp(u-u_{1,2})(\Delta/4u_{1,2}^2)^{1/3}$ and determining the normalization constants by means of (27), we obtain

$$F_{n',n}(u) = (-)^{(l+|l|)/2}(2/\pi)^{1/2}(2u_1\Delta)^{-1/6}\Phi(\xi_1), \tag{29a}$$

$$F_{n',n}(u) = (-)^{n'}(2/\pi)^{1/2}(2u_2\Delta)^{-1/6}\Phi(\xi_2), \tag{29b}$$

where $\Phi(\xi)$ is the Airy function. With allowance for the relation $\Phi(\xi>0) = K_{1/3}(2\xi^{3/2}/3)(\xi/3\pi)^{1/2}$ ($K_{1/3}(y)$ is a MacDonald function) we obtain from (29) the well-known expressions [6] for $F_{n',n}$ in the classically forbidden regions in the neighborhoods of the turning points. Note that the solutions (27a) and (29a) have meaning only for $|l| \gg 1$. Otherwise, in the region $u \lesssim u_1$ the quasiclassical solution is invalid, since the characteristic scales of variation of $F_{n',n}(u)$ in this region become comparable with its dimensions; however, at the same time formula (24b) holds.

In the classically allowed interval $u_1 < u < u_2$ not too near the turning points (for $u-u_1 \gg (u_1^2/\Delta)^{1/3}$ and $u_2-u \gg (u_2^2/\Delta)^{1/3}$) we obtain

$$F_{n',n}(u) = (\pi p u)^{-1/2}\cos(pu-\varphi), \quad \varphi = \frac{\nu}{2}\arcsin\frac{u_1+u_2-2u}{\Delta} - \frac{|l|}{2}\arcsin\frac{2u_1u_2-uu_1-uu_2}{u\Delta} - \frac{\pi}{4}(\nu+|l|+2l-1). \tag{30}$$

It can be shown that in the interval $0 < u < \infty$ the function $F_{n',n}(u)$ has $\min(n, n')$ nodes, and in the quasiclassical approximation all nodes are in the classically allowed region. In accordance with (30), $F_{n',n}^2(u)$ averaged over the oscillations is in this region equal to

$$\langle F_{n',n}^2(u)\rangle = \pi^{-1}(u-u_1)^{-1/2}(u_2-u)^{-1/2}. \tag{31}$$

In many problems [13-15], one is interested in the cases $n \gg n'$ and $n' \gg n$. At the same time, the smaller of the indices may be either larger or of the order of 1. To be specific, suppose $n \gg n'$ (for $n' \gg n$, see (8)). Then the distance Δ between the turning points $u_{1,2} \approx n \mp \sqrt{n(4n'+2)}$ satisfies the condition $\Delta \ll u_{1,2}$. By the substitution $\xi = (n-u)/\sqrt{n}$ for $|\xi| \ll \sqrt{n}$ Eq. (26) is reduced to the equation for the parabolic cylinder functions $D_{n'}(\xi)$ (which for integral n' can be expressed in terms of Hermite polynomials) and with allowance for (13) for the determination of the normalization factor and (25) for the determination of its sign gives the well-known expressions [13]

$$F_{n',n}(u) = \frac{D_{n'}(\xi)}{(2\pi n)^{1/4}(n'!)^{1/2}} = (2\pi n)^{-1/4}(2^{n'}n'!)^{-1/2}e^{-\xi^2/4}H_{n'}\left(\frac{\xi}{\sqrt{2}}\right). \tag{32}$$

Note that in the case $n \gg n'$ when $F_{n',n}^2(u)$ is integrated over u with a function that varies smoothly in the region $|u-n| \leqslant \sqrt{n(n'+1)}$, we can use the formula [16] $F_{n',n}^2(u) \approx n^{-1}\delta(1-u/n)$.

The formulas obtained in this section, together with Eqs. (23)-(25), make it possible to calculate $F_{n',n}(u)$ for all values of u if at least one of the indices n or n' is much greater than unity. The regions of applicability of the given expressions partly overlap, and in the regions of overlapping they give the same result.

E. Summation of Quadratic Combinations of $F_{n',n}(u)$. From the integral representation (14), we obtain

$$\sum_{n''=0} F_{n',n''}(u)F_{n,n''}(u) = \sum_{n''=0} F_{n'',n'}(u)F_{n'',n}(u) = \delta_{n,n'}, \tag{33}$$

and by means of (15) we obtain the well-known [6] formula

$$\sum_{n'=0}^{\infty} F_{n',n}^2(u)e^{i(n'-n)\varphi} = e^{iu\sin\varphi}F_{n,n}\left(4u\sin^2\frac{\varphi}{2}\right), \tag{34}$$

from which there follow the sum rules [6, 17]

$$\sum_{n'=0}^{\infty} F_{n',n}^2(u) = 1, \quad \sum_{n'=0}^{\infty}(n'-n)F_{n',n}^2(u) = u, \quad \sum_{n'=0}^{\infty}(n'-n)^2F_{n',n}^2(u) = u^2+2nu+u \quad \text{etc.} \tag{35}$$

When various processes in a nonrelativistic nondegenerate plasma are considered [9, 18-20], it is necessary to sum quadratic combinations of $F_{n',n}$ with factor $\exp(-nb)$ for fixed value of $l = n'-n$. Such summation can be done by means of the relation

$$\sum_{n=0}^{\infty} F_{n',n}(u)F_{n',n}(v)e^{-nb} = \left(2\,\mathrm{sh}\frac{b}{2}\right)^{-1} CI_l(\zeta), \tag{36}$$

which follows from the corresponding sum for the Laguerre polynomials [11]. In (36), $I_l(\zeta)$ is a modified Bessel function, and

$$\zeta=(uv)^{1/2}/\mathrm{sh}\frac{b}{2}, \quad C=\exp\left[\frac{b}{2}(l+1)-\frac{u+v}{2}\,\mathrm{cth}\frac{b}{2}\right]. \tag{37}$$

We give some sums:

$$\sum_{n=0}^{\infty} F_{n',n}(u)\frac{d}{dv}F_{n',n}(v)e^{-nb} = \left(2\,\mathrm{sh}\frac{b}{2}\right)^{-2} C\left[\sqrt{\frac{u}{v}}I_l'(\zeta)-\mathrm{ch}\frac{b}{2}I_l(\zeta)\right], \quad I_l'(\zeta)\equiv\frac{d}{d\zeta}I_l(\zeta); \tag{38}$$

$$\sum_{n=0}^{\infty}\sqrt{nn'}F_{n'-1,n}(u)F_{n',n-1}(u)e^{-nb} = \left(4\,\mathrm{sh}^2\frac{b}{2}\right)^{-1} u\exp\left[\frac{b}{2}l-u\,\mathrm{cth}\frac{b}{2}\right]\left[I_l'(\chi)-\mathrm{ch}\frac{b}{2}I_l(\chi)\right], \quad \chi=u\Big/\mathrm{sh}\frac{b}{2}. \tag{39}$$

3. Transport of Heat and Charge by Relativistic Electrons at Right Angles to a Magnetic Field

Using the above formulas, we can readily obtain simple general expressions for the transverse transport coefficients in a relativistic electron gas with quantizing magnetic field B.

We consider an electron gas in which there are weak gradients of the temperature T and the chemical potential μ, and also a weak electric field E, all at right angles to B. Then the conduction current density j and the heat flux q (see, for example, [1]) have the form

$$\mathbf{j}=\sigma_\perp\mathcal{E}+\sigma_\wedge\mathbf{b}\times\mathcal{E}+\lambda_\perp\nabla T+\lambda_\wedge\mathbf{b}\times\nabla T, \quad \mathbf{q}=-T\lambda_\perp\mathcal{E}-T\lambda_\wedge\mathbf{b}\times\mathcal{E}-\varkappa_\perp\nabla T-\varkappa_\wedge\mathbf{b}\times\nabla T. \tag{40}$$

Here, $\mathcal{E}=\mathbf{E}+\nabla\mu/e$, $\mathbf{b}=\mathbf{B}/B$; $\sigma_\perp$, $\lambda_\perp$, $\varkappa_\perp$ are the transport coefficients that determine the transport of heat and charge along the vectors $\mathcal{E}$ and ∇T; $\sigma_\wedge$, $\lambda_\wedge$, $\varkappa_\wedge$ determine the transport at right angles to these vectors and B. The quantities $\sigma_\perp$ and $\sigma_\wedge$ are the transverse and Hall conductivities, respectively, and the remaining coefficients determine the thermal conductivity and the specific thermoelectric force in the well-known manner (see, for example, [1]).

As a rule, the effects due to the quantizing nature of the magnetic field are manifested for $\omega_B^*\gg\nu_{\mathrm{eff}}$, where ω_B^* is the characteristic gyrofrequency of the electrons, and ν_{eff} is the effective frequency of electron collisions. In this case, $\sigma_\wedge$, $\lambda_\wedge$, $\varkappa_\wedge$ do not depend on the electron scattering mechanism and can be expressed in a universal manner in terms of the thermodynamic characteristics of the electron gas (see, for example, [1,21]; in particular, $\sigma_\wedge=en_e c/B$ and $\lambda_\wedge=cS/B$, where n_e and S are the electron concentrations and entropy per unit volume). Thus, it is sufficient to consider $\sigma_\perp$, $\lambda_\perp$, $\varkappa_\perp$.

We restrict ourselves to the frequently encountered case when the electron collisions can be regarded as elastic (this is the case if the characteristic change in the energy of an electron in a collision satisfies $\Delta\varepsilon\ll T$). Under these assumptions and for $\omega_B^*\gg\nu_{\mathrm{eff}}$ we have in the Born approximation (see, for example, [1])

$$\begin{Bmatrix}\sigma_\perp\\ \lambda_\perp\\ \varkappa_\perp\end{Bmatrix} = -\sum_{\alpha\alpha'}\frac{\partial f_0}{\partial\varepsilon}W_{\alpha\alpha'}\frac{(y_B-y_{B'})^2}{2}\begin{Bmatrix}e^2\\ e(\varepsilon-\mu)T^{-1}\\ (\varepsilon-\mu)^2T^{-1}\end{Bmatrix} \tag{41}$$

$$f_0=(e^{(\varepsilon-\mu)/T}+1)^{-1}, \quad W_{\alpha\alpha'}=\frac{2\pi}{\hbar}\delta(\varepsilon_\alpha-\varepsilon_{\alpha'})|V_{\alpha\alpha'}|^2. \tag{42}$$

Here, $W_{\alpha\alpha'}$ is the probability (in unit time) of an electron scattered on the potential V(r) making a transition from the state $\alpha=(p_z, p_x, n, s, \eta=1)$ to $\alpha'=(p_z', p_x', n', s', \eta=1)$ with shift of the y coordinate of the center of gyration of the electron by $y_B'-y_B$ (see Subsection 2A), $\varepsilon_\alpha=\varepsilon_{\alpha'}=\varepsilon$.

The matrix element $V_{\alpha\alpha'}$ of the potential V(r) can be calculated by means of (5) by expanding V(r) in a Fourier integral. Under the assumption that V(r) does not depend on the orientation of the electron's spin, the summation with respect to the spin variables s and s' in (41) can be readily performed by means of (11) and simplified by means of (22). The energy conservation law in (42) is satisfied by two values of p_z', which correspond to the two possible scattering channels. We denote the channel in which p_z and p_z' have

the same sign (forward scattering) by +, and the channel in which p_z and p_z' have opposite signs (backward scattering) by −. The momentum component $\hbar q_z = p_z' - p_z$ transferred during the scattering in these channels is

$$\hbar|q_z^{\pm}| = |p_n \mp p_{n'}|, \quad cp_n = (\varepsilon^2 - m^2c^4 - 2mc^2\hbar\omega_B n)^{1/2}. \tag{43}$$

Finally, from (41) we obtain

$$\begin{Bmatrix} \sigma_\perp \\ \lambda_\perp \\ \varkappa_\perp \end{Bmatrix} = -\frac{n_i m^3 \omega_B}{8\pi^3\hbar^4} \int_{mc^2}^{\infty} d\varepsilon \frac{\partial f_0}{\partial \varepsilon} \begin{Bmatrix} e^2 \\ e(\varepsilon-\mu)T^{-1} \\ (\varepsilon-\mu)^2 T^{-1} \end{Bmatrix} \sum_{\pm, n, n'} \int_0^{\infty} \frac{u\,du}{p_n p_{n'}} \times$$

$$\times |V_q|^2 \chi_{n',n}^{\pm}(u), \quad \chi_{n',n}^{\pm}(u) = \left[\left(\frac{\varepsilon}{mc^2}\right)^2 - \left(\frac{\hbar q_z^{\pm}}{2mc}\right)^2\right][F_{n',n}^2(u) + F_{n'-1,n-1}^2(u)] - \frac{u}{2}\frac{\hbar\omega_B}{mc^2}[F_{n',n-1}^2(u) + F_{n'-1,n}^2(u)], \tag{44}$$

where, as in (9), $u = \hbar q_\perp^2/2m\omega_B$; V_q is the Fourier transform of $V(r)$, and n_i is the concentration of the scatterers. The summation in (44) is over the values of n and n′ for which $\varepsilon > \varepsilon_n$ and $\varepsilon > \varepsilon_{n'}$, where $\varepsilon_n = (m^2c^4 + 2mc^2\hbar\omega_B n)^{1/2}$. For $\varepsilon = \varepsilon_n$ and $\varepsilon = \varepsilon_{n'}$, the integrand in (44) contains integrable square root singularities due to the singularities of the number density of electron states in the magnetic field. Exceptions are the terms with $n = n'$, in which the singularities degenerate into power singularities. To avoid logarithmic divergence of the integrals over ε, the domain of integration in these terms must be restricted to values $\varepsilon > \varepsilon_n + \gamma_n$ (see, for example, [1]; for more consistent elimination of divergence, see [22]). Small $\gamma_n \ll T$, on which the transport coefficients depend very weakly, can be determined by physical considerations and can be due to various factors, of which the main ones are (see, for example, [1, 22, 19]) a certain inelasticity of the electron scattering, collisional broadening of the Landau levels, and a possible violation of the applicability of the Born approximation for $\varepsilon \approx \varepsilon_n$. For $\hbar q \ll mc$ and $(\varepsilon - mc^2) \ll mc^2$

$$\chi_{n',n}^{\pm}(u) \approx F_{n'-1,n-1}^2(u) + F_{n',n}^2(u), \tag{45}$$

and then formulas (44) go over into the well-known nonrelativistic expressions (see, for example, [1]) for the transport coefficients. Note that in contrast to the ideal electron gas considered here, in which the spin splitting of the Landau levels is equal to the distance between the levels, the most interesting case in solid-state physics (see, for example, [23]) is when the spin splitting is negligibly small. The transition to such a case in (45) is trivial and reduces to the substitution $F_{n'-1,n-1} \to F_{n',n}$.

For a nondegenerate nonrelativistic gas in (44)

$$f_0 \approx e^{(\mu-\varepsilon)/T} = \exp\left\{\frac{\mu_0}{T} - \frac{p_z^2}{2mT} - bn\right\}, \quad e^{\mu_0/T} = \frac{(2\pi\hbar)^2}{m\omega_B}(2\pi mT)^{-1/2} n_e \operatorname{th}\frac{b}{2}, \tag{46}$$

where $b = \hbar\omega_B/T$. We use (45) and by means of (36) carry out a summation over n for fixed values of $l = n' - n$. We obtain

$$\sigma_\perp = \frac{e^2 n_i m^2 b}{16\pi^3\hbar^5} e^{\mu_0/T} \operatorname{cth}\frac{b}{2} \sum_{l=-\infty}^{\infty} \int_0^{\infty} d\zeta\, \zeta^{-1/2}(\zeta + \hbar\omega_B|l|)^{-1/2} \exp\left(-\frac{\zeta}{T} - \frac{b|l|}{2}\right) \int_0^{\infty} du\, u |V_q|^2 \exp\left(-u \operatorname{cth}\frac{b}{2}\right) I_l\left(u \Big/ \operatorname{sh}\frac{b}{2}\right),$$

$$\lambda_\perp = \frac{1}{e}\left[\frac{\partial}{\partial T}(T\sigma_\perp)\right]_{\mu_0}, \quad \varkappa_\perp = \frac{1}{e^2}\left\{\frac{\partial}{\partial T}\left[T^2\frac{\partial}{\partial T}(T\sigma_\perp)\right]\right\}_{\mu_0}; \tag{47}$$

and in the term with $l = 0$ in accordance with what we have said above the lower limit of integration with respect to ζ must be replaced by γ.

Equations (44) also strongly simplify for a strongly degenerate relativistic electron gas. In this case,

$$\lambda_\perp = \frac{\pi^2}{3e} T \frac{\partial}{\partial \mu}\sigma_\perp(\mu), \quad \varkappa_\perp = \frac{\pi^2}{3e^2} T\sigma_\perp, \tag{48}$$

and $\sigma_\perp$ is given by (44), in which it is sufficient to set $\partial f_0/\partial\varepsilon = -\delta(\varepsilon - \mu)$, so that the integration over ε can be performed. It should be noted that the conductivity $\sigma_\perp$ in a relativistic degenerate gas for Coulomb scattering of electrons by ions was considered in [24]. Although the general approach in [24] is basically correct (only the screening radius of the Coulomb potential was incorrectly chosen [25]), the expressions obtained there were not simplified and were given in a form far more cumbersome than (44). In [24], the numerical values of $\sigma_\perp$ are strongly overestimated (see, for example, [25]), presumably because of inaccuracies in the numerical calculations.

The expressions (44) also simplify in the limit of a magnetic field so strong that the plasma electrons populate only the Landau ground level. Then in the sum over n, n′ there remains the single term with n = n′ = 0, and by virtue of (16)

$$\chi_{0,0}^{\pm}(u)=\left[\left(\frac{\varepsilon}{mc^2}\right)^2-\left(\frac{\hbar q_z^{\pm}}{2mc}\right)^2\right]e^{-u}. \tag{49}$$

In the nonrelativistic limit, these formulas are well-known (see, for example, [1]); in the ultrarelativistic limit for the special case of a short-range potential with $|V_q|^2 = \text{const}$ a formula for $\sigma_\perp$ was obtained in the recent paper [26] (note that the value of $\sigma_\perp$ in [26] is two times too small).

Finally, we consider the transition to the limit of a nonquantizing magnetic field. In this case, the contribution to (44) is made by values $n, n' \gg 1$. We replace the sum over n and n′ by a corresponding double integral and go over to the variables of integration $P_z = (p_z' + p_z)/2$ and $q_z = (p_z' - p_z)/\hbar$. Then in (44),

$$\sum_{\pm,n,n'}\frac{1}{p_n p_{n'}}\chi_{n,n'}^{\pm}(u)\approx\frac{2}{\hbar m^2\omega_B^2}\int dq_z\int dP_z F_{n',n}^2(u)\left[\left(\frac{\varepsilon}{mc^2}\right)^2-\left(\frac{\hbar q}{2mc}\right)^2\right], \tag{50}$$

where $q^2 = q_z^2 + q_\perp^2$. For $F_{n',n}^2(u)$ it is sufficient to use the expression (31), which is rewritten in the form

$$F_{n',n}^2(u)=\frac{m\omega_B}{\pi}\left[q_\perp^2\left(p^2-\frac{\hbar^2q^2}{4}\right)-P_z^2q^2\right]^{-1/2},\qquad cp=(\varepsilon^2-m^2c^4)^{1/2}. \tag{51}$$

In (50) and (51), P_z vary in the range $P_z^2<(q_\perp/q)^2[p^2-(\hbar q/2)^2]$, where $\hbar q < 2p$, in which the function (31) is nonzero. In (50), we integrate over P_z in this range and use the fact that by virtue of symmetry considerations we can in (44) make the substitution $u\to\hbar q^2/3m\omega_B$. As a result, we find that in (44)

$$\sum_{\pm,n,n'}\frac{1}{p_n p_{n'}}\int_0^\infty du\, u\chi_{n,n'}^{\pm}(u)|V_q|^2\to\frac{4}{3}\frac{\hbar}{m^3\omega_B^3}\int_0^{2p/\hbar}q^3\,dq\,|V_q|^2[(\varepsilon/mc^2)^2-(\hbar q/2mc)^2], \tag{52}$$

which corresponds to the well-known expressions for a nonquantizing magnetic field (see, for example, [27, 28]).

Thus, we have obtained the simple expressions (44) for the transport coefficients in a relativistic electron gas. The use of these expressions is of great interest in not only astrophysics – for investigating the kinetic properties of the matter of neutron stars – but also in solid-state physics. In particular, for the latter great interest attaches [1] to the dependence of $\sigma_\perp$, $\lambda_\perp$, and $\varkappa_\perp$ on B and T for different electron scattering mechanisms in the limit of a quantizing magnetic field when the electrons populate only the Landau ground level. For a nonrelativistic electron gas (parabolic dispersion law), these dependences are given in [1]. In Table 1, we give the dependence of $\sigma_\perp$, $\lambda_\perp$, and $\varkappa_\perp$ on B and T in the limit of a quantizing magnetic field for different scattering mechanisms in the case of a linear dispersion law ($\varepsilon \approx p_z c \gg mc^2$). This case

TABLE 1

Scattering mechanism	$\lvert V_q\rvert^2$	Nondegenerate electrons, $\hbar\omega_B^* \gg T$			Degenerate electrons, $\hbar\omega_B^* \gg \mu$		
		$\sigma_\perp$	$\lambda_\perp$	$\varkappa_\perp$	$\sigma_\perp$	$\lambda_\perp$	$\varkappa_\perp$
Acoustic phonons:							
low T	$A_1 q$	$T^{-1}B^{1/2}$	$T^{-1}B^{3/2}$	$B^{1/2}$	$B^{3/2}$	$TB^{3/2}$	$TB^{3/2}$
high T	$A_1(2T/\hbar s)$	const	const	T	TB	T^2B^2	T^2B
Piezoelectric phonons:							
low T	A_3q^{-1}	$T^{-1}B^{-1/2}$	$T^{-1}B^{-1/2}$	$B^{-1/2}$	$B^{1/2}$	$TB^{3/2}$	$TB^{1/2}$
high T	$A_3q^{-2}(2T/\hbar s)$	B^{-1}	B^{-1}	TB^{-1}	T	T^2B	T^2
Optical phonons							
high T	$A_4q^{-2}(2T/\hbar\omega_{opt})$	B^{-1}	B^{-1}	TB^{-1}	T	T^2B	T^2
Point defects	A_2	T^{-1}	T^{-1}	const	B	TB^2	TB
Ionized impurities	$A_5(q^2+q_s^2)^{-2}$	$T^{-1}B^{-2}$	$T^{-1}B^{-2}$	B^{-2}	B^{-1}	T	TB^{-1}

is of interest for a number of semiconductors and metals, in which the behavior of the electrons is described by a Dirac equation (with effective mass and effective velocity of light; see, for example, [2]). In Table 1, as in [1], s is the velocity of sound, ω_{opt} is the frequency of an optical phonon, $A_{1,2,3,4,5}$ are constants, q_s is the screening constant, $\omega_B^*=eB/m^*c$ ($m^*\sim Tc^{-2}$ for nondegenerate electrons, and $m^*\sim\mu c^{-2}$ for degenerate electrons).

The ratios of the coefficients $\sigma_\perp$, $\lambda_\perp$, $\varkappa_\perp$ for linear law (see Table 1) to the same coefficients for quadratic law [1] in the case of nondegenerate statistics are proportional to $T^{-1/2}$. In the case of degenerate statistics, the corresponding $\sigma_\perp$ and $\varkappa_\perp$ ratios are proportional to B^{-2}, and the $\lambda_\perp$ ratio to B^{-3}.

Note that in the ultrarelativistic case the scattering in – channel is strongly suppressed.

Note that the original expressions (40) are written down in a coordinate system in which the matter as a whole is at rest. This system is generally used to study kinetic phenomena. For the use of (40) to study motions of matter with relativistic velocities (for astrophysical applications) see, for example, [29].

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